ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.12, 10591–10612
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.12.45

DISTANCE AND DISTANCE LAPLACIAN SPECTRUM OF THE ZERO-DIVISOR GRAPH ON THE RING OF INTEGERS MODULO n

P. M. MAGI[1], SR. MAGIE JOSE, AND A. KISHORE

ABSTRACT. For a commutative ring R with non-zero identity, let $Z^*(R)$ denote the set of non-zero zero-divisors of R. The zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple undirected graph with all non-zero zero-divisors as vertices and two distinct vertices $x, y \in Z^*(R)$ are adjacent if and only if $xy = 0$. In this paper, we describe the computation of distance, distance Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ by exploring its combinatorial structure as the joined union of its induced subgraphs.

1. INTRODUCTION

In this paper G denotes a simple, finite, undirected and connected graph with vertex set $V(G)$ and edge set $E(G)$. The order of a graph G is the cardinality of $V(G)$. If u and v are distinct vertices in a graph G, $d_G(u, v)$ denotes the distance between u and v; which is the length of a shortest path between u and v. Clearly $d_G(u, u) = 0$ and $d_G(u, v) = \infty$ if there is no path between u and v. If $u \in V(G)$, the open neighborhood of u; denoted by $N_G(u)$ is the set of vertices adjacent to u in G. The cardinality of $N_G(u)$ is the degree of u. In a connected graph G, the transmission degree of a vertex v is defined as $Tr(v) = \sum_{u \in V(G)} d_G(u, v)$. The adjacency matrix, $A(G)$ of a graph G of order n is a $0-1$ matrix of order $n \times n$ with entries a_{ij} such that a_{ij} is 1, if the $i-$th and $j-$th vertices are adjacent, and 0 otherwise.

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 05C50, 15A18.

Key words and phrases. distance matrix, distance Laplacian matrix, zero-divisor graph, block matrix.

For a graph G, The Laplacian matrix is defined as $L(G) = A(G) - Deg(G)$, and signless Laplacian matrix of G is defined as $Q(G) = A(G) + Deg(G)$ where $Deg(G)$ is the diagonal matrix of degree of verices. Note that $L(G)$ and $Q(G)$ are positive semi definite matrices. The distance matrix of a simple connected graph G of order n is the symmetric matrix $D = (d_{ij})_{n\times n}$, where $d_{i,j}$ denotes the distance between two distinct vertices u_i and u_j. In [17], M. Aouchiche and P. Hansen, initiated the study of distance Laplacian and distance signless Laplacian. For a connected graph G, the distance Laplacian matrix is given by $D^L(G) = Tr(G) - D(G)$ and distance signless Laplacian matrix is $D^Q(G) = Tr(G) + D(G)$; where $Tr(G)$ is the diagonal matrix of vertex transmission of G.

Let $Z^*(R) = Z(R)\backslash(0)$ be the set of non-zero zero-divisors of a commutative ring R, with non zero identity. In [12], Beck introduced the zero-divisor graph $G(R)$ of a commutative ring R. Anderson and Livingston redefined the concept of zero-divisor graph and introduced the subgraph $\Gamma(R)$(of $G(R)$) as zero-divisor graph whose vertices are the non-zero zero-divisor of R. In [26], the authors described the structure of $\Gamma(\mathbb{Z}_n)$ as the join of pairwise disjoint induced subgraphs which are regular. In [22] Magi P.M and et. al. has described the analysis of the adjacency matrix and some graph parameters of $\Gamma(\mathbb{Z}_{p^n})$ and described the computation of the eigen values of $\Gamma(\mathbb{Z}_{p^n})$ exploring its structure as the generalized join of its induced subgraphs. This paper aims to find the distance, distance Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$.

This paper is organized in the following manner. In Section 2, some basic definitions and notations are given. The role of Fiedler's result and its generalization (Linear Algebra), to the computation of the distance related spectrum of the generalized join of regular graphs, is described in Section 3. In Section 4, the distance spectrum of $\Gamma(\mathbb{Z}_n)$ is investigated and illustrated with example and in particular the distance spectrum of $\Gamma(\mathbb{Z}_{p^k})$ is described. In Section 5, the distance Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^k}), k \geqslant 3$ is found .

2. BASIC DEFINITIONS AND NOTATIONS

In this paper K_n denotes a complete graph on n vertices. The complement of K_n is a null graph and is denoted by $\overline{K_n}$. A partition $\{V_1, V_2, \ldots, V_k\}$ of the vertex set of $V(G)$ is said to be an equitable partition, if any two vertices in V_i have the same number of neighbours in V_j for $1 \leqslant i \leqslant j \leqslant k$. In [24], Sabidussi has defined the generalized join of a family of graphs $\{Y_x\}_{x\in X}$, indexed by $V(X)$, as the graph Z with $V(Z) = \{(x,y) : x \in X, y \in Y_x\}$ and $E(Z) = \{((x,y),(x',y')) : (x,x') \in E(X)$ or else $x = x'$ and $(y,y') \in E(Y_x)\}$. Let G be a finite graph with vertices labeled as $1,2,3,\ldots,n$ and let $H_1, H_2, \ldots, H_n$ be a family of vertex disjoint graphs. The generalized join of $H_1, H_2, \ldots, H_n$ denoted by $G[H_1, H_2, \ldots, H_n]$ is obtained by replacing each vertex i of G by the graph H_i and inserting all or none of the possible edges between H_i and H_j depending on whether or not i and j are adjacent in G. ie, $Z = G[H_1, H_2, \ldots, H_n]$ is obtained by taking the union of $H_1, H_2, \ldots, H_n$ and joining each vertex of H_i to all vertices of H_j if and only if $ij \in E(G)$. Refer [14, 21], for the basic definitions of graph theory . The eigenvalues of a square matrix M, are the roots of the characteristic polynomial, $det(xI - M)$ and the spectrum of M is the multi set of all the eigenvalues of M counted with multiplicities. An eigenvalue of a matrix is simple, if its algebraic multiplicity is 1. For a real symmetric matrix, all eigenvalues are real and the algebraic multiplicity of each eigenvalue is same as its geometric multiplicity. A graph is said to be integral if all the eigenvalues are integers.

There are many matrices associated to a graph G. The characteristic polynomial of a graph G is given $\Phi(G,x) = det(xI - A)$, where A is the adjacency matrix of G, and the spectrum of G is denoted by $Spec(G)$. Similarly $SpecD(G)$, $SpecD^L(G)$ and $SpecD^Q(G)$ denote the spectrum of G related to the distance, distance Laplacian and the distance signless Laplacian matrix of G respectively. Let us denote $det(xI - D(G)), det(xI - D^L(G))$ and $det(xI - D^Q(G))$ by $\Phi_D(G;x); \Phi_{D^L}(G;x)$ and $\Phi_{D^Q}(G;x)$ respectively. The distance spectrum of a connected graph G on n vertices, is denoted as $\partial_1 \geqslant \partial_2 \geqslant \cdots \geqslant \partial_n$ and $\partial_1^L \geqslant \partial_2^L \geqslant \cdots \geqslant \partial_n^L$ denotes the distance Laplacian spectrum.

As usual, $\phi(n)$ is the number of positive integers less than n and relatively prime to n. In this paper, J denotes an all-one matrix and O denotes a zero matrix. $\mathbf{1}_n$ denotes the all-one column vector of order $n \times 1$, and I_n denotes the unit matrix of order n.

Domingos M Cardoso and et.al [4, 5] have contributed much to the exploration of the graph spectrum and energy of complicated structures; especially the generalized join of regular graphs. The well known Fiedler's result plays a major role towards this end. See [19].

A wide literature of zero-divisor graphs can be found in [1,2,3,7,8,20,23,25].

3. Fiedler's result and its generalisation applied in graph Theory

Lemma 3.1. *[19] Let A and B be symmetric matrices of orders m and n, respectively, with corresponding eigenpairs $(\alpha_i, \boldsymbol{u}_i)$, $i = 1, 2, \ldots, m$ and $(\beta_i, \boldsymbol{v}_i)$, $i = 1, 2, \ldots, n$, respectively. Suppose that $\|\boldsymbol{u}_1\| = 1 = \|\boldsymbol{v}_1\|$. Then, for each arbitrary constant ρ, the matrix*

$$C = \begin{bmatrix} A & \rho \boldsymbol{u}_1 \boldsymbol{v}_1^T \\ \rho \boldsymbol{v}_1 \boldsymbol{u}_1^T & B \end{bmatrix}$$

has eigenvalues $\alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_n, \gamma_1, \gamma_2$, where γ_1, γ_2, are the eigenvalues of the matrix $\widehat{C} = \begin{bmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{bmatrix}$

In [4, 5], D.M. Cardoso et.al have extended the Fiedler's result to more than two block diagonal symmetric matrices and it was applied to the exploration of spectra of the generalised join of regular graphs.

3.1. Generalization of Fiedler's result.

For $j \in \{1, 2, \ldots, k\}$, let M_j be an $m_j \times m_j$ symmetric matrix, with corresponding eigenpairs $(\alpha_{rj}, \mathbf{u}_{rj})$, $1 \leqslant r \leqslant m_j$. Moreover, for $q \in \{1, 2, \ldots, k-1\}$ and $l \in \{q+1, \ldots, k\}$, let $\rho_{q,l}$ be arbitrary constants. let $\widehat{\alpha}$ be the $k-$tuple

$$\widehat{\alpha} = (\alpha_{i_1,1}, \ldots, \alpha_{i_k,k}), \tag{3.1}$$

where each $\alpha_{i_j,j}$ is chosen from the elements of $\{\alpha_{1,j}, \ldots, \alpha_{m_j,j}\}$ with $j \in \{1, 2, \ldots, k\}$. Then considering an arbitrary $\frac{k(k-1)}{2}$-tuple of reals

$$\widehat{\rho} = (\rho_{1,2}, \rho_{1,3}, \ldots, \rho_{1,k}, \rho_{2,3}, \ldots, \rho_{2,k}, \ldots, \rho_{k-1,k}), \tag{3.2}$$

consider the symmetric matrices

(3.3)

$$C_{\widehat{\alpha}}(\widehat{\rho}) = \begin{bmatrix} M_1 & \rho_{1,2}\mathbf{u}_{i_1,1}\mathbf{u}_{i_2,2}^T & \rho_{1,3}\mathbf{u}_{i_1,1}\mathbf{u}_{i_3,3}^T & \cdots & \rho_{1,k}\mathbf{u}_{i_1,1}\mathbf{u}_{i_k,k}^T \\ \rho_{1,2}\mathbf{u}_{i_2,2}\mathbf{u}_{i_1,1}^T & M_2 & \rho_{2,3}\mathbf{u}_{i_2,2}\mathbf{u}_{i_3,3}^T & \cdots & \rho_{2,k}\mathbf{u}_{i_2,2}\mathbf{u}_{i_k,k}^T \\ \rho_{1,3}\mathbf{u}_{i_3,3}\mathbf{u}_{i_1,1}^T & \rho_{2,3}\mathbf{u}_{i_3,3}\mathbf{u}_{i_2,2}^T & M_3 & \cdots & \rho_{3,k}\mathbf{u}_{i_3,3}\mathbf{u}_{i_k,k}^T \\ \vdots & \vdots & & \ddots & \vdots \\ \rho_{1,k-1}\mathbf{u}_{i_{k-1},k-1}\mathbf{u}_{i_1,1}^T & \rho_{2,k-1}\mathbf{u}_{i_{k-1},k-1}\mathbf{u}_{i_2,2}^T & \cdots & M_{k-1} & \rho_{k-1,k}\mathbf{u}_{i_{k-1},k-1}\mathbf{u}_{i_k,k}^T \\ \rho_{1,k}\mathbf{u}_{i_k,k}\mathbf{u}_{i_1,1}^T & \rho_{2,k}\mathbf{u}_{i_k,k}\mathbf{u}_{i_2,2}^T & \cdots & \cdots & M_k \end{bmatrix}$$

$$\widetilde{C}_{\widehat{\alpha}}(\widehat{\rho}) = \begin{bmatrix} \alpha_{i_1,1} & \rho_{1,2} & \cdots & \rho_{1,k-1} & \rho_{1,k} \\ \rho_{1,2} & \alpha_{i_2,2} & \cdots & \rho_{2,k-1} & \rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{1,k-1} & \rho_{2,k-1} & \cdots & \alpha_{i_{k-1},k-1} & \rho_{k-1,k} \\ \rho_{1,k} & \rho_{2,k} & \cdots & \rho_{k-1,k} & \alpha_{i_k,k} \end{bmatrix}_{k\times k}. \tag{3.4}$$

Theorem 3.1. *[18] For $j \in \{1,2,\dots,k\}$, let M_j be an $m_j \times m_j$ symmetric matrix, with eigenpairs $(\alpha_{rj}, \boldsymbol{u}_{rj}), \forall r \in I_j = \{1,2,\dots,m_j\}$ and suppose that for each j, the system of eigenvecotrs $\{\boldsymbol{u}_{rj}, r \in I_j\}$ is orthonormal. Consider a $\frac{k(k-1)}{2}$-tuple of scalars, $\widehat{\rho} = (\rho_{1,2}, \rho_{1,3}, \dots, \rho_{1,k}, \rho_{2,3}, \dots, \rho_{2,k}, \dots, \rho_{k-1,k})$ and the k-tuple $\widehat{\alpha} = (\alpha_{i_1,1}, \dots, \alpha_{i_k,k})$ as defined in* (3.1) *and* (3.2) *. Then the matrix $C_{\widehat{\alpha}}(\widehat{\rho})$ in* (3.3) *has the multiset of eigenvalues $\left(\cup_{j=1}^{k}\{\alpha_{1,j}, \dots, \alpha_{m_j,j}\}\backslash\{\alpha_{i_j,j}\}\right) \cup \{\gamma_1, \dots, \gamma_k\}$, where $\gamma_1, \gamma_2, \dots, \gamma_k$ are eigenvalues of the matrix $\widetilde{C}_{\widehat{\alpha}}(\widehat{\rho})$ in* (3.4).

4. Distance spectrum of $\Gamma(\mathbb{Z}_n)$

The concept of distance and transmission of vertices in a graph finds wide application in the design of communication networks. A wide survey of distance spectra of graphs can be found in [16]. In analogous with energy of graphs , the concept of distance energy and Laplacian energy of graphs was introduced in [10, 13]. In [5], the authors make use of the Fiedlers lemma to get the adjacency spectrum and Laplacian spectrum of the joined union of regular graphs. The same tool of Linear Algebra is used in [6] to get the distance related spectrum of the joined union of regular graphs. The structure of $\Gamma(\mathbb{Z}_n)$ as the joined union of the induced subgraphs $\Gamma(S(d_i)), i = 1, 2 \dots, s(n)$; which are either complete or null graphs, the vertex sets of which form an equitable partition for $V(\Gamma(\mathbb{Z}_n))$; makes the task of computing the spectrum easy. Refer[22]. In this section we describe the distance matrix and its eigenvalues of $\Gamma(\mathbb{Z}_n)$, for any n and especially for $n = p^k, k \geqslant 3$. We also show that -1 and -2 are the distance eigenvalues of $\Gamma(\mathbb{Z}_n)$, for any n, and count their multiplicities.

Consider $G[H_1, H_2, \dots, H_k]$ where G is a connected graph with vertices labeled as $1, 2, \dots, k$ and H_j is r_j- regular and $|V(H_j)| = n_j$, for every $j = 1, 2, \dots, k$. Let $A(H_j)$ denote the adjacency matrix of H_j.

Take $M_j = 2(J-I)_{n_j} - A(H_j)$. Then clearly $\alpha_{i_j,j} = 2(n_j-1) - r_j$ is the Perron eigenvalue for M_j for every $j = 1,2,\ldots,k$ with corresponding Perron eigenvector, $\mathbf{1}_{n_j}$. [Note that since H_j is r_j- regular, r_j is the perron eigenvalue of H_j with $\mathbf{1}_{n_j}$ as the corresponding eigenvector, for $j = 1,2,\ldots,k$]. Thus, since G is connected and H_j is regular, M_j, $j = 1,2,\ldots,k$ correspond to the diagonal blocks in the distance matrix of $G\left[H_1,H_2,\ldots,H_k\right]$.

As in (3.3), taking

$$M_j = 2(J-I)_{n_j} - A(H_j), (\alpha_{i_j,j}, \mathbf{u}_{i_j,j}) = \left(2(n_j-1) - r_j, \frac{1}{\sqrt{n_j}}\mathbf{1}_{n_j}\right)$$

and the real numbers $\rho_{l,q} = d_{l,q} \cdot \sqrt{n_l n_q}$, for $l \in \{1,2,\ldots,k-1\}$, $q \in \{l+1,\ldots,k\}$, where $d_{l,q} = d_{q,l} = d_G(l,q)$, is the distance between the vertices l and q in the connected graph G, it can be seen that the distance matrix of $G\left[H_1,H_2,\ldots,H_k\right]$ is obtained as in the following theorem.

Theorem 4.1. *[6] Consider $G\left[H_1,H_2,\ldots,H_k\right]$, where G is a connected graph with vertices labeled as $1,2,\ldots,k$ and H_j is r_j- regular and $|V(H_j)| = n_j$, for every $j = 1,2,\ldots,k$,and let $d_{l,q}$ denote the distance between the distinct vertices l snd q in G for $l \in \{1,2,\ldots,k-1\}$, $q \in \{l+1,\ldots,k\}$. Let $A(H_j)$ denote the adjacency matrix of H_j and $M_j = 2(J-I)_{n_j} - A(H_j)$. Then, the distance matrix of the generalized G-join of the graphs $H_1,H_2,\ldots,H_k$ is given by,*

$D(G\left[H_1,H_2,\ldots,H_k\right])=$

$$\begin{bmatrix} M_1 & d_{1,2}J_{n_1\times n_2} & d_{1,3}J_{n_1\times n_3} & \ldots & d_{1,k}J_{n_1\times n_k} \\ d_{1,2}J^T_{n_1\times n_2} & M_2 & d_{2,3}J_{n_2\times n_3} & \ldots & d_{2,k}J_{n_2\times n_k} \\ d_{1,3}J^T_{n_1\times n_3} & d_{2,3}J^T_{n_2\times n_3} & M_3 & \ldots & d_{3,k}J_{n_3\times n_k} \\ \vdots & \vdots & & \ddots & \vdots \\ d_{1,k-1}J^T_{n_1\times n_{k-1}} & d_{2,k-1}J^T_{n_2\times n_{k-1}} & \ldots & M_{k-1} & d_{k-1,k}J_{n_{k-1}\times n_k} \\ d_{1,k}J^T_{n_1\times n_k} & d_{2,k}J^T_{n_2\times n_k} & \ldots & \ldots & M_k \end{bmatrix}. \tag{4.1}$$

Also, applying Theorem 3.1, the distance spectrum of $D(G\left[H_1,H_2,\ldots,H_k\right])$ is given by the following theorem.

Theorem 4.2. *[6] Consider $G\left[H_1,H_2,\ldots,H_k\right]$, where G is a connected graph with vertices labeled as $1,2,\ldots,k$ and H_j is r_j- regular and $|V(H_j)| = n_j$, for every $j = 1,2,\ldots,k$. Let $A(H_j)$ denote the adjacency matrix of H_j and $M_j = 2(J-I)_{n_j} - A(H_j)$.*

Then, the distance spectrum of $G[H_1, H_2, \ldots, H_k]$ *is given by,*

$$\sigma_D(G[H_1, H_2, \ldots, H_n]) = \left(\bigcup_{j=1}^{k} \sigma(M_j) \backslash \{\, 2(n_j - 1) - r_j \} \right) \cup \sigma(\widetilde{C}) \tag{4.2}$$

where

$$\widetilde{C} = \begin{bmatrix} 2(n_1 - 1) - r_1 & d_{1,2}\sqrt{n_1 n_2} & \ldots & d_{1,k}\sqrt{n_1 n_k} \\ d_{1,2}\sqrt{n_1 n_2} & 2(n_2 - 1) - r_2 & \ldots & d_{2,k}\sqrt{n_2 n_k} \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,k}\sqrt{n_1 n_k} & d_{2,k}\sqrt{n_2 n_k} & \ldots & 2(n_k - 1) - r_k \end{bmatrix}.$$

Consider the graph G as a vertex weighted graph by assigning the weight $n_j = |V(H_j)|$ to the vertex j of G for $j = 1, 2, \ldots, k$ and consider the diagonal matrix of vertex weights,

$$W = \begin{bmatrix} n_1 & 0 & \ldots & 0 \\ 0 & n_2 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & n_k \end{bmatrix}.$$

Let $T_D(G)$ be the combinatorial (vertex weighted) distance matrix of G,(see[9]) given by

$$T_D(G) = \begin{bmatrix} 2(n_1 - 1) - r_1 & d_{1,2}n_2 & \ldots & d_{1,k}n_k \\ d_{1,2}n_1 & 2(n_2 - 1) - r_2 & \ldots & d_{2,k}n_k \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,k}n_1 & d_{2,k}n_2 & \ldots & 2(n_k - 1) - r_k \end{bmatrix}.$$

Remark 4.1. *Since* $T_D(G) = W^{-\frac{1}{2}}\widetilde{C}W^{\frac{1}{2}}$, *it follows that* $\widetilde{C}$ *and* $T_D(G)$ *are similar. Thus* $\sigma(\widetilde{C}) = \sigma(T_D(G))$. *Thus the distance spectrum of* $G[H_1, H_2, \ldots, H_n]$ *is completely determined by the matrices* M_j *for* $j = 1, 2, \ldots, k$ *and the combinatorial distance matrix* $T_D(G)$ *associated to* G.

4.1. **The zero-divisor graph $\Gamma(\mathbb{Z}_n)$.** As usual, let $\mathbb{Z}_n$ denote the commutative ring of integers modulo n. When n is a prime, $\mathbb{Z}_n$ is an integral domain and has no zero divisors. Thus to avoid triviality, we assume that n is not a prime. Also we note that if $n = p^2, 2^3$; $\Gamma(\mathbb{Z}_n)$ is complete. So, through out this paper we assume that $n \neq p^2, 2^3$. We recall that in any finite commutative ring with unity, every non-zero element is either a unit or a zero-divisor. The number of non-zero zero-divisors of $\mathbb{Z}_n$ is $n - \phi(n) - 1$. [22].

In [26], Sriparna Chattopadhyay et.al. describe the structure of $\Gamma(\mathbb{Z}_n)$ as the generalised join of its induced subgraphs .

By a proper divisor of n, we mean a positive divisor d such that d/n , $1 < d < n$. Let $s(n)$ denote the number of proper divisors of n. Then, $s(n) = \sigma_0(n) - 2$, where $\sigma_k(n)$ is the sum of k powers of all divisors of n, including n and 1.

If $n = p_1^{n_1} \cdot p_2^{n_2} \cdots p_r^{n_r}$, where $p_1, p_2, \ldots, p_r$ are distinct primes, and $n_1, n_2, \ldots, n_r$ are positive integers,

$$s(n) = \prod_{i=1}^{r}(n_i + 1) \ - 2.$$

Let $S(d) = \{k \in \mathbb{Z}_n : gcd(k, n) = d\}$. Clearly $\{S(d_1), S(d_2), \ldots, S(d_{s(n)})\}$ is a collection of pairwise disjoint sets of vertices and is an equitable partition for the vertex set of $\Gamma(\mathbb{Z}_n)$ such that $S(d_i) \cap S(d_j) = \phi, i \neq j$, and any two vertices in $S(d_i)$ have the same number of neighbours in $S(d_j)$ for all divisors d_i, d_j of n. Using elementary number theory, it can be seen that

$$|S(d_i)| = \phi(n/d_i), \quad \text{for every} \quad i = 1, 2, \ldots, n.$$

Let the subgraph of $\Gamma(\mathbb{Z}_n)$, induced by $S(d_i)$ be denoted by $\Gamma(S(d_i))$ for $i = 1, 2, \ldots, s(n)$. Then,

$$\Gamma(S(d_i)) = \begin{cases} \overline{K}_{\phi(\frac{n}{d_i})} & \text{if } n \not| \ d_i^2 \\ K_{\phi(\frac{n}{d_i})} & \text{if } n/d_i^2 \end{cases}.$$

It is obvious that $\Gamma(S(d_i))$ is regular for each $i = 1, 2, \ldots, s(n)$. For example, in $\Gamma(\mathbb{Z}_{p^3})$, $S(p)$ induces $\overline{K}_{p(p-1)}$ and $S(p^2)$ induces K_{p-1}. In $\Gamma(\mathbb{Z}_{p^2q})$, $S(p), S(q), S(p^2)$ induce $\overline{K}_{(p-1)(q-1)}, \overline{K}_{p(p-1)}, \overline{K}_{q-1}$ respectively while $S(pq)$ induces K_{p-1}.

The compressed zero-divisor graph, denoted by Υ_n (in fact a subgraph of $\Gamma(\mathbb{Z}_n)$) is a simple connected graph associated with $\Gamma(\mathbb{Z}_n)$ with vertices labeled as $d_1, d_2, \ldots, d_{s(n)}$. See[3]. In [11], the authors designate this graph as proper divisor graph, where as in [20] it is named as projection graph. The vertices d_i and d_j are adjacent in Υ_n if and only if n/d_id_j [26]. For example the compressed zero-divisor graph of $\Gamma(\mathbb{Z}_{36})$ is given in figure 1.

Thus the zero divisor graph on the ring of integers modulo n can be constructed as

$$\Gamma(\mathbb{Z}_n) = \Upsilon_n[\Gamma(S(d_1)), \Gamma(S(d_2)), \ldots, \Gamma(S(d_{s(n)}))] \tag{4.3}$$

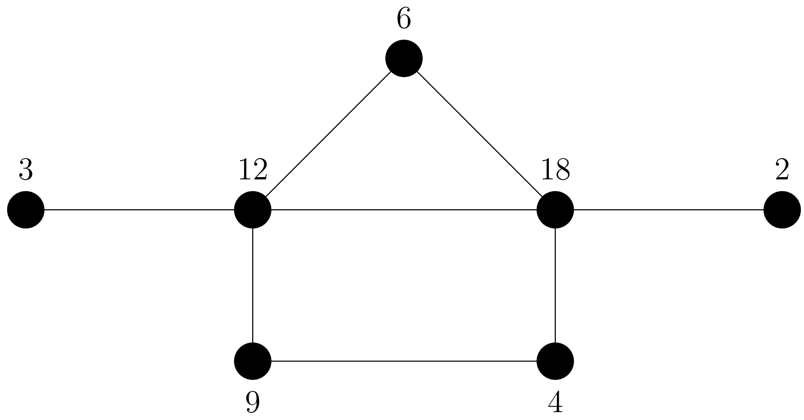

FIGURE 1. Υ_{36}

and the study of spectrum of $\Gamma(\mathbb{Z}_n)$ can be facilitated through the compressed zero-divisor graph in a better way.

Throughout this paper we use $d_{i,j}$ to denote the distance between the vertices d_i and d_j in Υ_n. That is $d_{i,j} = d_{\Upsilon_n}(d_i, d_j)$ We have the following lemmas which are used in the main theorem of this section.

Lemma 4.1. *For any two distinct vertices d_i and d_j in the compressed zero-divisor graph Υ_n ,*

$$d_{i,j} = \begin{cases} 1 & \textit{if } n/d_id_j, \\ 2 & \textit{if } n \nmid d_id_j, \quad \gcd(d_i, d_j) \neq 1, \\ 3 & \textit{otherwise.} \end{cases}$$

Proof. $n = p_1^{n_1} \cdot p_2^{n_2} \cdots p_r^{n_r}$, where $p_1, p_2, \ldots, p_r$ are distinct primes, and $n_1, n_2, \ldots, n_r$ are positive integers. Let $d_1, d_2, \ldots, d_{s(n)}$ be the proper divisors of n.

Case (i) is trivial.

Case (ii): Let $n \nmid d_id_j$, and let $\gcd(d_i, d_j) = g > 1$. Clearly, $n/(\frac{n}{g})d_i$ and $n/(\frac{n}{g})d_j$. Thus $\frac{n}{g} \sim d_i$ and $\frac{n}{g} \sim d_j$. Hence $\frac{n}{g}$ is a common neighbor of d_i and d_j in Υ_n.

Conversely we prove that if $n \nmid d_id_j$ and d_i and d_j have a common neighbor in Υ_n, then $\gcd(d_i, d_j) > 1$. Let d_k be the common neighbor of d_i and d_j .Then n/d_id_k and n/d_jd_k . Thus it is obvious that $(\frac{n}{d_k})/d_i$ and $(\frac{n}{d_k})/d_j$. Hence $\frac{n}{d_k}$ is a common divisor of d_i and d_j. Thus the $\gcd(d_i, d_j) \geqslant \frac{n}{d_k} > 1$. Thus when $n \nmid d_id_j$, d_i and d_j have a common neighbor in Υ_n, iff $\gcd(d_i, d_j) > 1$.

Case (iii): Let $n \nmid d_i d_j$, and $\gcd(d_i, d_j) = 1$. Then, as proved in the above case, d_i and d_j do not have a common neighbor. Now,

$$d_i \sim \frac{n}{d_i}, \quad d_j \sim \frac{n}{d_j}.$$

Since n divides $\frac{n}{d_i} \cdot \frac{n}{d_j}$, it follows that $\frac{n}{d_i} \sim \frac{n}{d_j}$. Thus, $d_i \sim \frac{n}{d_i} \sim \frac{n}{d_j} \sim d_j$ is a shortest path between d_i and d_j. Thus $d_{\Upsilon_n}(d_i, d_j) = 3$, in this case. Thus the result immediately follows since for any commutative ring R, $diam(\Gamma(R)) \leqslant 3$. [27] □

Lemma 4.2. *Let $n = p_1^{n_1} \cdot p_2^{n_2} \cdots p_r^{n_r}$, where $p_1, p_2, \ldots, p_r$ are distinct primes, and $n_1, n_2, \ldots, n_r$ are positive integers. Then, the number of proper divisors d of n such that $n \nmid d^2$ is*

$$\prod_{i=1}^{r} \left(\lceil \frac{n_i}{2} \rceil + 1 \right).$$

Proof. The number of proper divisor of n is given by

$$s(n) = \prod_{i=1}^{r} (n_i + 1).$$

The number of proper divisors d of n such that $n \nmid d^2$ is exactly the number of proper divisors of $p_1^{\lceil \frac{n_1}{2} \rceil} \cdot p_2^{\lceil \frac{n_2}{2} \rceil} \cdots p_r^{\lceil \frac{n_r}{2} \rceil}$ which is $\prod_{i=1}^{r} \left(\lceil \frac{n_i}{2} \rceil + 1 \right) - 2$. □

The following lemma is the immediate consequence of Lemma 4.2.

Lemma 4.3. *The number of proper divisors d of n such that n divides d^2, is*

$$\prod_{i=1}^{r} (n_i + 1) - \prod_{i=1}^{r} \left(\lceil \frac{n_i}{2} \rceil + 1 \right).$$

The next theorem describes the distance spectrum of $\Gamma(\mathbb{Z}_n)$ for any n.

4.2. Distance matrix of $\Gamma(\mathbb{Z}_n)$.

Theorem 4.3. *Let $d_1, d_2, \ldots, d_{s(n)}$ be the proper divisors of n. Then, the distance spectrum of $\Gamma(\mathbb{Z}_n)$ is given by*

$$D(\Gamma(\mathbb{Z}_n)) = \begin{bmatrix} M_1 & d_{1,2} J_{\phi(\frac{n}{d_1}) \times \phi(\frac{n}{d_2})} & \cdots & d_{1,s(n)} J_{\phi(\frac{n}{d_1}) \times \phi(\frac{n}{d_{s(n)}})} \\ d_{1,2} J^T_{\phi(\frac{n}{d_1}) \times \phi(\frac{n}{d_2})} & M_2 & \cdots & d_{2,s(n)} J_{\phi(\frac{n}{d_2}) \times \phi(\frac{n}{d_{s(n)}})} \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,s(n)} J^T_{\phi(\frac{n}{d_1}) \times \phi(\frac{n}{d_{s(n)}})} & d_{2,s(n)} J^T_{\phi(\frac{n}{d_2}) \times \phi(\frac{n}{d_{s(n)}})} & \cdots & M_k \end{bmatrix}$$

where,

$$M_j = \begin{cases} 2(J-I)_{\phi(\frac{n}{d_j})} & \textit{if } n \nmid d_j^2 \\ (J-I)_{\phi(\frac{n}{d_j})} & \textit{if } n/d_j^2 \end{cases}$$

and for $l \in \{1., 2, \ldots, s(n)-1\}$ *and* $q \in \{l+1, \ldots, s(n)\}$, $l \neq q$,

$$d_{lq} = \begin{cases} 1 & \textit{if } n/d_l d_q \\ 2 & \textit{if } n \nmid d_l d_q,\ \gcd(d_l, d_q) \neq 1. \\ 3 & \textit{otherwise} \end{cases}.$$

Proof. For $n = p^2$ for any prime p or $n = 8$, the zero divisor graph $\Gamma(\mathbb{Z}_n)$ is a complete graph and the result is trivial. By (4.3), $\Gamma(\mathbb{Z}_n)$ is the Υ_n - join of $\Gamma(S(d_1)), \Gamma(S(d_2)), \ldots, \Gamma(S(d_{s(n)}))$. Also, the adjacency matrix of $\Gamma(S(d_j))$ is given by,

$$A(\Gamma(S(d_j))) = \begin{cases} O_{\phi(\frac{n}{d_j})} & \text{if } n \nmid d_j^2 \\ (J-I)_{\phi(\frac{n}{d_j})} & \text{if } n/d_j^2. \end{cases}$$

Thus, taking $G = \Upsilon_n$, and $H_j = \Gamma(S(d_j))$ and $M_j = 2(J-I)_{\phi(\frac{n}{d_j})} - A(\Gamma(S(d_j)))$, the conclusion is an immediate consequence of Theorem 4.1 and Lemma 4.1. □

4.3. **Distance spectrum of $\Gamma(\mathbb{Z}_n)$.** For the proper divisors $d_1, \ldots, d_{s(n)}$, and the matrices,

$$M_j = \begin{cases} 2(J-I)_{\phi(\frac{n}{d_j})} & \text{if } n \nmid d_j^2 \\ (J-I)_{\phi(\frac{n}{d_j})} & \text{if } n/d_j^2, \end{cases}$$

the distance spectrum of $\Gamma(\mathbb{Z}_n)$ are completely determined by the matrices M_j, for $j = 1, 2, \ldots, s(n)$ and the combinatorial (vertex weighted) distance matrix of Υ_n, as in (4.1) and (4.2). The spectrum of M_j as described above are, if $n \nmid d_j^2$,

$$\sigma(M_j) = \begin{pmatrix} -2 & 2(\phi(\frac{n}{d_j})-1) \\ \phi(\frac{n}{d_j})-1 & 1 \end{pmatrix} \tag{4.4}$$

and if n/d_j^2,

$$\sigma(M_j) = \begin{pmatrix} -1 & \phi(\frac{n}{d_j})-1 \\ \phi(\frac{n}{d_j})-1 & 1 \end{pmatrix}. \tag{4.5}$$

Also we note that the subgraphs $\Gamma(S(d_j))$, for $j = 1, 2, \ldots, s(n)$ are r_j- regular, where

$$r_j = \begin{cases} \phi(\frac{n}{d_j}) - 1 & \text{if } n/d_j^2 \\ 0 & \text{if } n \nmid d_j^2. \end{cases}$$

While taking the union of all eigenvalues of M_j as described above in (4.4) and (4.5), the multiplicity of -2 as the distance eigenvalue of $\Gamma(\mathbb{Z}_n)$ is $\sum_{d/n, n \nmid d^2}(\phi(\frac{n}{d}) - 1)$, where the Σ runs over all divisors d of n such that d/n and $n \nmid d^2$. By Lemma 4.2, this count amounts to $\sum_{d/n, n \nmid d^2} \phi(\frac{n}{d}) - \prod_{i=1}^{r} \left(\lceil \frac{n_i}{2} \rceil + 1\right) + 2$. Similarly, while taking the union of all eigenvalues of M_j, the multiplicity of -1 as the distance eigenvalue of $\Gamma(\mathbb{Z}_n)$ is $\sum_{d/n, n/d^2}(\phi(\frac{n}{d}) - 1)$, which counts to $\sum_{d/n, n/d^2} \phi(\frac{n}{d}) - \prod_{i=1}^{r}(n_i + 1) + \prod_{i=1}^{r} \left(\lceil \frac{n_i}{2} \rceil + 1\right)$, by Lemma 4.3. Thus, applying Theorem 4.2 and Remark 4.1, we have the following theorem.

Theorem 4.4. *For any $n = p_1^{n_1} \cdot p_2^{n_2} \cdots p_r^{n_r}$, the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ has distance eigen values -2 and -1 with multiplicities $\sum_{d/n, n \nmid d^2} \phi(\frac{n}{d}) - \prod_{i=1}^{r} \left(\lceil \frac{n_i}{2} \rceil + 1\right) + 2$ and $\sum_{d/n, n/d^2} \phi(\frac{n}{d}) - \prod_{i=1}^{r}(n_i + 1) + \prod_{i=1}^{r} \left(\lceil \frac{n_i}{2} \rceil + 1\right)$ respectively and the remaining distance eigenvalues are the eigenvalues of the vertex weighted distance matrix of Υ_n, as follows*

$$T_D(\Upsilon_n) = \begin{bmatrix} t_1 & d_{1,2}\phi(\frac{n}{d_2}) & \ldots & d_{1,s(n)}\phi(\frac{n}{d_{s(n)}}) \\ d_{1,2}\phi(\frac{n}{d_1}) & t_2 & \ldots & d_{2,s(n)}\phi(\frac{n}{d_{s(n)}}) \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,s(n)}\phi(\frac{n}{d_1}) & d_{2,s(n)}\phi(\frac{n}{d_2}) & \ldots & t_k \end{bmatrix}, \tag{4.6}$$

where

$$t_j = \begin{cases} 2(\phi(\frac{n}{d_j}) - 1) & \textit{if } n \nmid d_j^2 \\ \phi(\frac{n}{d_j}) - 1 & \textit{if } n/d_j^2, \end{cases}$$

and for $i \in \{1, 2, \ldots, s(n) - 1\}, j \in \{i + 1, \ldots, s(n)\}$, $i \neq j$,

$$d_{i,j} = \begin{cases} 1 & \textit{if } n/d_i d_j \\ 2 & \textit{if } n \nmid d_i d_j, \ \gcd(d_i, d_j) \neq 1. \\ 3 & \textit{otherwise.} \end{cases}$$

Remark 4.2. *The matrix $T_D(\Upsilon_n)$ as in (4.6) can be described as the vertex weighted distance matrix of the compressed zero-divisor graph Υ_n. Refer[9]. Thus the distance* *eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_n)$ are completely determined by this vertex weighted distance matrix $T_D(\Upsilon_n)$. Thus we have the following corollory.*

Corollary 4.1. *$\mathit{\Gamma}(\mathbb{Z}_n)$ is distance integral if and only if $T_D(\Upsilon_n)$ is integral.*

Example 1. $SpecD(\mathit{\Gamma}(\mathbb{Z}_{pq})) =$

$$\begin{pmatrix} -2 & p+q-4+\sqrt{p^2+q^2-pq-(p+q)+1} & p+q-4-\sqrt{p^2+q^2-pq-(p+q)+1} \\ p+q-4 & 1 & 1 \end{pmatrix}$$

Consider $\mathit{\Gamma}(\mathbb{Z}_{pq})$, where $p < q$ are distinct primes. Counting the number of non-zero zero-divisors of $\mathbb{Z}_{pq}$, it can be easily seen that the zero-divisor graph $\mathit{\Gamma}(\mathbb{Z}_{pq})$, has $p + q - 2$ vertices. The proper divisors of pq are p and q and the compressed zero divisor graph $\Upsilon_{pq} \cong K_2$, with vertices labeled as p and q. Clearly $\mathit{\Gamma}(\mathbb{Z}_{pq}) = K_2[\Gamma(S(p)), \Gamma(S(q))]$, where $\Gamma(S(p)) = \overline{K}_{q-1}$ and $\Gamma(S(q)) = \overline{K}_{p-1}$. Using Theorem 4.3, the distance matrix of $\mathit{\Gamma}(\mathbb{Z}_{pq})$ is given by

$$D(\mathit{\Gamma}(\mathbb{Z}_{pq})) = \left[\begin{array}{c|c} 2(J-I)_{(q-1)\times(q-1)} & J_{(q-1)\times(p-1)} \\ \hline J_{(p-1)\times(q-1)} & 2(J-I)_{(p-1)\times(p-1)} \end{array}\right].$$

Note that $n = pq$ has no proper divisor d such that n/d^2. Also it is obvious that a square matrix M has eigenvalue -1 if and only if the matrix $M + I$ has nullity at least one. Thus -1 is not an eigenvalue of $\mathit{\Gamma}(\mathbb{Z}_{pq})$ for any primes $p < q$. Thus using Theorem 4.4, we see that -2 is an eigenvalue of $\mathit{\Gamma}(\mathbb{Z}_{pq})$ with multiplicity $p+q-4$. And by Theorem 4.4, the other distance eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_{pq})$ is determined by its vertex weighted distance matrix,

$$T_D(\Upsilon_{pq}) = \begin{bmatrix} 2(q-2) & p-1 \\ q-1 & 2(p-2) \end{bmatrix}.$$

Thus the remaining two distance eigenvalues of this graph are determined by the polynomial, $Q(x) = x^2 - 2x(p + q - 4) + 3pq - 7(p + q) + 15$.

Remark 4.3. *We note that for $n = pq$, where $p < q$ are distinct primes, -1 is not a distance-eigenvalue of $\mathit{\Gamma}(\mathbb{Z}_n)$. Also, since $\mathit{\Gamma}(\mathbb{Z}_n)$ for $n = p^2, 2^3$; is a complete graph (hence adjacency matrix and distance matrix are equal), -2 is not a distance eigenvalue of $\mathit{\Gamma}(\mathbb{Z}_n)$. For all other values of n, both -1 and -2 are distance eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_n)$.*

4.4. **Distance spectrum of $\Gamma(\mathbb{Z}_{p^k}), k \geqslant 3$.** The proper divisors of p^k are $p, p^2, \ldots, p^{k-1}$ and $\{S(p), S(p^2), \ldots, S(p^{k-1})\}$ forms an equitable partition for $V(\Gamma(\mathbb{Z}_{p^k}))$. The order of the graph $\Gamma(\mathbb{Z}_{p^k})$ is $p^{k-1}-1$. The analysis of the adjacency matrix and some graph parameters of $\Gamma(\mathbb{Z}_{p^k})$ can be seen in [22],

$$\Gamma(S(p^j)) = \begin{cases} \overline{K}_{\phi(p^{k-j})} & \text{if } j < \lceil \frac{k}{2} \rceil \\ K_{\phi(p^{k-j})} & \text{if } j \geqslant \lceil \frac{k}{2} \rceil, \end{cases}$$

where $\overline{K}_{\phi(p^{k-j})}$ is a null graph which is 0- regular and $K_{\phi(p^{k-j})}$ is a complete graph which is $\phi(p^{k-j})-1$- regular. In the compressed zero-divisor graph Υ_{p^k}, $p^i \sim p^j$ for distinct i and j, if and only if $i+j \geqslant k$. Also the vertex p^{k-1} is adjacent to every other vertices of Υ_{p^k}.Thus Υ_{p^k} is a connected graph with diameter 2. [11]. Also, for $i \in \{1, 2, \ldots, k-2\}, j \in \{i+1, \ldots, k-1\}$, $i \neq j$,

$$d_{i,j} = \begin{cases} 1 & \text{if } i+j \geqslant k \\ 2 & \text{otherwise} \end{cases}.$$

The zero-divisor graph,

$$\Gamma(\mathbb{Z}_{p^k}) = \Upsilon_{p^k}[\Gamma(S(p)), \Gamma(S(p^2)), \ldots, \Gamma(S(p^{k-1}))] \tag{4.7}$$

For $j = 1, 2, \ldots, k-1$, we take, as in Theorem 4.1,

$$M_j = \begin{cases} 2(J-I)_{\phi(p^{k-j})} & \text{if } j < \lceil \frac{k}{2} \rceil \\ (J-I)_{\phi(p^{k-j})} & \text{if } j \geqslant \lceil \frac{k}{2} \rceil \end{cases}.$$

Then, it is obvious that, if $j < \lceil \frac{k}{2} \rceil$,

$$\sigma(M_j) = \begin{pmatrix} -2 & 2\left((\phi(p^{k-j})-1\right) \\ \phi(p^{k-j})-1 & 1 \end{pmatrix} \tag{4.8}$$

and if $j \geqslant \lceil \frac{k}{2} \rceil$,

$$\sigma(M_j) = \begin{pmatrix} -1 & \phi(p^{k-j})-1 \\ \phi(p^{k-j})-1 & 1 \end{pmatrix}. \tag{4.9}$$

Taking the union of the eigenvalues of M_j, for $j = 1, 2, \ldots, k-1$, the number of times -2 is counted as an eigenvalue is, from (4.8) and (4.9),

$$\Sigma_{j<\lceil \frac{k}{2} \rceil} \left(\phi(p^{k-j})-1\right) = \Sigma_{j=1}^{\lceil \frac{k}{2} \rceil -1} \left(\phi(p^{k-j})-1\right) = p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor} - \lceil \frac{k}{2} \rceil + 1.$$

similarly the eigenvalue -1 is counted $p^{\lfloor \frac{k}{2} \rfloor} - \lfloor \frac{k}{2} \rfloor - 1$ times while taking the union of M_j, for $j = 1, 2, \ldots, k-1$. Thus considering (4.8) and (4.9) and applying Theorem 4.1 and Theorem 4.2, we have the following theorem.

Theorem 4.5. *For $k \geqslant 3$, the zero-divisor graph $\Gamma(\mathbb{Z}_{p^k})$ has distance eigenvalue -2 and -1 with multiplicities $p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor} - \lceil \frac{k}{2} \rceil + 1$ and $p^{\lfloor \frac{k}{2} \rfloor} - \lfloor \frac{k}{2} \rfloor - 1$ respectively and the remaining distance eigenvalues are the eigenvalues of the vertex weighted distance matrix of order $k-1$,*

$$T_D(\Upsilon_{p^k}) = \begin{bmatrix} t_1 & d_{1,2}\phi(p^{k-2}) & \ldots & d_{1,k-1}\phi(p) \\ d_{1,2}\phi(p^{k-1}) & t_2 & \ldots & d_{2,k-1}\phi(p) \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,k-1}\phi(p^{k-1}) & d_{2,k-1}\phi(p^{k-2}) & \ldots & t_{k-1} \end{bmatrix},$$

where, $d_{i,j} = \begin{cases} 1 & \text{if } i+j \geqslant k \\ 2 & \text{otherwise} \end{cases}$ *and* $t_j = \begin{cases} 2\left((\phi(p^{k-j}) - 1\right) & \text{if } j < \lceil \frac{k}{2} \rceil \\ \phi(p^{k-j}) - 1 & j \geqslant \lceil \frac{k}{2} \rceil \end{cases}.$

Example 2. *The distance spectrum of $\Gamma(\mathbb{Z}_{p^3})$ is given by,*

$$SpecD(\Gamma(\mathbb{Z}_{p^3})) = \begin{pmatrix} -2 & -1 & \frac{2p^2-p-4+\sqrt{4p^4-8p^3+p^2+4p}}{2} & \frac{2p^2-p-4-\sqrt{4p^4-8p^3+p^2+4p}}{2} \\ p^2-p-1 & p-2 & 1 & 1 \end{pmatrix}.$$

The number of non-zero zero-divisors of $\mathbb{Z}_{p^3}$ is p^2-1. The proper divisors of p^3 are p and p^2 and the compressed zero-divisor graph Υ_{p^3} is isomorphic to K_2. The distance matrix of $\Gamma(\mathbb{Z}_{p^3})$, for $p \neq 2$ is given by,

$$D(\Gamma(\mathbb{Z}_{p^3})) = \left[\begin{array}{c|c} 2(J-I)_{p(p-1)\times p(p-1)} & J_{p(p-1)\times(p-1)} \\ \hline J_{(p-1)\times p(p-1)} & (J-I)_{(p-1)\times(p-1)} \end{array}\right].$$

The distance eigenvalues of $D(\Gamma(\mathbb{Z}_{p^3}))$ are -2 and -1 with multiplicities p^2-p-1 and $p-2$ respectively and the remaining two distance eigenvalues are the eigenvalues of the vertex weighted distance matrix $T_D(\Upsilon_{p^3})$ given by,

$$T_D(\Upsilon_{p^3}) = \begin{bmatrix} 2(p^2-p-1) & p-1 \\ p^2-p & p-2 \end{bmatrix}.$$

5. DISTANCE LAPLACIAN SPECTRUM OF $\Gamma(\mathbb{Z}_n)$

Let $G = \Gamma(\mathbb{Z}_n)$, $n \neq p^2, 2^3$. In this section we investigate the distance Laplacian matrix of $\Gamma(\mathbb{Z}_n)$ and describe the explicit way of computing its distance Laplacian eigenvalues. Also the spectrum of the distance Laplacian matrix of $\Gamma(\mathbb{Z}_{p^k}), k \geqslant 3$ is explored. It can be noted that the distance Laplacain matrix of any connected graph is positive semi-definite and the least distance Laplacian eigenvalue, $0,$ is of multiplicity $1.$ [18]

Let $d_1, d_2, \ldots, d_k$ be the proper divisors of n such that $\Gamma(S(d_1)), \Gamma(S(d_2)), \ldots,$ $\Gamma(S(d_k))$ are null graphs and let $\Gamma(S(d_{k+1})), \ldots, \Gamma(S(d_{s(n)}))$ be complete subgraphs of $\Gamma(\mathbb{Z}_n)$. Then, since $\Gamma(\mathbb{Z}_n)$ is a generalised join of its induced subgraphs which are either null (corresponding to the divisors $d_1, d_2, \ldots, d_k$) or complete (corresponding to the divisors $d_{k+1}, \ldots, d_{s(n)}$); the distance between any two distinct vertices of $S(d_j)$ for fixed $j \in \{1, 2, \ldots, k\}$ is 2 and the distance between any two distinct vertices of $S(d_J)$, for fixed $j \in \{k+1, \ldots, s(n)\}$ is 1. Let M_j be the matrix as described in section:4, and let $\lambda_1(M_j)$ be the Perron eigenvalue of M_j for $j = 1, 2, \ldots, s(n)$. It is easy to see that,

$$\lambda_1(M_j) = \begin{cases} 2(\phi(\frac{n}{d_j}) - 1) & \text{if } j \leqslant k \\ \phi(\frac{n}{d_j}) - 1 & \text{if } j \geqslant k+1 \end{cases}.$$

Then for any vertex $v_{d_1} \in S(d_1)$, the transmission degree of v_{d_1} is given by

$$\begin{aligned} Tr(v_{d_1}) &= \Sigma_{u \in V(G)} d_G(u, v_{d_1}) = 2(\phi(\frac{n}{d_1}) - 1) + d_{1,2}\phi(\frac{n}{d_2}) + d_{1,3}\phi(\frac{n}{d_3}) \\ &\quad + \ldots + d_{1,s(n)}\phi(\frac{n}{d_{s(n)}}) \\ &= \lambda_1(M_1) + \Sigma_{j \neq 1} d_{1,j}\phi(\frac{n}{d_j}). \end{aligned}$$

Similar results hold for $Tr(v_{d_2}), \ldots, Tr(v_{d_k})$ and we have for any vertex $v_i \in S(d_i)$,

$$Tr(v_{d_i}) = \lambda_1(M_i) + \Sigma_{j \neq i} d_{i,j}\phi(\frac{n}{d_j}), \quad i = 1, 2, \ldots, k.$$

Since, $S(d_{k+1}), \ldots, S(d_{s(n)})$ induce complete subgraphs, we see that the transmission degree of any vertex $v_{d_i} \in S(d_i), i = k+1, \ldots, s(n)$ is given by

$$Tr(v_{d_i}) = \phi(\frac{n}{d_i}) - 1 + \Sigma_{j \neq i} d_{i,j}\phi(\frac{n}{d_j}) = \lambda_1(M_i) + \Sigma_{j \neq i} d_{i,j}\phi(\frac{n}{d_j}), i = k+1, \ldots, s(n).$$

Thus the diagonal matrix of vertex transmission of G is given by,

$$Tr(\Gamma(\mathbb{Z}_n)) = \begin{bmatrix} T_1 & O & \dots & O \\ O & T_2 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & T_{s(n)} \end{bmatrix},$$

where the diagonal blocks T_i is given by

$$T_i = (\lambda_1(M_i) + \tau_i)\, I_{\phi(\frac{n}{d_i})}, \tag{5.1}$$

where $\tau_i = \Sigma_{j\neq i} d_{i,j}\phi(\frac{n}{d_j}), i = 1, 2, \dots, s(n)$.

Since the distance laplacian matrix of any connected graph G is $Tr(G) - D(G)$, it follows that,

$$D^L(\Gamma(\mathbb{Z}_n)) =$$
$$\begin{bmatrix} L_1 & -d_{1,2}J_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_2})} & \dots & -d_{1,s(n)}J_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_{s(n)}})} \\ -d_{1,2}J^T_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_2})} & L_2 & \dots & -d_{2,s(n)}J_{\phi(\frac{n}{d_2})\times\phi(\frac{n}{d_{s(n)}})} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{1,s(n)}J^T_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_{s(n)}})} & -d_{2,s(n)}J^T_{\phi(\frac{n}{d_2})\times\phi(\frac{n}{d_{s(n)}})} & \dots & L_k \end{bmatrix},$$

where the diagonal blocks L_i is given by, using (5.1),

$$L_i = T_i - M_i = (\lambda_1(M_i) + \tau_i)\, I_{\phi(\frac{n}{d_i})} - M_i, \quad i = 1, 2, \dots, s(n).$$

Since T_i and M_i commute each other, for $i = 1, 2, \dots, s(n)$, it can be easily seen that, each eigenvalue $\lambda(L_i)$ is given by

$$\lambda(L_i) = \lambda(T_i) - \lambda(M_i), i = 1, 2, \dots, s(n).$$

Also, $\lambda_1(M_i) + \tau_i$ is the Perron eigenvalue of T_i such that

$$T_i\mathbf{1}_{\phi(\frac{n}{d_i})} = (\lambda_1(M_i) + \tau_i)\, \mathbf{1}_{\phi(\frac{n}{d_i})}.$$

Thus, τ_i is the Perron eigenvalue of L_i for $i = 1, 2, \dots, s(n)$. Thus applying Theorem 4.2, we arrive at the following theorem.

Theorem 5.1. *For $n \neq p^2, 2^3$, the distance Laplacain spectrum of $G = \Gamma(\mathbb{Z}_n)$ is given by*

$$\sigma_{D^L}(G) = \bigcup_{i=1}^{s(n)} \{\sigma(L_i)\backslash\tau_i\} \cup \sigma(\widetilde{C}),$$

where $\widetilde{C}$ is the vertex weighted distance Laplacian matrix of the compressed zero-divisor graph Υ_n, given by

$$\widetilde{C} = \begin{bmatrix} \tau_1 & -d_{1,2}\phi(\frac{n}{d_2}) & \dots & -d_{1,s(n)}\phi(\frac{n}{d_{s(n)}}) \\ -d_{1,2}\phi(\frac{n}{d_1}) & \tau_2 & \dots & -d_{2,s(n)}\phi(\frac{n}{d_{s(n)}}) \\ \vdots & \vdots & \ddots & \vdots \\ -d_{1,s(n)}\phi(\frac{n}{d_1}) & -d_{2,s(n)}\phi(\frac{n}{d_2}) & \dots & \tau_k \end{bmatrix}.$$

For example, $\mathit{\Gamma}(\mathbb{Z}_{pq}) = \Upsilon_{pq}[\Gamma(S(p)), \Gamma(S(q))]$, where $\Upsilon_{pq} \cong K_2$. In this case, $T_1 = (p+2q-5)I_{q-1}, M_1 = 2(J-I)_{q-1}, T_2 = (2p+q-5)I_{p-1}, M_2 = 2(J-I)_{p-1}$ and thus the distance Laplacian matrix of $\mathit{\Gamma}(\mathbb{Z}_{pq})$ is given by

$$D^L(\mathit{\Gamma}(\mathbb{Z}_{pq})) = \left[\begin{array}{c|c} (p+2q-5)I - 2(J-I)_{(q-1)\times(q-1)} & -J_{(q-1)\times(p-1)} \\ \hline -J_{(p-1)\times(q-1)} & (2p+q-5)I - 2(J-I)_{(p-1)\times(p-1)} \end{array}\right].$$

Here

$$L_1 = (p+2q-3)I_{q-1} - 2J_{q-1}$$
$$L_2 = (2p+q-3)I_{p-1} - 2J_{p-1}.$$

Clearly, $\tau_1 = \lambda_1(L_1) = p-1$ and $\tau_2 = \lambda_1(L_2) = q-1$;

$$\sigma(L_1) = \begin{pmatrix} p-1 & p+2q-3 \\ 1 & q-2 \end{pmatrix} \quad \text{and} \quad \sigma(L_2) = \begin{pmatrix} q-1 & 2p+q-3 \\ 1 & p-2 \end{pmatrix}.$$

Thus we see that $p+2q-3$ and $2p+q-3$ are distance Laplacian eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_{pq})$ with multiplicities $q-2$ and $p-2$ respectively and the remaining distance Laplacian eigenvalues are the eigenvalues of the vertex weighted distance Laplacian matrix of Υ_{pq} given by

$$T_D^L(\Upsilon_{pq}) = \begin{bmatrix} p-1 & -(p-1) \\ -(q-1) & q-1 \end{bmatrix}.$$

The characteristic of the above matrix is $x^2 - (p+q-2)x$ and has eigenvalues 0 and $p+q-2$. Thus the distance Laplacian spectrum of this graph is obtained as

$$SpecD^L(\mathit{\Gamma}(\mathbb{Z}_{pq})) = \begin{pmatrix} 0 & p+q-2 & p+2q-3 & 2p+q-3 \\ 1 & 1 & q-2 & p-2 \end{pmatrix}.$$

5.1. **Distance Laplacian spectrum of $\boldsymbol{\Gamma(\mathbb{Z}_{p^k}), k \geqslant 3}$.** For any connected graph G with n vertices, the distance Laplacian matrix $D^L(G)$ is positive semi definite and the least distance Laplacain eigenvalue is 0 with multiplicity 1. ie $\partial_1^L \geqslant \partial_2^L \geqslant \cdots > \partial_n^L = 0$. refer[18]. The the distance Laplacian eigenvalues of a graph of diameter at most 2, can be expressed in terms of its Laplacian eigenvalues as in the following theorem.

Theorem 5.2. *[15] Let G be a connected graph on n vertices with diameter $d \leqslant 2$. Let $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) = 0$, be the Laplacian eigenvalues of G. Then the distance Laplacian eigenvalues of G are*

$$2n - \mu_{n-1}(G) \geqslant 2n - \mu_{n-2}(G) \geqslant \cdots \geqslant 2n - \mu_1(G) > \partial_n^L = 0.$$

The next theorem explores the Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^k}), k \geqslant 3$.

Theorem 5.3. *[26] Consider $\Gamma(\mathbb{Z}_{p^k}), k \geqslant 3$. Then the following hold*

(i) If $k = 2m$ for some $m \geqslant 2$, then the Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^k})$ is given by

$$\begin{pmatrix} p^{2m-1}-1 & p^{2m-2}-1 & \cdots & p^{m+1}-1 & p^m-1 & p^{m-1}-1 & \cdots & p-1 & 0 \\ \phi(p) & \phi(p^2) & \cdots & \phi(p^{m-1}) & \phi(p^m)-1 & \phi(p^{m+1}) & \cdots & \phi(p^{2m-1}) & 1 \end{pmatrix}$$

(ii) If $k = 2m+1$ for some $m \geqslant 1$, then the Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^k})$ is given by

$$\begin{pmatrix} p^{2m}-1 & p^{2m-1}-1 & \cdots & p^{m+1}-1 & p^m-1 & p^{m-1}-1 & \cdots & p-1 & 0 \\ \phi(p) & \phi(p^2) & \cdots & \phi(p^m) & \phi(p^{m+1})-1 & \phi(p^{m+2}) & \cdots & \phi(p^{2m}) & 1 \end{pmatrix}.$$

Thus we have the following theorem.

Theorem 5.4. *Let $k \geqslant 3$.*

(i) If $k = 2m$ for some $m \geqslant 2$, then the distance Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^k})$ is given by

$$\begin{pmatrix} \begin{array}{c} 2p^{2m-1} \\ -p-1 \end{array} & \begin{array}{c} 2p^{2m-1} \\ -p^2-1 \end{array} & \cdots & \begin{array}{c} 2p^{2m-1} \\ -p^{m-1}-1 \end{array} & \begin{array}{c} 2p^{2m-1} \\ -p^m-1 \end{array} & \begin{array}{c} 2p^{2m-1} \\ -p^{m+1}-1 \end{array} & \cdots & p^{2m-1}-1 & 0 \\ \phi(p^{2m-1}) & \phi(p^{2m-2}) & \cdots & \phi(p^{m+1}) & \phi(p^m)-1 & \phi(p^{m-1}) & \cdots & \phi(p) & 1 \end{pmatrix}.$$

(ii) If $k = 2m+1$ for some $m \geqslant 1$, then the distance Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^k})$ is given by

$$\begin{pmatrix} \begin{array}{c} 2p^{2m} \\ -p-1 \end{array} & \begin{array}{c} 2p^{2m} \\ -p^2-1 \end{array} & \cdots & \begin{array}{c} 2p^{2m} \\ -p^{m-1}-1 \end{array} & \begin{array}{c} 2p^{2m} \\ -p^m-1 \end{array} & \begin{array}{c} 2p^{2m} \\ -p^{m+1}-1 \end{array} & \cdots & p^{2m}-1 & 0 \\ \phi(p^{2m}) & \phi(p^{2m-1}) & \cdots & \phi(p^{m+2}) & \phi(p^{m+1})-1 & \phi(p^m) & \cdots & \phi(p) & 1 \end{pmatrix}.$$

Proof. The compressed zero-divisor graph Υ_{p^k} is a connected graph of diameter 2 for $k \geqslant 4$, since $p^{k-1} \sim p^j, \forall j = 1, 2, \ldots, k-2$. And for $k = 3$, $\Upsilon_{p^k} \cong K_2$ and hence it is of diameter 1. Thus from equation (4.7), it follows that $\Gamma(\mathbb{Z}_{p^k})$ is of diameter at most 2 for $k \geqslant 3$. The number of vertices in $\Gamma(\mathbb{Z}_{p^k})$ is $p^{k-1} - 1$. Thus the conclusion follows from Theorem 5.2 and Theorem 5.3. [Kindly note that, for convenience, each eigenvalue is spread into first two rows where as the multiplicity is given against each eigenvalue, in the third row.] □

6. CONCLUSION

The special combinatorial structure as well as the typical block structure of the distance related matrices of the zero-divisor graph on the ring of integers modulo n plays a major role to motivate us to inculcate the tools of matrices in the computation of the spectrum. We conclude that the distance related spectrum of $\Gamma(\mathbb{Z}_n)$ is completely determined by its compressed zero-divisor graph.

REFERENCES

[1] A. HAOUAOUI, A. BENHISSI: *Zero-divisors and zero-divisor graphs of power series rings* , Ricerche di Mathematica, **65** (2015), 1-13.

[2] D. F. ANDERSON, P. S. LIVINGSTON: *The zero-divisor graph of a commutative ring*, J. Algebra, **217**(2) (1999), 434-447.

[3] D. F. ANDERSON, J. D. LANGRANGE: *Some remarks on the compressed zero-divisor graphs*, J. Algebra, **447** (2016), 297-321.

[4] D. M CARDOSO, I. GUTMAN, E. A. MARTINS, M. ROBBIANO: *A generalization of Fiedler's lemma and some applications*, Linear and Multilinear Algebra, **59**(8) (2011), 929-942.

[5] D. M. CARDOSO, M. A. DE FREITA, E. A. MARTIN, M. ROBBIANO: *spectra of graphs obtained by a generalization of the join graph operation*, Discrete Mathematics, **313** (2013), 733-741.

[6] D. M. CARDOSO, R.C. DÍAZ, O. ROJO: *Distance matrices on the H-join of graphs: a general result and applications*, Linear Algebra and its Applications, **559** (2018), 34-53.

[7] D. CVETKOVIĆ, P. ROWLINSON, S. SIMIĆ: *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, 2010.

[8] E. A. OBASA, S. AL-ADDASI, B. AL KHAMAISEH: *Some properties of the zero-divisor graph for the ring of Gaussian integers modulo n*, Glasgow Mathematical Journal, **53**(2) (2011), 391-399.

[9] F. R. K. CHUNG, R. P. LANGLANDS: *A combinatorial Laplacian with vertex weights*, Journal of Combinatorial Theory, series A, **75** (1996), 316-327.

[10] G. INDULAL, I. GUTMAN, A. VIJAYAKUMAR: *On distance energy of graphs* , MATCH Commun. Math. Comput. Chem., **60** (2008), 461-472.

[11] H. KUMAR, K. L. PATRA, B. K. SAHOO: *Proper divisor graph of a positive integer*, arXiv.2005.O4441v1[math.Co], May 2020.

[12] I. BECK: *Coloring of a commutative ring*, J. Algebra, **116**(1) (1988), 208-226.

[13] I. GUTMAN, B. ZHOU: *Laplacian energy of a graph*, Linear Algebra Appl., **414**(2006), 29-37.

[14] J. A. BONDY, U. S. R. MURTY: *Graph Theory*, Springer, 2008.

[15] K. C. DAS, M. AOUCHICHE, P. HANSEN: *On distance Laplacian and distance signless Laplacian eigenvalues of graphs*, Linear and Multilinear Algebra, **67**(11) (2019), 2307-2324.

[16] M. AOUCHICHE, P. HANSEN: *Distance spectra of graphs, A survey*, Linear Algebra Appl., **458** (2014), 301-386.

[17] M. AOUCHICHE, P. HANSEN: *Two Laplacian for the distance matrix of a graph*, Linear Algebra Appl., **439**(1) (2013), 21-33.

[18] M. AOUCHICHE, P. HANSEN: *Some properties of the distance Laplacian eigenvalues of a graph*, Czechoslovak mathematical Journal, **139**(64) (2014), 751-761.

[19] M. FIEDLER: *Eigenvalues of nonnegative symmetric matrices*, Linear Algebra Appl., **9** (1974), 119-142.

[20] M. YOUNG: *Adjacency matrices of zero-divisor graphs of integers modulo n*, Involve: A Journal of Mathematics, **8**(5) (2015), 753-761.

[21] N. BIGGS: *Algebraic Graph Theory*, Cambridge University Press, 1974.

[22] P. M. MAGI, SR. MAGIE JOSE, A. KISHORE: *Spectrum of the zero-divisor graph on the ring of integers modulo n*, J. Math. Comput. Sci., **10** (2020), 1643-1666.

[23] P. SHARMA, A. SHARMA, R. K. VATS: *Analysis of Adjacency Matrix and Neighborhood Associated with Zero Divisor Graph of Finite Commutative Rings*, International Journal of Computer Applications, **14**(3) (2011), 38-42.

[24] R. L. HEMMINGER: *The Group of an X-join of Graphs*, Journal of Combinatorial Theory, **5** (1968), 408-418.

[25] S. AYKAC, N. AKGUNES: *Analysis of graph parameters associated with zero divisor graphs of commutative rings*, NTMSCI, **6**(2) (2018), 144-149.

[26] S. CHATTOPADHYAY, K. L. PATRA, B. K. SAHOO: *Laplacian eigenvalues of the zero divisor graph of the ring Zn*, Linear Algebra and its Applications, **584** (2020), 267-286.

[27] T. TAMIZH CHELVAM, T. ASIR: *Distances in zero-divisor and total graphs from commutative rings- A survey*, AKCE International Journal of Graphs and Combinatorics, **13** (2016), 290-298.

DEPARTMENT OF MATHEMATICS, ST. MARY'S COLLEGE, THRISSUR, KERALA, INDIA
Email address: tcr.maggie81@gmail.com

DEPARTMENT OF MATHEMATICS, ST. MARY'S COLLEGE, THRISSUR, KERALA, INDIA
Email address: srmargaretmary.cmc@gmail.com

DEPARTMENT OF MATHEMATICS, VIMALA COLLEGE, THRISSUR, KERALA, INDIA
Email address: anjalykishor@gmail.com