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DETOUR PEBBLING IN GRAPHS

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ABSTRACT. Given a distribution of pebbles on the vertices of a connected graph G, a pebbling move is defined as the removal of two pebbles from some vertex and the placement of one of those pebbles on an adjacent vertex. The *pebbling number of a vertex* v in a graph G is the smallest number $f(G,v)$ such that for every placement of $f(G,v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. The *pebbling number* of G is the smallest number, $f(G)$ such that from any distribution of $f(G)$ pebbles, it is possible to move a pebble to any specified target vertex by a sequence of pebbling moves. Thus $f(G)$ is the maximum value of $f(G,v)$ over all vertices v. In this paper, we define detour pebbling number of a vertex $(f^*(G),v)$ and detour pebbling number for a graph G. Also we compute the detour pebbling numbers for some standard graphs.

1. INTRODUCTION

Graph Pebbling was first suggested by Lagarias and Sags for finding an intutive proof for a number theoritical conjecture. Chung [1] used this graph pebbling tool successfully and introduced graph pebbling into the literature. Some of the applications of graph pebbling can be found in computational complexity, compiler theory, graph searching, sparse matrix factorization, and computational geometry. Hulbert published a survey of graph pebbling [4]. For the past two decades, graph pebbling is an essential tool for transportation of consumable resources.

Graph pebbling is an optimization model of the network for the transport of resources consumed in transit. As it moves from one place to another, electricity,

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heat, or other resources may dissipate, oil tankers may use some of the oil it transports, information may be lost as it passes through its medium, or military forces may be lost when travelling through a area.

In pebbling, a pebbling move is defined as a removal of two pebbles from one vertex and placing a pebble on an adjacent vertex. It is an intersting question and eager of thirst that how many pebbles are needed to reach any vertex of the graph by using a sequence of pebbling moves for any initial distribution of pebbles on the graph. Graph pebbling model resembles as the network optimization for military troops or any other resources that will vanish gradually when it transmits. However if there is any blocks or any distrubance while transmitting through a particular medium then there is a natural question for guarenteed reaching of any node with the available resources in a connected environment. Thus it motivates the concept of detour in graph pebbling to ensure the reachability of any vertex in the connected graph.

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Consider a connected graph with fixed number of pebbles distributed on its vertices. A *pebbling move* consists of the removal of two pebbles from a vertex and placement of one of those pebbles at an adjacent vertex. The *pebbling number of a vertex* v in a graph G is the smallest number $f(G, v)$ such that for every placement of $f(G, v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. Then the *pebbling number* of G is the smallest number, $f(G)$ such that from any distribution of $f(G)$ pebbles, it is possible to move a pebble to any specified target vertex by a sequence of pebbling moves. Thus $f(G)$ is the maximum value of $f(G, v)$ over all vertices v. The pebbling number of a graph was extended to $t-$pebbling number of a graph and there are so many articles with regard to $t-$pebbling numbers [2], [3], [6], [7], [8].

In this paper we define detour pebbling number of a graph and compute the detour pebbling number for some standard graphs. This paper organises as follows: In the second section, we define the detour pebbling. In the third section we compute the detour pebbling number for some standard graphs like path, cycle, complete graph, star graph, fan graph and wheel graphs.

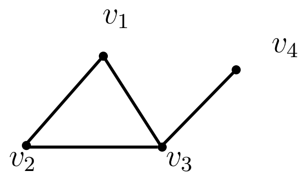

FIGURE 1. A Graph with 4 Vertices.

2. DETOUR PEBBLING

For any pair of vertices $u, v \in V(G)$, the standard distance $d(u, v)$ is the length of the shortest $u - v$ path and the detour distance $d^*(u, v)$ is the length of the longest $u - v$ path between u and v in G. Chartrand et.al, [5], were the first to introduce the notion of detour distances in graphs and was subsequently explored by many researchers.

Definition 2.1. *A detour pebbling number of a vertex v of a graph G is the smallest number $f^*(G, v)$ such that for any placement of $f^*(G, v)$ pebbles on the vertices of G it is possible to move a pebble to v using a detour path by a sequence of pebbling moves. The detour pebbling number of a graph is denoted by $f^*(G)$, is the maximum $f^*(G, v)$ over all the vertices of G.*

Consider the Figure 1 graph with 4 vertices.

We can reach v_1 from v_4 using the shortest path v_1, v_3, v_4. But the detour path from v_1 to v_4 is v_1, v_2, v_3, v_4. The detour path for v_1 to v_3 is v_1, v_2, v_3 and so on. Thus the detour pebbling number for v_1 is $f^*(G, v_1) = 8$. By placing 7 pebbles on v_4 we cant reach v_1. Also however 8 pebbles are placed on the vertices of the graph the detour path from v_1 to any vertex of G has length at most 3, using $2^3 = 8$ pebbles we can reach v_1. Similarly the detour pebbling number for v_2 is $f^*(G, v_2) = 8$, for v_3 is $f^*(G, v_3) = 4$ and $f^*(G, v_4) = 8$. The maximum over all vertices of G is 8. Thus the detour pebbling number for G is $f^*(G) = 8$.

Theorem 2.1. *[1] The pebbling number for a path P_n with n vertices, $f(P_n) = 2^{n-1}$.*

3. DETOUR PEBBLING FOR SOME STANDARD GRAPHS

In this section we provide the detour pebbling number for some standard graphs like paths, cycles, complete grpahs, star graphs, fan graphs and wheel graphs.

Theorem 3.1. *For any path P_n with n vertices, the detour pebbling number is $f^*(P_n) = 2^{n-1}$.*

Proof. Placing $2^{n-1}-1$ pebbles on the vertex v_n, we cannot reach v_1. Thus $f^*(P_n) \geq 2^{n-1}$. To prove the sufficient part, let D be any distribution of 2^{n-1} pebbles on the vertices of P_n. Let any end vertex be the target vertex. Without loss of generality v_1 be the target vertex. The detour distance from v_1 to any vertex in P_n is at most $n-1$, using 2^{n-1} pebbles on the path (by Theorem 2.1) we can move a pebble to the target. Let v_k be the target vertex, where $1 < k < n$.

Since the detour distance of v_k to v_i where $k < i \leq n$ is at most $n-k$, by Theorem 2.1, if $p(< P_n - \{v_k, v_{k+1}, \dots, v_n\} >)$ contains at least 2^{n-k} pebbles we can reach the target. Otherwise $p(< \{v_1, v_2, \dots, v_k\} >)$ has at least 2^{k-1} pebbles. Since the detour distance from v_k to v_j for any $1 \leq j < k$ is at most $k-1$, we can use 2^{k-1} pebbles to reach the target. □

Theorem 3.2. *For cycles C_n with n vertices, the detour pebbling number is $f^*(C_n) = 2^{n-1}$.*

Proof. Let the vertex set of C_n be denoted as $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Placing $2^{n-1}-1$ pebbles on the vertex v_n we cannot reach v_1, since the detour path from v_n to v_1 has length $n-1$. Thus $f^*(C_n) \geq 2^{n-1}$. For proving the sufficiency part, let D be any distribution of 2^{n-1} pebbles on the vertices of C_n. Let v_1 be the target vertex. Assume $p(v_1) = 0$. We now consider the spanning subpath P, which is the detour path from our target v_1 to v_n as $P : v_1, v_2, v_3, \dots, v_n$. Since 2^{n-1} pebbles are distributed on this path of length $n-1$, by Theorem 3.1, we can move a pebble to v_1 through the detour path. By symmetry we can pebble any vertex in the graph C_n. □

Theorem 3.3. *For Complete graph K_n with n vertices $(n > 1)$, the detour pebbling number is $f^*(K_n) = 2^{n-1}$.*

Proof. Let the vertex set of K_n be denoted as $V(K_n) = \{v_1, v_2, v_3, \dots, v_n\}$. Placing $2^{n-1}-1$ pebbles on v_2 we cannot reach the vertex v_1, since the detour path from v_2 to v_1 has length $n-1$. Thus $f^*(K_n) \geq 2^{n-1}$. For proving the sufficiency part, let D be any distribution of 2^{n-1} pebbles on the vertices of the graph K_n. Let v_1 be the target vertex. Assume $p(v_1) = 0$. In the complete graph, we can see a detour path of length $n-1$ consists of the all vertices $\{v_n, v_{n-1}, v_{n-2}, \dots, , v_2, v_1\}$. Since 2^{n-1}

pebbles are distributed in these vertices by Theorem 3.1 we can reach the target. By symmetry we can reach all the vertices of the graph K_n. □

Theorem 3.4. *Let $K_{1,n}$ be an n-star where $n > 1$. The detour pebbling number for the $n-$star graph is $f^*(K_{1,n}) = n + 2$.*

Proof. Let the vertex set of $K_{1,n}$ be denoted as $V(K_{1,n}) = \{v_1, u_1, u_2, \ldots, u_n\}$, where $u_1, u_2, \ldots, u_n$ are the pendant vertices of the star graph. Placing 3 pebbles on u_n and each pebble on all other u_i's except v_1, we cannot reach u_1. Thus $f^*(K_{1,n}) \geq n + 2$. For proving the sufficiency part, let D be any distribution of $n+2$ pebbles on the vertices of the graph $K_{1,n}$. Let u_1 be the target vertex. Assume that $p(u_1) = 0$. Since $n + 2$ pebbles are distributed in $n + 1$ vertices, by Pigeon hole principle, there exits a vertex with at least 2 pebbles. Suppose $p(v_1) \geq 2$. Since there is only one path from v_1 to u_1 obviously it is the detour path and hence we can reach u_1. Therefore assume $p(v_1) \leq 1$. Then at least $n + 1$ pebbles are distributed in the remaining n pendant vertices. Thus either any one pendant vertex contains at least 4 pebbles or two pendant vertices contain at least 2 pebbles. Anyhow we can put two pebbles to the vertex v_1 and hence we can reach the target.

Let v_1 be the target vertex. Assume that $p(v_1) = 0$. Since $n + 2$ pebbles are distributed in the n pendant vertices, by Pigeon hole principle, there exits a pendant vertex with at least 2 pebbles. Since u_i to v_1 for some i, is a detour path, we are done. □

Theorem 3.5. *For the fan graph F_n with n vertices, the detour pebbling number is $f^*(F_n) = 2^{n-1}$.*

Proof. The fan graph $F_n = K_1 \cup P_{n-1}$. Let the vertex set of K_1 be denoted as $\{v_1\}$ and P_{n-1} be denoted as $V(P_{n-1}) = \{u_1, u_2, \ldots, u_{n-1}\}$. Thus the vertex set of F_n is $\{v_1, u_1, u_2, \ldots, u_{n-1}\}$. Placing $2^{n-1} - 1$ pebbles on the vertex u_{n-1} we cannot reach v_1 as the detour path from u_{n-1} to v_1 has length $n - 1$. Thus $f^*(F_n) \geq 2^{n-1}$. For proving the sufficiency part, let D be any distribution of 2^{n-1} pebbles on the vertices of the graph F_n. Let v_1 be the target vertex. Assume $p(v_1) = 0$. Then 2^{n-1} pebbles are distributed on the vertices $\{u_1, u_2, \ldots, u_{n-1}\}$, we can have a spanning path of length $n - 1$. Using this spanning path of length $n - 1$, by Theorem 3.1 we can reach the target. Let u_1 be the target vertex. Assume $p(u_1) = 0$. Since 2^{n-1} pebbles are distributed on the spanning subpath $P : u_2, u_3, \ldots, u_{n-1}, v_1, u_1$, which is the detour path from u_2 to v_2 of lenth $n-1$. By Theorem 3.1, using 2^{n-1} pebbles we can reach v_1. By symmetry we can reach all the vertices of the graph. $\square$

Theorem 3.6. *For the wheel graph W_n with $n+1$ vertices, the detour pebbling number is $f^*(W_n) = 2^n$.*

Proof. The Wheel graph $F_n = K_1 \cup C_n$. Let the vertex set of K_1 be denoted as $V(K_1) = \{v_1\}$ and C_n be denoted as $V(C_n) = \{u_1, u_2, \ldots, u_n\}$. Placing $2^n - 1$ pebbles on the vertex u_n, we cannot reach u_1 as the detour path from u_n to u_1 has length n. Thus $f^*(W_n) \geq 2^n$. For proving the sufficiency part, let D be any distribution of 2^n pebbles on the vertices of the graph W_n. Let v_1 be the target vertex. Assume $p(v_1) = 0$. Then 2^n pebbles are distributed on the vertices of the cycle C_n. Thus we can see the spanning subpath from u_n to v_1 as $P : u_n, u_{n-1}, \ldots, u_2, u_1, v_1$ which is a detour path of length n. Since this path consists of 2^n pebbles by Theorem 3.1 we can reach the target.

Let $u_k, 1 \leq k \leq n$ be the target vertex. Assume $p(u_k) = 0$. Then 2^n pebbles are distributed on the remaining vertices. Consider the spanning subpath from u_k to v_1 as $P : u_k, u_{k+1}, \ldots, u_n, u_1, \ldots, u_{k-1}, v_1$ which is a detour path of length n. Since this path consists of 2^n pebbles by Theorem 3.1 we can reach the target. By symmetry, we can reach any vertex. $\square$

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