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SOME RESULTS ON PAIRWISE LOCALLY COMPACT BITOPOLOGICAL SPACES

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ABSTRACT. In this paper, it is proved that pairwise local compactness in bitopological spaces is a topological property. Further, necessary and sufficient condition for one point pairwise compactification of a bitopological space to be pairwise Hausdorff is obtained. Furthermore, an alternative proof for a pairwise Hausdorff and pairwise locally compact bitopological space to be pairwise regular is given. Finally, it is accomplished that one point pairwise compactification of a pairwise Hausdorff and pairwise locally compact bitopological space is pairwise normal.

1. INTRODUCTION

In the year 1963, a new theory, namely the theory of bitopological spaces commenced, when two arbitrary topologies on a non-empty set have been systematically studied by Kelly [1]. This new notion of bitopological spaces has been happened to be very effective for the investigation of non-symmetric functions which introduce two arbitrary topologies on a non-empty set. In this study, it has been also accomplished how various concepts and results of the classical topology can be generalized and extended in bitopological spaces. To demonstrate the same, the definitions of selective separation properties from the traditional topology have been generalized in bitopological spaces and named as pairwise Hausdorff, pairwise regular and pairwise normal. Taking inspiration from this fundamental paper,

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the most part of the theory of bitopological spaces is devoted for the generalizations or extensions of the concepts and results of the traditional topology. Specifically, the idea of compactness from the classical topology has been generalized, extended and investigated in various forms in bitopological spaces by Kim [2], Fletcher et al. [3], Swart [5], Saegrove [4], Datta [6] and Cooke and Reilly [8]. Also, by generalizing the idea of local compactness in the traditional topology, the concepts of pairwise local compactness and one point pairwise compactification in bitopological spaces have been introduced by Reilly [7]. This notion of pairwise local compactness has been further used to generalize some well-known results of the traditional topology.

In this paper, we will provide another definition of pairwise local compactness in bitopological spaces along with weaker form of pairwise local compactness in bitopological space and also use these concepts of pairwise local compactness to generalize and prove some additional results in bitopological spaces. In particular, it is obtained that image of a pairwise locally compact bitopological space under a pairwise continuous, pairwise open and pairwise onto map is pairwise locally compact and hence, pairwise local compactness in bitopological spaces is a topological property. Further, it is obtained that necessary and sufficient condition for one point pairwise compactification of a bitopological space to be pairwise Hausdorff is that bitopological space is pairwise Hausdorff and pairwise locally compact. Also, an alternative proof is given to prove that a pairwise Hausdorff and pairwise locally compact bitopological space is pairwise regular. Finally, it is proved that if a bitopological space is pairwise Hausdorff and pairwise locally compact, then its one point pairwise compactification is pairwise normal.

2. PRELIMINARIES

If X is a non-empty set and τ_1, τ_2 are two arbitrary topologies defined on X, then a triplet of the form (X, τ_1, τ_2) denotes a bitopological space. For any subset A of a bitopological space (X, τ_1, τ_2), notations τ_1-cl(A) and τ_2-cl(A) are used to denote closure of A with respect to τ_1 and τ_2 respectively. Similarly, τ_1-open (τ_1-closed) is used to represent open (closed) set with respect to τ_1, equivalently τ_2-open (τ_2-closed) is used to denote open (closed) set with respect to τ_2 in a bitopological space (X, τ_1, τ_2). Arbitrary subset of a bitopological space (X, τ_1, τ_2) is termed as pairwise open (pairwise closed) if and only if it is open (closed) with respect to both the individual topologies τ_1 and τ_2. In addition, two arbitrary bitopological spaces (X, τ_1, τ_2) and (Y, τ_1', τ_2') are said to be pairwise homeomorphic if and only if there exists a pairwise homeomorphism f between (X, τ_1, τ_2) and (Y, τ_1', τ_2'), i.e., $f : (X, \tau_1) \to (Y, \tau_1')$ is a homeomorphism and also $f : (X, \tau_2) \to (Y, \tau_2')$ is a homeomorphism.

Definition 2.1 (Pairwise T_1 bitopological space). *[4] A bitopological space (X, τ_1, τ_2) is called pairwise T_1 if for any two different members x and y of X, there exists a τ_1-open set U and a τ_2-open set V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.*

Definition 2.2 (Pairwise T_2 bitopological space). *[1] A bitopological space (X, τ_1, τ_2) is known as pairwise Hausdorff or pairwise T_2 if for any two distinct members x and y, there exists a τ_1-open set G containing x and a τ_2-open set H containing y with $G \cap H = \emptyset$.*

Definition 2.3 (Pairwise regular bitopological space). *[1] In a bitopological space (X, τ_1, τ_2), τ_1 is termed as regular with respect to τ_2 if for any member x of X and for arbitrary τ_1-closed set F not containing x, there exists a τ_1-open set G containing x and a τ_2-open set H containing F with $G \cap H = \emptyset$. If both the topologies of a bitopological space are simultaneously regular with respect to one another, then bitopological space is known as pairwise regular.*

Definition 2.4 (Pairwise normal bitopological space). *[1] A bitopological space (X, τ_1, τ_2) is known as pairwise normal if for arbitrary τ_1-closed set M and arbitrary τ_2-closed set N with $M \cap N = \emptyset$, there exists a τ_1-open set G containing N and a τ_2-open set H containing M with $G \cap H = \emptyset$.*

Definition 2.5 (Subspace of a bitopological space). *[9] For a bitopological space (X, τ_1, τ_2) if $Y \subseteq X$, then $\tau_1^Y = \{H \cap Y : H \in \tau_1\}$ and $\tau_2^Y = \{G \cap Y : G \in \tau_2\}$ are two arbitrary topologies on Y. Bitopological space (Y, τ_1^Y, τ_2^Y) is termed as a subspace of the bitopological space (X, τ_1, τ_2).*

In [9], it has been also proved that property of pairwise regularity in bitopological space is a hereditary property. On the same lines, it can also be accomplished that pairwise Hausdoorff property in bitopological spaces is also a hereditary property.

Definition 2.6 (Pairwise compact bitopological space). *[3] For a bitopological space (X, τ_1, τ_2), a cover $\mathcal{G}$ of X such that $\mathcal{G} \subseteq \tau_1 \cup \tau_2$ with $\tau_1 \cap \mathcal{G} \neq \emptyset$ and $\tau_2 \cap \mathcal{G} \neq \emptyset$*

is known as pairwise open cover. Bitopological space (X, τ_1, τ_2) *is termed as pairwise compact if every pairwise open cover of* X *possesses a finite subcover.*

Definition 2.7 (Pairwise locally compact bitopological space)**.** *[7] In a bitopological space* (X, τ_1, τ_2)*,* τ_1 *is known as locally compact with respect to* τ_2 *if* τ_2*-closure of any* τ_1*-open set containing arbitrary member of* X *is pairwise compact. If both the topologies of a bitopological space are locally compact with respect to one another, then bitopological space is termed as pairwise locally compact.*

Definition 2.8 (One point pairwise compactification in a bitopological space)**.** *[7] Consider* $X^* = X \cup \{\infty\}$*, for a non pairwise compact bitopological space* (X, τ_1, τ_2) *and for arbitrary point* ∞ *not in* X*. Evidently,* $\tau_1^* = \{A :$ *either* $A \in \tau_1$ *or if* $\infty \in A,$ *then* $X^* - A$ *is* τ_1 *compact,* τ_1 *closed, pairwise compact and* τ_2*-compact*$\}$ *and* $\tau_2^* = \{A :$ *either* $A \in \tau_2$ *or if* $\infty \in A,$ *then* $X^* - A$ *is* τ_2*-compact,* τ_2*-closed, pairwise compact and* τ_1*-compact*$\}$ *are topologies on* X*. Then, bitopological space* $(X^*, \tau_1^*, \tau_2^*)$ *is pairwise compact, in which bitopological space* (X, τ_1, τ_2) *can be densely embedded. Bitopological space* $(X^*, \tau_1^*, \tau_2^*)$ *is known as one point pairwise compactification of the bitopological space* (X, τ_1, τ_2)*.*

3. Some results on pairwise locally compact bitopological spaces

This section begins with the introduction of an alternative definition of pairwise local compactness in bitopological space. Also, as obvious from the previously available literature of bitopological spaces, it is quite natural to introduce weak form of pairwise local compactness in bitopological spaces. Subsequently, results related to local compactness from the classical topology are generalized and extended for pairwise local compactness in bitopological spaces.

Definition 3.1 (Pairwise locally compact bitopological space)**.** *In a bitopological space* (X, τ_1, τ_2)*, if arbitrary* τ_1*-nhd and arbitrary* τ_2*-nhd of any member of* X *are both pairwise compact, then bitoopological space is known as pairwise locally compact.*

Definition 3.2 (Weak pairwise locally compact bitopological space)**.** *If in a bitopological space* (X, τ_1, τ_2)*, either* τ_2*-closure of any* τ_1*-open set containing arbitrary member of* X *is pairwise compact or* τ_1*-closure of any* τ_2*-open set containing arbitrary member of* X *is pairwise compact, then bitopological space is known as weak pairwise locally compact.*

Obviously, a pairwise locally compact bitopological space is weak pairwise locally compact.

Theorem 3.1. *Necessary and sufficient condition for a pairwise Hausdorff bitopological space* (X, τ_1, τ_2) *to be pairwise locally compact is that an arbitrary point of* X *is a* τ_1*-interior (equivalently,* τ_2*-interior) point of some pairwise compact subspace of the given bitopological space.*

Proof. Suppose that pairwise Hausdorff bitopological space (X, τ_1, τ_2) is pairwise locally compact. For an arbitrary $x \in X$, we can find a τ_1-open set (equivalently, τ_2-open set) G containing x such that $\tau_2 - cl(G)$ (equivalently, $\tau_1 - cl(G)$) is pairwise compact. Evidently, $x \in G \subseteq \tau_2 - cl(G) = Y$. Therefore, x is a τ_1-interior (equivalently, τ_2-interior) point of a pairwise compact subspace (Y, τ_1^Y, τ_2^Y) of (X, τ_1, τ_2).

Conversely, for arbitrary $x \in X$ let (G, τ_1^G, τ_2^G) is a pairwise compact subspace of (X, τ_1, τ_2) such that x is a τ_1-interior point of G. As G is pairwise compact in a pairwise Hausdorff bitopological space (X, τ_1, τ_2). Thus, G is pairwise closed. Consequently, $\tau_2 - cl(G) = G$ is pairwise compact. Hence, required result follows. □

Next we will prove that pairwise local compactness is preserved under a pairwise onto, pairwise open and pairwise continuous mapping. To accomplish the same, first it is obtained that pairwise compactness is preserved under a pairwise onto and pairwise continuous map.

Theorem 3.2. *If a bitopological space is pairwise compact, then its pairwise continuous and pairwise onto image is also pairwise compact.*

Proof. For a pairwise compact bitopological space (X, τ_1, τ_2), consider a pairwise onto and pairwise continuous map $f : (X, \tau_1, \tau_2) \to (Y, \tau_1', \tau_2')$. Let $\mathcal{G}$ be an arbitrary pairwise open cover of Y, here $\mathcal{G} \subseteq \tau_1' \cup \tau_2'$ with $\tau_1' \cap \mathcal{G} \neq \emptyset$ and $\tau_2' \cap \mathcal{G} \neq \emptyset$. Evidently, $f^{-1}(\bigcup_{G \in \mathcal{G}} G) = f^{-1}(Y)$, i. e., $\bigcup_{G \in \mathcal{G}} f^{-1}(G) = X$. As $f^{-1}(G)$ is a τ_1-open set or a τ_2-open set and collection $\mathcal{H} = \{f^{-1}(G) : G \in \mathcal{G}\}$ with $\mathcal{H} \subseteq \tau_1 \cup \tau_2$ also $\tau_1 \cap \mathcal{H} \neq \emptyset$ and $\tau_2 \cap \mathcal{H} \neq \emptyset$ is a pairwise open cover of X. Consequently, $X = \bigcup_{i=1}^{n} f^{-1}(G_i)$, i. e., $f^{-1}(\bigcup_{i=1}^{n} G_i) = X$ or $\bigcup_{i=1}^{n} G_i = Y$. □

Theorem 3.3. *A pairwise continuous, pairwise open and pairwise onto image of a pairwise locally compact bitopological space is also pairwise locally compact.*

Proof. For a pairwise locally compact bitopological space (X, τ_1, τ_2), suppose that map $f : (X, \tau_1, \tau_2) \to (Y, \tau_1', \tau_2')$ is pairwise continuous, pairwise open and pairwise onto. For arbitrary $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Evidently, one can find a τ_1-open set G containing x such that $\tau_2 - cl(G)$ is pairwise compact and also there exists a τ_2-open set H which contains x and also $\tau_1 - cl(H)$ is pairwise compact. Then, $f(G)$ is a τ_1'-open set containing $f(x)$, also $f[\tau_2 - cl(G)] \subseteq \tau_2' - cl[f(G)]$. Being pairwise continuous and pairwise open, f is pairwise closed. Consequently, $\tau_2' - cl[f[\tau_2 - cl(G)]] = f[\tau_2 - cl(G)]$. Further, $G \subseteq \tau_2 - cl(G)$ means $f(G) \subseteq f[\tau_2 - cl(G)]$ or $\tau_2' - cl[f(G)] \subseteq \tau_2' - cl[f[\tau_2 - cl(G)]]$ or $\tau_2' - cl[f(G)] \subseteq f[\tau_2 - cl(G)]$. Therefore, $\tau_2' - cl[f(G)] = f[\tau_2 - cl(G)]$. As $\tau_2 - cl(G)$ is pairwise compact, consequently $f[\tau_2 - cl(G)] = \tau_2' - cl[f(G)]$ is also pairwise compact. It is accomplished that if $y \in Y$ be arbitrary, then we can find a τ_1'-open set $f(G)$ containing y such that $\tau_2' - cl[f(G)]$ is pairwise compact. Similarly, for arbitrary $y \in Y$, we can find a τ_2'-open set $f(H)$ containing y such that $\tau_1' - cl[f(H)]$ is pairwise compact. □

Remark 3.1. *Pairwise compactness and pairwise local compactness in bitopological spaces are topological properties.*

The concept of pairwise local compactness is not a hereditary property. But if an additional condition of pairwise closed is imposed on the subset, then the following result [7] can be obtained.

Theorem 3.4. *If a bitopological space is pairwise locally compact, then any pairwise closed subset of this bitopological space is also pairwise locally compact.*

Proof. Choose Y as a pairwise closed subset of a pairwise locally compact bitopological space (X, τ_1, τ_2). For arbitrary $y \in Y$, we can find a τ_1-open set G_1 containing y such that $\tau_2 - cl(G_1)$ is pairwise compact and also one can find a τ_2-open set H_1 containing y in a way that $\tau_1 - cl(H_1)$ is pairwise compact. Evidently, $G_2 = G_1 \cap Y$ is a τ_1^Y-open set containing y. As Y is pairwise closed, therefore G_2 is pairwise closed in $\tau_2 - cl(G_1)$. Consequently, G_2 is pairwise compact in (Y, τ_1^Y, τ_2^Y). □

In the succeeding theorem, it is proved that if a bitopological space is pairwise Hausdorff and pairwise locally compact, then its one point pairwise compactification is pairwise Hausdorff and vice-versa.

Theorem 3.5. *Necessary and sufficient condition for one point pairwise compactification* $(X^*, \tau_1^*, \tau_2^*)$ *of a bitopological space* (X, τ_1, τ_2) *to be pairwise Hausdorff is that bitopological space* (X, τ_1, τ_2) *is pairwise Hausdorff and pairwise locally compact.*

Proof. Suppose that $(X^*, \tau_1^*, \tau_2^*)$ is pairwise Hausdorff. (X, τ_1, τ_2) being subspace of $(X^*, \tau_1^*, \tau_2^*)$ is also pairwise Hausdorff. For any $x \in X$, we have $x \in X^*$ and $x \neq \infty$. Therefore, we can find a τ_1^*-open set G^* and a τ_2^*-open set H^* such that $x \in G^*$ and $\infty \in H^*$ with $G^* \cap H^* = \emptyset$. Since, $X^* - H^*$ is a pairwise compact subset of (X, τ_1, τ_2) and also $x \in G^* \subseteq X^* - H^*$. Consequently, $X^* - H^*$ constitutes a τ_1^*-nhd of x which is pairwise compact. Similarly, τ_2^*-nhd of x which is pairwise compact can be obtained.

Conversely, assume that bitopological space (X, τ_1, τ_2) is pairwise locally compact and pairwise Hausdorff. Let $x \neq y$ are arbitrary members of X^*. Choosing $x \neq \infty$ and $y \neq \infty$. Then, $x \neq y$ are members of X which is pairwise Hausdorff. Consequently, $(X^*, \tau_1^*, \tau_2^*)$ is pairwise Hausdorff. Alternatively, when $x \in X$ and $y = \infty$. Then, one can find a τ_1-open set G containing x such that $\tau_2 - cl(G)$ is pairwise compact. Evidently, $\infty \in X^* - (\tau_2 - cl(G))$ and $X^* - (X^* - (\tau_2 - cl(G)))$ is τ_2-closed and also pairwise compact in (X, τ_1, τ_2). Consequently, $X^* - (\tau_2 - cl(G)) \in \tau_2^*$, also $G \cap (X^* - (\tau_2 - cl(G))) = \emptyset$. On the same lines, when $y \in X$ and $x = \infty$, it can be proved that $(X^*, \tau_1^*, \tau_2^*)$ is pairwise Hausdorff. □

Further, we will provide an alternative proof for the corollary of proposition 3 [7]. To accomplish the said result, first a lemma is proved.

Lemma 3.1. *If arbitrary point* x *of a pairwise Hausdorff and pairwise compact bitopological space* (X, τ_1, τ_2) *is not a member of any* τ_1*-closed set* F*, then it is possible to find a* τ_1*-open set* G *and a* τ_2*-open set* H *such that* $x \in G$ *and* $F \subseteq H$ *with* $G \cap H = \emptyset$*.*

Proof. For arbitrary $y \in F$, we can find a τ_1-open set G_y which contains x and a τ_2-open set H_y which contains y with $G_y \cap H_y = \emptyset$. As $\{H_y : y \in F\}$ is τ_2-open cover of F and F, being τ_1-closed in a pairwise compact bitopological space (X, τ_1, τ_2), is τ_2-compact [Lemma 3, [3]]. As a result, $F \subseteq \bigcup_{i=1}^{n} H_{y_i}$. Correspondingly, suppose that $G_{y_1}, G_{y_2}, ..., G_{y_n}$ are τ_1-open sets such that $x \in G_{y_i}$ and $G_{y_i} \cap H_{y_i} = \emptyset$ for each i. Choose $G = \bigcap_{i=1}^{n} G_{y_i}$, a τ_1-open set and $H = \bigcup_{i=1}^{n} H_{y_i}$, a τ_2-open set. Evidently, $x \in G$ and $F \subseteq H$ and also $G \cap H = \emptyset$. □

Theorem 3.6. *If a bitopological space (X, τ_1, τ_2) is pairwise Hausdorff and pairwise locally compact, then it is pairwise regular.*

Proof. Consider arbitrary τ_1^*-closed set F^* not containing any point x of X^*, where $(X^*, \tau_1^*, \tau_2^*)$ is one point pairwise compactification of (X, τ_1, τ_2). Since, bitopological space $(X^*, \tau_1^*, \tau_2^*)$ is pairwise Hausdorff and pairwise compact. Therefore, there exists a τ_1^*-open set G^* and a τ_2^*-open set H^* such that $x \in G^*$ and $F^* \subseteq H^*$ with $G^* \cap H^* = \emptyset$. Thus, $(X^*, \tau_1^*, \tau_2^*)$ is pairwise regular. As property of pairwise regular is a hereditary property [9, Theorem 2]. Hence, bitopological space (X, τ_1, τ_2) is pairwise regular. □

Finally, it is obtained that if a given bitopological space is pairwise Hausdorff and pairwise locally compact, then its one point pairwise compactification will be pairwise normal.

Theorem 3.7. *One point pairwise compactification $(X^*, \tau_1^*, \tau_2^*)$ of a pairwise Hausdorff and pairwise locally compact bitopological space (X, τ_1, τ_2) is pairwise normal.*

Proof. Consider arbitrary τ_1^*-closed set E^* and arbitrary τ_2^*-closed set F^* such that $E^* \cap F^* = \emptyset$. If $x \in F^*$ be arbitrary, then $x \notin E^*$. As E^* is a τ_1^*-closed subset of a pairwise Hausdorff and pairwise compact bitopological space $(X^*, \tau_1^*, \tau_2^*)$. Therefore, we can find a τ_1^*-open set G_x^* and a τ_2^*-open set H_x^* such that $x \in G_x^*$ and $E^* \subseteq H_x^*$ with $G_x^* \cap H_x^* = \emptyset$. It is evident that $F^* \subseteq \bigcup_{i=1}^{n} G_{x_i}^*$. Correspondingly, suppose that $H_{x_1}^*, H_{x_2}^*, ..., H_{x_n}^*$ are τ_2^*-open sets such that $E^* \subseteq H_{x_i}^*$ and $G_{x_i}^* \cap H_{x_i}^* = \emptyset$ for each i. Choose $G^* = \bigcup_{i=1}^{n} G_{x_i}^*$, a τ_1^*-open set and $H^* = \bigcap_{i=1}^{n} H_{x_i}^*$, a τ_2^*-open set. It is evident that $F^* \subseteq G^*$ and $E^* \subseteq H^*$ and also $G^* \cap H^* = \emptyset$. □

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