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ON A COMPREHENSIVE CLASS OF P-VALENT FUNCTIONS DEFINED BY GENERALIZED AL-OBOUDI DIFFERENTIAL OPERATOR

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ABSTRACT. The main purpose of this research is to introduce a new class $\mathcal{H}^m_{t,p}(\sigma,\mu,\beta,\lambda)$ define by Opoola differential operator involving functions $f(z) \in \mathfrak{A}_p$. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also determined.

1. INTRODUCTION AND DEFINITION

Let $\mathfrak{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{\kappa=1}^{\infty} a_{p+\kappa} z^{p+\kappa}, \quad (p \in \mathfrak{N} = \{1,2,3,\cdots\}, \ \ z \in \mathfrak{U}), \tag{1.1}$$

which are holomorphic in the open unit disc $\mathfrak{U} = \{z \in \mathcal{C} : |z| < 1\}$.

Let $\mathcal{S}_p \subset \mathfrak{A}_p$ denote the class of functions p-valent in $\mathfrak{U}$ and by $S^*_p(\sigma) \subset \mathcal{S}_p$ denote the class of p-valently starlike functions of order σ, $0 \le \sigma < p$ which satisfy the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \sigma \qquad (z \in \mathfrak{U}).$$

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Also, a function $f(z)$ belonging to $\mathcal{K}_p(\sigma) \subset \mathcal{S}(p)$ is said to be p-valently convex functions of order σ, $0 \leq \sigma < p$ in $\mathfrak{U}$, if and only if

$$\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \sigma \qquad (z \in \mathfrak{U})$$

and by $\mathcal{R}_p(\sigma) \subset \mathcal{S}_p$ denote the class of p-valently bounded turning functions which satisfy the condition

$$\Re\{f'(z)\} > \sigma \quad (z \in \mathfrak{U}).$$

It is well known that $\mathcal{K}_p(\sigma) \subset S_p^*(\sigma) \subset \mathcal{S}_p$.

Motivated by the work of Shaba et al [9], we introduce new subclass of the p-valent function class $\mathfrak{A}_p$ associated with the Opoola differential operator and to obtain some of its properties.

Definition 1.1. *[6] The Opoola differential operator $D^{m,\mu}_{\beta,\lambda,p}$ for p-valently functions is define as follows:*

$$D^{0,\mu}_{\beta,\lambda,p}f(z) = f(z),$$

$$D^{1,\mu}_{\beta,\lambda,p}f(z) = (1 + (\beta - \mu - 1)\lambda)f(z) + z^p(\mu - \beta)\lambda + z\lambda f'(z) + z^p(p-1)(1-\lambda) = D^{\mu}_{\beta,\lambda,p}f(z), \tag{1.2}$$

$$D^{2,\mu}_{\beta,\lambda,p}f(z) = D^{\mu}_{\beta,\lambda,p}(D^{1,\mu}_{\beta,\lambda,p}f(z)),$$

$$D^{m,\mu}_{\beta,\lambda,p}f(z) = D^{\mu}_{\beta,\lambda,p}(D^{m-1,\mu}_{\beta,\lambda,p}f(z)) \quad m \in \mathfrak{N}, \tag{1.3}$$

if $f(z)$ is given by (1.1)*, then by* (1.2) *and* (1.3)*, we see that*

$$D^{m,\mu}_{\beta,\lambda,p}f(z) = p^m z^p + \sum_{\kappa=1}^{\infty}(1 + (p + \kappa + \beta - \mu - 1)\lambda)^m a_{p+\kappa}z^{p+\kappa}, \tag{1.4}$$

where $0 \leq \mu \leq \beta, \lambda \geq 0$ *and* $m \in \mathfrak{N}_0 = \{0, 1, 2, 3 \ldots\}$.

Remark 1.1. *We have the following remarks:*

(1) *When $\mu = \beta = 1$ and $\lambda = p = 1$, $D^m f(z)$ is the Salegean differential operator [8].*
(2) *When $\mu = \beta = p = 1$, $D^m_\lambda f(z)$ is the Al-Oboudi differential operator [1].*
(3) *When $p = 1$, $D^m_\lambda f(z)$ is the Opoola differential operator [7].*

Remark 1.2. *It follows from the* (1.4) *that*

$$D^{m+1,\mu}_{\beta,\lambda,p}f(z) = (1 + (\beta - \mu - 1)\lambda)D^{m,\mu}_{\beta,\lambda,p}f(z) + z^p(\mu - \beta)\lambda + z^p(p-1)(1-\lambda) + z\lambda(D^{m,\mu}_{\beta,\lambda,p}f(z))', \tag{1.5}$$

for $m \in \mathfrak{N}_0$ *and* $z \in \mathfrak{U}$.

Lemma 1.1. *[5] Let* $q(z)$ *be holomorphic in* $\mathfrak{U}$ *with* $q(0) = 1$ *and suppose that*

$$\Re\left(1 + \frac{zq'(z)}{q(z)}\right) > \frac{3\sigma - 1}{2\sigma}, \quad (z \in \mathfrak{U}).$$

Then $\Re(q(z)) > \sigma$ *in* $\mathfrak{U}$ *and* $\frac{1}{2} \leq \sigma < 1$.

2. Main result

Definition 2.1. *We say that a function* $f(z) \in \mathfrak{A}_p$ *is in the class* $\mathcal{H}^m_{t,p}(\sigma, \mu, \beta, \lambda)$ *if*

$$\left|\frac{D^{m+1,\mu}_{\beta,\lambda,p} f(z)}{z^p} \left(\frac{z^p}{D^{m,\mu}_{\beta,\lambda,p} f(z)}\right)^t - p\right| < p - \sigma \quad (z \in \mathfrak{U}),$$

where $0 \leq \mu \leq \beta, t \geq 0, \lambda \geq 0$, $m \in \mathfrak{N}_0 = \{0, 1, 2, 3 \cdots\}$ *and* $0 \leq \sigma < 1$.

Remark 2.1. *The family* $\mathcal{H}^m_{t,p}(\sigma, \mu, \beta, \lambda)$ *is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. In place of "equivalence" we are going to take "contained in" as it was discussed in [2]. For example,*

(1) For $m = 0$ and $t = 1$, we have the class $\mathcal{H}^0_{1,p}(\sigma, \mu, \beta, \lambda)$ contained in $S^*_p(\sigma)$.
(2) For $m = 1$, $\mu = \beta = 1$ and $t = 1$, we have the class $\mathcal{H}^1_{1,p}(\sigma, 1, 1, 1)$ contained in $\mathcal{K}_p(\sigma)$.
(3) For $m = 0$ and $t = 0$, we have the class $\mathcal{H}^0_{0,p}(\sigma, \mu, \beta, \lambda)$ contained in $\mathcal{R}_p(\sigma)$.
(4) For $\mu = \beta = 1$ and $p = 1$, we have the class

$$\mathcal{H}^m_{t,1}(\sigma, 1, 1, \lambda) = \mathcal{H}^m_t(\sigma, \lambda).$$

introduced by Catas and Lupas [3].
(5) For $m = 0$ and $p = 1$, the class

$$\mathcal{B}(t, \sigma) = \left\{f(z) \in \mathfrak{A} : \left|f'(z)\left(\frac{z}{f(z)}\right)^t - 1\right| < 1 - \sigma; t \geq 0, 0 \leq \sigma < 1,\ z \in \mathfrak{U}\right\}$$

intoduced by Frasin and Jahangiri [5].
(6) For $m = 0$, $p = 1$ and $t = 2$, the class

$$\mathcal{B}(\sigma) = \left\{f(z) \in \mathfrak{A} : \left|\frac{z^z f'(z)}{f^2(z)} - 1\right| < 1 - \sigma; 0 \leq \sigma < 1,\ z \in \mathfrak{U}\right\}$$

intoduced by Frasin and Darus [4].

Theorem 2.1. *If for the function* $f(z) \in \mathfrak{A}_p$*,* $0 \leq \mu \leq \beta, t \geq 0, \lambda \geq 0$*,* $m \in \mathfrak{N}_0 = \{0, 1, 2, 3 \ldots\}$ *and* $\frac{1}{2} \leq \sigma < 1$*, we have*

$$\Re\Bigg(\frac{1}{\lambda}\frac{D^{m+2,\mu}_{\beta,\lambda,p}f(z)}{D^{m+1,\mu}_{\beta,\lambda,p}f(z)} - \frac{t}{\lambda}\frac{D^{m+1,\mu}_{\beta,\lambda,p}f(z)}{D^{m,\mu}_{\beta,\lambda,p}f(z)} - \frac{z^p(\mu-\beta)}{D^{m+1,\mu}_{\beta,\lambda,p}f(z)} + \frac{tz^p(\mu-\beta)}{D^{m,\mu}_{\beta,\lambda,p}f(z)}$$
$$-\frac{z^p(p-1)(1-\lambda)}{\lambda D^{m+1,\mu}_{\beta,\lambda,p}f(z)} + \frac{tz^p(p-1)(1-\lambda)}{\lambda D^{m,\mu}_{\beta,\lambda,p}f(z)} - \frac{(1-t)(1+\lambda\beta-\lambda\mu)}{\lambda}$$
$$+\frac{(\lambda-\lambda t)(1-p)}{\lambda} + 1\Bigg) > \frac{3\sigma-1}{2\sigma},$$

then $f(z) \in \mathcal{H}^m_{t,p}(\sigma, \mu, \beta, \lambda)$.

Proof. Consider

$$q(z) = \frac{D^{m+1,\mu}_{\beta,\lambda,p}f(z)}{z^p}\left(\frac{z^p}{D^{m,\mu}_{\beta,\lambda,p}f(z)}\right)^t,$$

then $q(z)$ is analytic in $\mathbb{U}$ with $q(0) = 1$, by simplification we have

$$\ln(q(z)) = \ln(D^{m+1,\mu}_{\beta,\lambda,p}f(z)) - p\ln(z) + tp\ln(z) - t\ln(D^{m,\mu}_{\beta,\lambda,p}f(z)).$$

A simple differentiation yields

$$\frac{zq'(z)}{q(z)} = \frac{1}{\lambda}\frac{D^{m+2,\mu}_{\beta,\lambda,p}f(z)}{D^{m+1,\mu}_{\beta,\lambda,p}f(z)} - \frac{t}{\lambda}\frac{D^{m+1,\mu}_{\beta,\lambda,p}f(z)}{D^{m,\mu}_{\beta,\lambda,p}f(z)} - \frac{z^p(\mu-\beta)}{D^{m+1,\mu}_{\beta,\lambda,p}f(z)} + \frac{tz^p(\mu-\beta)}{D^{m,\mu}_{\beta,\lambda,p}f(z)}$$
$$-\frac{z^p(p-1)(1-\lambda)}{\lambda D^{m+1,\mu}_{\beta,\lambda,p}f(z)} + \frac{tz^p(p-1)(1-\lambda)}{\lambda D^{m,\mu}_{\beta,\lambda,p}f(z)} - \frac{(1-t)(1+\lambda\beta-\lambda\mu)}{\lambda}$$
$$+\frac{(\lambda-\lambda t)(1-p)}{\lambda}.$$

By the hypothesis of the Theorem 2.1, we have

$$\Re\left(1 + \frac{zq'(z)}{q(z)}\right) > \frac{3\sigma-1}{2\sigma}, \quad (z \in \mathfrak{U}).$$

Hence, by Lemma 1.1, we have

$$\Re\left(\frac{D^{m+1,\mu}_{\beta,\lambda,p}f(z)}{z^p}\left(\frac{z^p}{D^{m,\mu}_{\beta,\lambda,p}f(z)}\right)^t\right) > \sigma, \quad (z \in \mathfrak{U}),$$

and therefore, $f(z) \in \mathcal{H}^m_{t,p}(\sigma, \mu, \beta, \lambda)$ by Definition 2.1. □

As consequences of the above theorem we have the following corollaries.

Choosing $m = 1$, $t = 1$, $\sigma = \frac{1}{2}$ and $\beta = \lambda = \mu = p = 1$, we have

Corollary 2.1. *If* $\psi(z) \in \mathfrak{A}$ *and*

$$\Re\left(\frac{2zf''(z)+z^2f'''(z)}{zf''(z)+f'(z)} - \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \quad (z \in \mathfrak{U}).$$

Then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{1}{2}, \quad (z \in \mathfrak{U}).$$

That is $f(z)$ *is convex of order* $\frac{1}{2}$.

Choosing $m = 0$, $t = 0$, $\sigma = \frac{1}{2}$ and $\beta = \lambda = \mu = p = 1$, we have

Corollary 2.2. *If* $f(z) \in \mathfrak{A}$ *and*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{1}{2}, \quad (z \in \mathfrak{U}).$$

Then

$$\Re\left[f'(z)\right] > \frac{1}{2}, \quad (z \in \mathfrak{U}).$$

In another words, if the function $f(z)$ *is convex of order* $\frac{1}{2}$ *then* $f(z) \in \mathcal{H}^0_{0,1}(\sigma,1,1,1)$ *which is contained in* $\mathfrak{R}(\frac{1}{2})$.

Choosing $m = 0$, $t = 1$, $\sigma = \frac{1}{2}$ and $\beta = \lambda = \mu = p = 1$, we have

Corollary 2.3. *If* $f(z) \in \mathfrak{A}$ *and*

$$\Re\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) > -\frac{3}{2}, \quad (z \in \mathfrak{U}).$$

Then

$$\Re\left[\frac{zf'(z)}{f(z)}\right] > \frac{1}{2}, \quad (z \in \mathfrak{U}).$$

That is $f(z)$ *is starlike of order* $\frac{1}{2}$.

Choosing $m = 1$, $t = 0$, $\sigma = \frac{1}{2}$ and $\beta = \lambda = \mu = p = 1$, we have

Corollary 2.4. *If* $f(z) \in \mathfrak{A}$ *and*

$$\Re\left(\frac{2zf''(z)+z^3f'''(z)}{z^2f''(z)+zf'(z)}\right) > -\frac{1}{2}, \quad (z \in \mathfrak{U}).$$

Then

$$\Re\left[zf''(z)+f'(z)\right] > \frac{1}{2}, \quad (z \in \mathfrak{U}).$$

3. Conclusions

In this research, by applying Opoola differential operator, we define a comprehensive subclass of p-valent functions and established some of its properties. Results obtained provide properties of certain well-known subclasses of p-valent functions.

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