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GENERALIZATION OF ZUMKELLER NUMBERS

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ABSTRACT. In this paper, we attempt to generalize Zumkeller numbers to k-Zumkeller numbers, k-half-Zumkeller numbers, near-k-Zumkeller numbers and r-near-k-Zumkeller numbers and study various properties of these numbers.

1. INTRODUCTION

If the sum of all the proper positive divisors of a positive integer is equal to the number, then the number is called perfect number. Many generalizations of perfect numbers are seen in [1], [2]. Generalizing the concept of perfect number, R. H. Zumkeller defined a new type of number as Zumkeller number.

Definition 1.1. *A positive integer n is Zumkeller if the set positive divisors of n can be partitioned into two subsets such that sum of each subset is equal to $\frac{\sigma(n)}{2}$, where $\sigma(n)$ is the sum of all positive divisors of n [4].*

Clark et. al in [3] and Y. Peng, K.P.S. Bhaskara Rao in [5] established several results and conjectures on Zumkeller numbers.

In this paper, we define k-Zumkeller numbers, k-half-Zumkeller numbers, near-k-Zumkeller numbers and r-near-k-Zumkeller numbers and study properties of these numbers.

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2. Generalization Of Zumkeller Numbers

2.1. k-Zumkeller Numbers.

Definition 2.1. *A positive integer n is called a k-Zumkeller number if the set of positive divisors of n can be partitioned into k disjoint subsets of equal sum.*

A k-Zumkeller partition for a k-Zumkeller number n is a partition $\{A_1, A_2, \ldots, A_k\}$ of the set of positive divisors of n so that each of $A_1, A_2, \ldots, A_k$ sums to the same value $\frac{\sigma(n)}{k}$.

Example 1. *The 2-Zumkeller numbers are the Zumkeller numbers. The numbers 120, 180, 240, 360, 420, 480, 504, 540, 600, 660, 672,... are the first few 3-Zumkeller numbers. One of the example of 4-Zumkeller numbers is 30240.*

Proposition 2.1. *If n is a k-Zumkeller number then,*

(a) $\sigma(n) \geq kn$.
(b) $k|\sigma(n)$.

Proof. The Proof is identical to proof in [5]. □

Proposition 2.2. *There is no k-Zumkeller number of the form 2^m, where $m \geq 2$ is an integer.*

Proposition 2.3. *There is no k-Zumkeller number of the form $p_1 p_2 \cdots p_m$, where $p_i^{'s}$ are distinct primes.*

Proposition 2.4. *If n is a k-Zumkeller number, and p is a prime such that $(n, p) = 1$ then np^l is also a k-Zumkeller number for any positive integer l.*

Proof. If $\{A_1, A_2, \ldots, A_k\}$ is the k-Zumkeller partition for n then clearly $\{A_1 \cup (pA_1) \cup \ldots \cup (p^l A_1), A_2 \cup (pA_2) \cup \ldots \cup (p^l A_2), \ldots, A_k \cup (pA_k) \cup \ldots \cup (p^l A_k)\}$ is the k-Zumkeller partition for the integer np^l. □

Corollary 2.1. *If n is a k-Zumkeller number and m is relatively prime to n, then mn is also a k-Zumkeller number.*

Proposition 2.5. *If the integer $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ (where $p_i^{'s}$ are distinct primes) is a k-Zumkeller number then the integer $\prod_{i=1}^{r} p_i^{\alpha_i + \beta_i(\alpha_i+1)}$ is also a k-Zumkeller number, where $\beta_1, \beta_2, \ldots, \beta_r$ are non-negative integers.*

Proof. The Proof is identical to proof in [5]. □

Example 2. $120 = 2^3 \times 3 \times 5$ *is a* 3*-Zumkeller number therefore taking* $\beta_1 = 2$*,* $\beta_2 = 2$ *and* $\beta_3 = 1$ *we have* $2^{11} \times 3^5 \times 5^3$ *is also a* 3*-Zumkeller number.*

Proposition 2.6. *If* m *is* k_1*-Zumkeller number and* n *is* k_2*-Zumkeller number and also* $(m, n) = 1$ *then* mn *is a* k_1k_2*-Zumkeller number.*

Proof. Let $\{M_1, M_2, \ldots, M_{k_1}\}$ be the k_1-Zumkeller partition for the integer m and $\{N_1, N_2, \ldots, N_{k_2}\}$ be the k_2-Zumkeller partition for the integer n. Since, $(m, n) = 1$, clearly $\{M_iN_j | 1 \leq i \leq k_1, 1 \leq j \leq k_2\}$ is a k_1k_2-Zumkeller partition for the integer mn. Thus, mn is a k_1k_2-Zumkeller number. □

We can generalize the above proposition as given below.

Corollary 2.2. *If* n_i *(* $i = 1, 2, \ldots, m$ *) is a* k_i*-Zumkeller number then* $n_1n_2 \cdots n_m$ *is a* $(k_1k_2 \cdots k_m)$*-Zumkeller number for* $(n_i, n_j) = 1$*;* $2 \leq i \neq j \leq m$*.*

2.2. *k*-Half-Zumkeller Numbers.

Definition 2.2. *A positive integer* n *is called a* k*-half-Zumkeller number if the set of proper positive divisors of* n *can be partitioned into* k *disjoint subsets of equal sum.*

A k-half-Zumkeller partition for a k-Zumkeller number n is a partition $\{A_1, A_2, \ldots, A_k\}$ of the set of positive divisors of n so that each of $A_1, A_2, \ldots, A_k$ sums to the same value $\frac{\sigma(n)-n}{k}$.

Example 3. *The* 2*-half-Zumkeller numbers are the half-Zumkeller numbers. The numbers 24, 60, 72, 90, 96, 120 ... are the first few* 3*-half-Zumkeller numbers. 120 is the first* 4*-half-Zumkeller number.* 945 *is an example of odd* 3*-half-Zumkeller number.*

Proposition 2.7. *If* n *is a* k*-half-Zumkeller number then,*

(a) $\sigma(n) - n \geq k \times \frac{n}{m}$*, where* $m \geq 2$ *is the smallest positive integer s.t.* $m|n$*;*
(b) $k|\{\sigma(n) - n\}$.

Proposition 2.8. *If* m *and* n *are* k*-half-Zumkeller numbers with* $(m, n) = 1$*, then* mn *is also* k*-half-Zumkeller number.*

Proof. The proof is identical to the proof of proposition 25 of [5]. □

2.3. Near-k-Zumkeller Numbers.

Definition 2.3. *For any positive integer $k > 1$, a positive integer n is called a near-k-Zumkeller number if we can partition the set of all the positive divisors of n into k disjoint subsets of equal sum, except for one of the divisor d where $1 < d < n$. The divisor d is called the redundant divisor.*

A near-k-Zumkeller partition for a positive integer n is a partition $\{A_1, A_2, \dots, A_k\}$ of the set of positive divisors of n so that each of $A_1, A_2, \dots, A_k$ sums to the same value $\frac{\sigma(n)-d}{k}$.

Example 4. *The near-2-Zumkeller numbers are the near-Zumkeller numbers. The numbers $180, 240, 360, 420$ are the first few near-3-Zumkeller numbers.*

Proposition 2.9. *If n is a near-k-Zumkeller number with redundant divisor d then*

(a) $\sigma(n) \geq kn + d$
(b) $k|\{\sigma(n) - d\}$.

Proposition 2.10. *There is no near-k-Zumkeller number of the form 2^l, where $l \geq 1$ is a positive integer.*

Proposition 2.11. *There is no near-k-Zumkeller number of the form p_1p_2, where p_1, p_2 are distinct odd prime numbers.*

Proposition 2.12. *For $k \geq 3$ there is no near-k-Zumkeller number of the form $2^\alpha p$ where p is an odd prime and $\alpha \geq 1$ is a positive integer.*

Proof. If $n = 2^\alpha p$ is a near-k-Zumkeller number ($k \geq 3$) with redundant divisor d then

$$kn + d \leq \sigma(2^\alpha p) = \left(2^{\alpha+1} - 1\right)(p+1)$$
$$\Rightarrow \quad k + \frac{d}{n} \leq \left(2 - \frac{1}{2^\alpha}\right)\left(1 + \frac{1}{p}\right) \leq 2\left(1 + \frac{1}{p}\right) < 3,$$

which is a contradiction. Hence, there is no near-k-Zumkeller number of the form $2^\alpha p$ where $k \geq 3$. □

Proposition 2.13. *There is no near-k-Zumkeller number of the form $2^\alpha p_1 p_2$ for $k \geq 4$, where p_1, p_2 are odd primes, $p_1 < p_2$ and $\alpha \geq 1$ is a positive integer.*

Proof. Proceeding similarly as proposition 2.12 we get,

$$\Rightarrow k + \frac{d}{n} \leq 2\left(1 + \frac{1}{p_1}\right)\left(1 + \frac{1}{p_2}\right) < 4,$$

which is a contradiction. Hence, there is no near-k-Zumkeller number of the form $2^\alpha p_1 p_2$ for $k \geq 4$. □

Corollary 2.3. *The near-3-Zumkeller number of the form $2^\alpha p_1 p_2$ exists where p_1, p_2 are odd primes, $p_1 < p_2$ and $\alpha \geq 1$ is a positive integer, only for $p_1 = 3$ and $p_2 = 5$ or $p_2 = 7$.*

Proof. If $n = 2^\alpha p_1 p_2$ is near-3-Zumkeller number with redundant divisor d we have,

$$3 + \frac{d}{n} \leq 2\left(1 + \frac{1}{p_1}\right)\left(1 + \frac{1}{p_2}\right).$$

The above inequality exists only for $p_1 = 3$ and $p_2 = 5$ or $p_2 = 7$. □

Remark 2.1. *If $p_1 = 3$ and $p_2 = 5$ in the above proposition then the redundant divisor d must be a multiple of 3.*

Proposition 2.14. *There is no near-k-Zumkeller number of the form $2^\alpha p_1^2 p_2$ for $k \geq 4$, where p_1, p_2 are odd primes, $p_1 < p_2$ and $\alpha \geq 1$ is a positive integer.*

Proof. If $n = 2^\alpha p_1^2 p_2$ is near-3-Zumkeller number with redundant divisor d we have,

$$k + \frac{d}{n} \leq 2\left(1 + \frac{1}{p_1} + \frac{1}{p_1^2}\right)\left(1 + \frac{1}{p_2}\right) < 4,$$

which is a contradiction. Hence, there is no near-k-Zumkeller number of the form $2^\alpha p_1^2 p_2$ for $k \geq 4$. □

Corollary 2.4. *The near-3-Zumkeller number of the form $2^\alpha p_1^2 p_2$ exists where p_1, p_2 are odd primes, $p_1 < p_2$ and $\alpha \geq 1$ is a positive integer, only for $p_1 = 3$ and $p_2 = 5$ or $p_2 = 7$ or $p_2 = 11$ or $p_2 = 13$ or $p_2 = 17$ or $p_2 = 19$ or $p_2 = 23$.*

Remark 2.2. *If $p_1 = 3$ and $p_2 = 5$ or 11 or 17 or 23 in the above proposition then the redundant divisor d must be a multiple of 3.*

2.4. r-Near-k-Zumkeller Numbers.

Definition 2.4. *A positive integer n is called r-near-k-Zumkeller number if we can partition the set of all the positive divisors of n into k disjoint subsets of equal sum, except for r number of positive divisors of n say $d_1, d_2, \ldots, d_r$, where $1 < d_i < n, \forall i = 1, 2, \ldots, r$. The divisors $d_1, d_2, \ldots, d_r$ are called the redundant divisors.*

A r-near-k-Zumkeller partition for a r-near-k-Zumkeller number n is a partition $\{A_1, A_2, \ldots, A_k\}$ of the set of positive divisors of n so that each of $A_1, A_2, \ldots, A_k$ sums to the same value

$$\frac{\sigma(n) - \sum_{i=1}^{r} d_i}{k}.$$

Example 5. *The 1-near-k-Zumkeller numbers are the near-k-Zumkeller numbers. The numbers $240, 360, 420, \ldots$ are the 2-near-3-Zumkeller numbers. $240, 360, 420, \ldots$ are the 3-near-3-Zumkeller numbers.*

Proposition 2.15. *If n is a r-near-k-Zumkeller number with redundant divisor $d_1, d_2, \ldots, d_r$ then*

(a) $\sigma(n) \geq kn + \sum_{i=1}^{r} d_i$;
(b) $k | \{\sigma(n) - \sum_{i=1}^{r} d_i\}$.

Proposition 2.16. *There is no r-near-k-Zumkeller number of the form 2^l, where $l \geq 1$ is a positive integer.*

Proposition 2.17. *There is no r-near-k-Zumkeller number of the form p_1p_2, where p_1, p_2 are distinct odd prime numbers.*

Proposition 2.18. *For $k \geq 3$ there is no r-near-k-Zumkeller number of the form $2^\alpha p$ where p is an odd prime and $\alpha \geq 1$ is a positive integer.*

Proposition 2.19. *There is no r-near-k-Zumkeller number of the form $2^\alpha p_1p_2$ for $k \geq 4$, where p_1, p_2 are odd primes, $p_1 < p_2$ and $\alpha \geq 1$ is a positive integer.*

Corollary 2.5. *The r-near-3-Zumkeller number of the form $2^\alpha p_1p_2$ exists where p_1, p_2 are odd primes, $p_1 < p_2$ and $\alpha \geq 1$ is a positive integer, only for $p_1 = 3$ and $p_2 = 5$ or $p_2 = 7$.*

Remark 2.3. *If $p_1 = 3$ and $p_2 = 5$ in the above proposition then $3 \Big| \sum_{i=1}^{r} d_i$.*

Proposition 2.20. *There is no r-near-k-Zumkeller number of the form $2^\alpha p_1^2 p_2$ for $k \geq 4$, where p_1, p_2 are odd primes, $p_1 < p_2$ and $\alpha \geq 1$ is a positive integer.*

Corollary 2.6. *The r-near-3-Zumkeller number of the form $2^\alpha p_1^2 p_2$ exists where p_1, p_2 are odd primes, $p_1 < p_2$ and $\alpha \geq 1$ is a positive integer, only for $p_1 = 3$ and $p_2 = 5$ or $p_2 = 7$ or $p_2 = 11$ or $p_2 = 13$ or $p_2 = 17$ or $p_2 = 19$ or $p_2 = 23$.*

Remark 2.4. *If $p_1 = 3$ and $p_2 = 5$ or 11 or 17 or 23 in the above proposition then $3 | \sum_{i=1}^{r} d_i$.*

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