

A NEW TECHNIQUE TO INVESTIGATE THE ORIENTATION OF THE PERIODIC POINTS OF PERIOD 2^n OF SOME UNIMODAL MAPS GOVERNING ONE DIMENSIONAL DISCRETE DYNAMICAL SYSTEMS

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ABSTRACT. Period doubling scenario occurring in logistic like models is very popular and has a useful aspect while studying population dynamics. It has been established in the literature that unimodal maps follow a universal behavior. However, the orientation of the relative positions of the periodic points of period $2^n, n = 1, 2, 3, \dots$ of some unimodal maps has been investigated using the Horizontal Visibility Graph technique for the first time. Satisfactory results have been obtained explaining the degree pattern of the corresponding Horizontal visibility graph (HVG) of the said periodic points. The results show that periodic points have some uniform properties, which is passed on as the periodicity advances.

1. INTRODUCTION

First-order difference equations governed by unimodal maps have an essential role in describing many practical situations arising in various aspects [1]. Logistic like models are very popular among them due to its simple nature yet containing much detailed information. May [10] has popularized the mathematical scenario of logistic difference equation by showing the period-doubling scenario to chaos. Feigenbaum [9] has shown that the period-doubling scenario in unimodal maps follows a universal way to chaos, thus making a good prediction to periodic points of the logistic model. The graph-theoretic aspect of dynamical systems has been

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a recent interest in the literature. A strong motivation has been observed to lay a bridge between two giant fields, dynamical system and graph theory [7]. One of the simplified notions is Horizontal Visibility Graph [2, 3, 6]. The main motive is to analyze the time series analysis of the dynamical system with a complex network. Some fruitful results regarding distribution functions of periodic points and chaotic bands for logistic like models are some of the many important facts discussed in the literature. Gutin et al. [6] discuss some general properties regarding the HVG graph. Lacasa et al. [8] discuss the characterization of chaotic, uncorrelated, and correlated stochastic processes by using a horizontal visibility algorithm. Luque et al. [4, 5] highlight the degree distribution and some general properties of Feigenbaum graphs using horizontal visibility graphs. Dutta et al. [11] discuss some of the aspects of HVG. Their vertices sets represent a relative position in one-dimensional state space with respect to time of the periodic points of the n th bifurcation points. The paper has discussed the positional aspects of the key vertices of the set. In this paper, some more graph-theoretic characteristics of the HVGs of the said vertex mode l [11] has been considered. The graph-theoretic view strengthens the predictive of various kinds of qualitative and quantitative behavior viz. positional aspect of the periodic points etc. which is not reflected easily from the set of periodic points.

2. PRELIMINARIES

The difference equation $x_{n+1} = ax_n(1 - x_n)$ follows a period doubling route to chaos as control parameter a is increased. The bifurcation diagram of the logistic map is as follows. If a'_n be the parameter at which n th bifurcation occurs. Then 2^n periodic points are obtained. The smallest periodic point is taken and iterated $2^n - 1$ times, which is nothing but the rearrangement of the previous periodic set. Let (i, j) denote the i th periodic point whose value is in the j th position when the periodic points in the set are arranged in increasing order. It has been observed that if the periodic points are arranged in the said order, then it follows the following model (2.1). Let $V_1 = \{(0, 0), (1, 1)\}$. Let V_{n-1} be the set containing

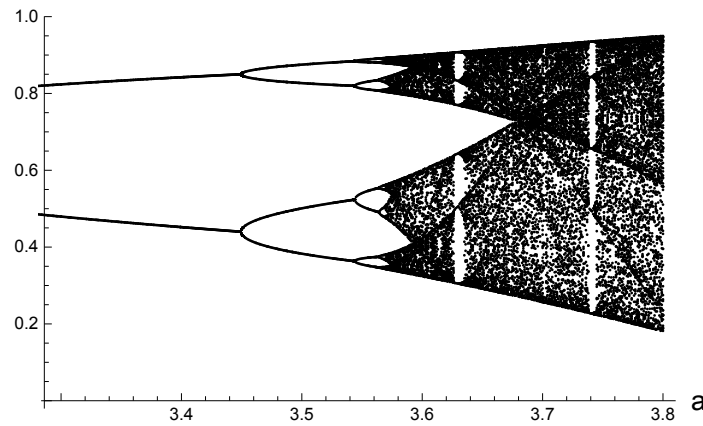


FIGURE 1. Bifurcation diagram of the logistic map having control parameter(a) in the abscissa and x in the ordinate.

2^{n-1} elements, then we consider the following collections:

$$\begin{aligned} V'_{n,1} &= \{(2k, i) | (k, i) \in V_{n-1}, i = 0, 1, 2, \dots, 2^{n-2} - 1\}, \\ V'_{n,2} &= \{(2k + 4, i) | (k, i) \in V_{n-1}, i = 2^{n-2}, \dots, 2^{n-1} - 2, \text{ and } n > 2\}, \\ V'_{n,3} &= \{(2k + 4 - 2^n, 2^{n-1} - 1) | (k, 2^{n-1} - 1) \in V_{n-1}\}, \\ V'_{n,4} &= \{(2k + 1, 2^{n-1} + i) | (k, i) \in V_{n-1}, i = 0, 1, 2, \dots, 2^{n-1} - 1\}. \end{aligned}$$

Then

$$(2.1) \quad V_n = \bigcup_{i=1}^4 V'_{n,i}.$$

A horizontal visibility graph $G_n = (V_n, E_n)$ is formed whose vertices sets are V_n . The edge set E_n is defined as $E_n = \{((n_1, i), (n_2, j)) | N((n_1, i), (n_2, j)) = 0\}$, where $N((n_1, i), (n_2, j))$ represents the number of elements of the form (k, k_1) such that $k_1 > i$ or j and $n_1 < k < n_2$.

Theorem 2.1. [11] If (k, i) exists, where $\sum_{t=1}^r 2^{n-t} \leq i < \sum_{t=1}^{r+1} 2^{n-t}$, then $k = \sum_{t=0}^{r-1} 2^t + 2^{r+1}m$, $m = 0, 1, 2, 3, \dots, 2^{n-r-1} - 1$ for V_n .

3. MAIN RESULTS

Definition 3.1 (Left Degree and Right Degree). Let (p, j) be a point of V_n . We define L_p and R_p for (p, j) as follows:

$$L_p = \{((n_1, i), (p, j)) : N((n_1, i), (p, j)) = 0 \text{ for } k_1 > i, j \text{ and } n_1 < p\},$$

$$R_p = \{((n_2, i), (p, j)) : N((n_2, i), (p, j)) = 0 \text{ for } k_1 \geq i, j \text{ and } p < n_2\},$$

where $N((n_1, i), (p, j))$ represents the number of elements of the form (k, k_1) such that $k_1 > i$ or j and $n_1 < k < p$ and $N((n_2, i), (p, j))$ represents the number of elements of the form (k, k_1) such that $k_1 > i$ or j and $n_2 < k < p$.

Left degree of (p, i) in V_n is defined as the total number of elements in L_p . Similarly right degree of a particular point (p, i) in a set V_n is defined as the total number of elements in R_p , i.e the cardinality of R_p . Total degree of a particular point (p, j) is defined as $T_D = N(L_p) + N(R_p)$.

Definition 3.2 (Level q ($q \leq n + 1$)). From Theorem 2.1 we get the following partitioned sets for the class $V_n - (2^n - 1, 2^n - 1)$:

$$\begin{aligned} V_{n,1} &= \left\{ (k, i) \mid k = 2m, 0 < m < 2^{n-1} - 1, 0 < i < 2^{n-1} - 1 \right\} \\ V_{n,2} &= \left\{ (k, i) \mid k = 4m + 1, 0 < m < 2^{n-2} - 1, 2^{n-1} < i < 2^{n-1} + 2^{n-1} - 1 \right\} \\ V_{n,3} &= \left\{ (k, i) \mid k = 8m + 3, 0 < m < 2^{n-3} - 1, 2^{n-1} + 2^{n-2} < i < 2^{n-1} + 2^{n-1} \right. \\ &\quad \left. + 2^{n-3} - 1 \right\} \\ &\vdots \\ V_{n,p} &= \left\{ (k, i) \mid k = 2^p m + \sum_{t=0}^{p-2} 2^t, 0 < m < 2^{n-p} - 1, 2^{n-1} + 2^{n-2} + \dots \right. \\ &\quad \left. + 2^{n-(p-1)} < i < 2^{n-1} + 2^{n-2} + \dots + 2^{n-p} - 1 \right\} \\ &\vdots \\ V_{n,n} &= \{(k, i) \mid k = 2^n m + \sum_{t=0}^{n-2} 2^t, m = 0, i = 2^n - 2\} \end{aligned}$$

Thus $V_{n,1}, V_{n,2}, \dots, V_{n,n}$ are called level1 set, level2 set, ..., level n sets of V_n .

Also we can say that the element $(2^n - 1, 2^n - 1) \in \text{level } (n + 1) \text{ set}$. Actually, we can say that there exist $(n + 1)$ level sets of V_n .

Lemma 3.1.

- (a) If $((m_1, i_1), (p, i)) \in L_p$ and $i_1 > i$ then for $m < m_1$, $((m, i_2), (p, i)) \notin L_p$.
- (b) If $((m_2, i_3), (q, i)) \in R_q$ and $i_3 > i$ then for $m > m_2$, $((m, i_2), (q, i)) \notin R_q$.

Proof.

- (a) Let $((m_1, i_1), (p, i)) \in L_p$. We consider (m, i_2) such that $m < m_1$. Then $N((m, i_2), (p, i)) \geq 1$ as $m < m_1 < p$ and $i_1 > i$. Therefore $((m, i_2), (p, i)) \notin L_p$.
- (b) Let $((m_2, i_3), (q, i)) \in R_q$. We consider (m, i_2) such that $m > m_2$. Then $N((m, i_2), (q, i)) \geq 1$ as $m > m_2 > q$. Therefore $((m, i_2), (q, i)) \notin R_q$.

□

Theorem 3.1. For any vertex set V_n and any positive integer m , degree of an element $(2m, i)$ is always two.

Proof. The proof will be done by finding left degree and right degree of $(2m, i)$ and adding the two results. Clearly from the definition of level sets $(2m, i) \in \text{level } 1 \text{ set}$. Since $2m - 1$ is odd so $(2m - 1, i_1) \in \text{level } p \text{ set}$ where $p > 1$. Therefore $((2m - 1, i_1), (2m, i)) \in L_{2m}$. Since $2m - 1 < 2m$ and $i_1 > i$ so by Lemma 3.1 (a), it can be said that left degree of $(2m, i)$ is one. Again $2m + 1$ so $(2m + 1, i_2) \in \text{level } q \text{ set}$ where $q > 1$. Therefore $((2m + 1, i_2), (2m, i)) \in R_{2m}$. Since $2m + 1 > 2m$ and $i_2 > i$ so by lemma 3.1(b), it can be said that right degree of $(2m, i)$ is one. So total degree of $(2m, i)$ is two. Hence the theorem. □

Lemma 3.2. In a HVG between two elements of level p set say (a, b) and (c, d) where a and c are consecutive ($a > c$), $\exists$ a unique element (r, s) ($c < r < a$) such that $(r, s) \in \text{level } q \text{ set}$ ($q > p$) and $p, q (\leq n)$ are positive integers.

Proof. Let us take two elements (a, b) and (c, d) ($a > c$) of level p set for some suitable b and d where a and c are consecutive. a can be expressed as $2^p m + \sum_{t=0}^{p-2} 2^t$, $0 < m < 2^{n-p} - 1$ (see definition 3.2).

Also, c can be expressed as $2^p m + \sum_{t=0}^{p-2} 2^t - 2^p = 2^p (m - 1) + \sum_{t=0}^{p-2} 2^t$.

Now we consider the element

$$\begin{aligned}
 (3.1) \quad 2^p m + \sum_{t=0}^{p-2} 2^t - 2^{p-1} &= 2^p m + (1 + 2 + 2^2 + \dots + 2^{p-2}) - 2^{p-1} \\
 &= 2^p m + (1 + 2 + 2^2 + \dots + 2^{p-2} + 2^{p-1}) - 2^p \\
 &= 2^p(m - 1) + (1 + 2 + 2^2 + \dots + 2^{p-2}) + 2^{p-1}.
 \end{aligned}$$

When m is odd clearly $(2^p(m - 1) + 1 + 2 + 2^2 + \dots + 2^{p-2}) + 2^{p-1}$, b_1 belongs to level $(p+1)$ set for some suitable b_1 . When m is even $m - 1$ is odd. So $m - 1$ is of the form $2m_1 - 1$. So (3.1) is taking the form

$$\begin{aligned}
 (3.2) \quad &2^p [2m_1 - 1] + (1 + 2 + 2^2 + \dots + 2^{p-2}) + 2^{p-1} \\
 &= 2^{p+1}m_1 - 2^p + (1 + 2 + 2^2 + \dots + 2^{p-1})
 \end{aligned}$$

and (3.2) is taking the same form as (3.1), in place of p here is $p + 1$. So with the same argument we can say that when m_1 is odd, $(2^{p+1}m_1 - 2^p + (1 + 2 + 2^2 + \dots + 2^{p-1})$, b_2) belongs to level $(p + 2)$ set for some suitable b_2 when m_1 is even then $m_1 = 2m_2$ and we can repeat the same.

Let at r th step it will become $2^{p+r}.0 + 1 + 2 + 2^2 + \dots + 2^{p+r-1}$ and it will belong to level $(p+r)$ set. Hence the proof.

To prove the uniqueness of such an element, we consider $a - x$. If x is odd then $(a - x, b_1) \in$ level 1 set. When x is even then x can be written in the form $x = x_1 2 + x_2 2^2 + x_3 2^3 + \dots + x_{p-1} 2^{p-1}$, where x_i is either 0 or 1 and at least one $x_i = 1$, $i < p - 1$ and if $x > 2^{p-1}$, $x_{p-1} = 1$, otherwise 0.

Now,

$$\begin{aligned}
 a - x &= 2^p m + (1 + 2 + 2^2 + \dots + 2^{p-2}) - (x_1 2 + x_2 2^2 + \dots + x_{p-1} 2^{p-1}) \\
 &= 2^p m + (1 - x_1) 2 + (1 - x_2) 2^2 + \dots + (1 - x_{p-2}) 2^{p-2} - x_{p-1} 2^{p-1} + 1.
 \end{aligned}$$

Let k be the smallest number such that $x_k = 1$, $k < p - 1$:

$$\begin{aligned}
 a - x &= 2^p m + 2 + 2^2 + \dots + 2^{k-1} + 2^{k+1} + \dots + 2^{p-2} - 2^{p-1} + 1 \\
 &= 2^p m + 2^{k+1} + 2^{k+2} + \dots + 2^{p-2} - 2^{p-1} + 1 + 2 + 2^2 + \dots + 2^{k-1} \\
 &= 2^{k+1} t + \sum_{i=0}^{k-1} 2^i,
 \end{aligned}$$

and $k + 1 < p$ and the element is in level $(k + 1)$ set. Hence proved. $\square$

Lemma 3.3. *An element belonging to any level p set can not be incident with any elements of same level set.*

Proof. On the contrary let us assume that an element (a, b) belonging to level p set be adjacent with an element of (c, d) of level p set. So two cases arise, $a > c$ or $a < c$. Using Lemma 3.2 in any cases $\exists$ an element (e, f) belonging to level q ($q > p$) set such that (a, b) is adjacent with (e, f) . So (a, b) can not be incident with (c, d) . Thus there does not exist any element in level p set by which (a, b) will be adjacent. $\square$

Theorem 3.2. *Any element of level p set is adjacent with atmost two elements of any level q set .*

Proof. Let $(a, b) \in$ level p set. If (a, b) is not adjacent with any element of level q set then the theorem holds. Let (a, b) be adjacent with (c, d) , an element of level q set ($q \neq p$). Two cases arise. These are $c < a$ or $c > a$.

CASE 1 ($c < a$)

We want to show that in level p set when $c < a$, there exist at most one element by which (a, b) is adjacent. If possible let there exist an element (e, f) ($e < a$) such that (a, b) is adjacent with (e, f) . Then two cases arise. $e < c$ or $e > c$. If $e < c$ then by lemma 3.2 there exist an element (g, h) in level t set such that $h > s$ and $g > e$. So (a, b) is adjacent with (g, h) . It can not be adjacent with (e, f) but with (c, d) . If $e > c$ then applying the same lemma 3.2, it can be said that (a, b) is adjacent with (e, f) but not (c, d) . In any cases there exist at most one element.

CASE 2 ($c > a$)

If (a, b) is adjacent with (c, d) then by using similar arguments of case 1 we have (a, b) will be adjacent with atmost one element of the form (c, d) .

Further in the same level (a, b) will not be adjacent with any element and hence the theorem. $\square$

Theorem 3.3. *The element say (a, b) , where a is of the form $\sum_{t=0}^{p-2} 2^t$, of level p ($p > 1$) set is not incident with any element of level q set, $q > p$ so far as left degree is concerned.*

Proof. We have for all $m_1, m_2, m_3, m_4, \dots, m_i, m_j \geq 0$ and $m = 0$, $4m + 1 < 8m_1 + 3 < 16m_3 + 7 < 32m_4 + 15 < \dots < 2^{n-1}m_i + \sum_{t=0}^{n-3} 2^t < 2^n m_j + \sum_{t=0}^{n-2} 2^t$.

Therefore (a, b) will not be adjacent with any element (p, q) where $a < p$. $\square$

Theorem 3.4. *In a $HVG(V_n, E_n)$ any element of level p set is incident with two elements each from its previous levels.*

Proof. Let the elements (a, b) be in level p set. Then $a = 2^p m + \sum_{t=0}^{p-2} 2^t$. We shall show that (a, b) will be adjacent with the element $(a - 2^r, b_{k_r})$ and $(a + 2^r, b_{l_r})$ for some suitable b_{k_r} and b_{l_r} and $r = 0, 1, 2, 3, \dots, p-2$

First let us show that $(a - 2^r, b_{k_r}) \in (r+1)$ th level set for each $r = 0, 1, 2, 3, \dots, (p-2)$ and there exist edge between (a, b) and $(a - 2^r, b_{k_r})$ for each r .

When $r = 0$,

$$\begin{aligned} a - 2^0 &= a - 1 = 2^p m + (1 + 2 + 2^2 + \dots + 2^{p-2}) - 1 \\ &= 2^p m + (2 + 2^2 + \dots + 2^{p-2}) \\ &= 2[2^{p-1} m + 1 + 2 + 2^2 + \dots + 2^{p-3}], \end{aligned}$$

$(a - 1, b_{k_0}) \in$ level 1 set. Again, when $r = 1$,

$$\begin{aligned} a - 2^1 &= a - 2 = 2^p m + (1 + 2 + 2^2 + \dots + 2^{p-2}) - 2 \\ &= 2^p m + (1 + 2^2 + \dots + 2^{p-2}) \\ &= 2^2 [2^{p-2} m + 1 + 2 + \dots + 2^{p-4}] + 1 \end{aligned}$$

$(a - 2, b_{k_1}) \in$ level 2 set. Now for $r = p-2$ we have

$$\begin{aligned} a - 2^{p-2} &= 2^p m + (1 + 2 + 2^2 + \dots + 2^{p-2}) - 2^{p-2} \\ &= 2^p m + (1 + 2 + 2^2 + \dots + 2^{p-2}) - 2^{p-2} \\ &= 2^p m + (1 + 2 + 2^2 + \dots + 2^{p-3}) \\ &= 2^{p-1} \cdot 2m + (1 + 2 + 2^2 + \dots + 2^{p-3}) \end{aligned}$$

We can say that $(a - 2^{p-2}, b_{k_{p-2}})$ is in level $(p-1)$ set.

We want to show now (a, b) is adjacent with $(a - 2^r, b_{k_r})$ for $r = 0, 1, 2, 3, \dots, (p-2)$. Since $(a - 2^r, b_{k_r}) \in (r+1)$ th level set. So another element of the same level set will be of the form $(a - 2^r - 2^{r+1}, b_s)$ or $(a - 2^r + 2^{r+1}, b_t)$.

Now $a - 2^r - 2^{r+1} < a - 2^r$. Again $a - 2^r + 2^{r+1} = a + 2^r (2 - 1) = a + 2^r$, which is greater than a . Hence $(a - 2^r, b_{k_r}) \in L_a$ and is of level $(r+1)$ set.

Again for $r = p-2$ we have,

$$\begin{aligned} a + 2^{p-2} &= 2^p m + (1 + 2 + 2^2 + \dots + 2^{p-2}) + 2^{p-2} \\ &= 2^p m + (1 + 2 + 2^2 + \dots + 2^{p-3}) + 2^{p-1} \\ &= 2^{p-1} (2m + 1) + (1 + 2 + \dots + 2^{p-3}), \end{aligned}$$

which is in level $(p-1)$ set.

Another element in this level is of this form $(a + 2^{p-2} - 2^{p-1}, b_u)$ or $(a + 2^{p-2} + 2^{p-1}, b_v)$ and $a + 2^{p-2} - 2^{p-1} = a - 2^{p-2} < a$.

So, $(a + 2^{p-2}, b_{k_{(p-2)}}) \in R_{a,0}$, (a, b) is incident with $(a + 2^{p-2}, b_{k_{(p-2)}})$.

Hence the theorem follows. $\square$

Theorem 3.5. In a $HVG(V_n, E_n)$ the total degree of (a, b) belonging to any level p ($p \leq n$) set where a is of the form $\sum_{t=0}^{p-2} 2^t$ is $2p - 1$.

Proof. By Theorem 3.4, (a, b) is incident with two elements each from its previous $(p - 1)$ level sets. Also by Theorem 3.3, any element (c, d) where $c < a$ belonging to level q set, $q > p$ is not incident with (a, b) . Now we consider $(a + 2^{p-1}, b_1)$ for some b_1 .

Then $a + 2^{p-1} = 2^p 0 + \sum_{t=0}^{p-2} 2^t + 2^{p-1} = 2^{p+1} 0 + \sum_{t=0}^{p-2} 2^t + 2^{p-1} \in \text{level}(p + 1)$ set. Clearly if $(c, d) \in \text{level}(p + 1)$ set, then $c > a + 2^{p-1}$, thus in level $(p + 1)$, there does not exist any element say (r, s) such that $a < r < a + 2^{p-1}$. So (a, b) will be adjacent with $(a + 2^{p-1}, b_1)$. Hence the theorem. $\square$

Theorem 3.6. In a $HVG(V_n, E_n)$ the total degree of (a, b) belonging to any level p set where a is of the form $2^p m + \sum_{t=0}^{p-2} 2^t$ is $2p$, $m \neq 0$.

Proof. The element (a, b) is incident with two elements from each of the previous level. Again $(a - 2^{p-1}, b_1) \in$ higher level than p and that is the only element with which (a, b) is adjacent. A similar way may be followed to show that $(a + 2^{p-1}, b_1) \in$ higher level than p and that is the only element with which (a, b) is adjacent. Hence the theorem. $\square$

4. CONCLUSION

This paper gives information about the orientation of the periodic points. For example, without looking at the periodic points, it can be said that 2^9 th periodic point of period 1024 will see 9 periodic points (each of them smaller) strictly increasing in positional aspect from left side and 10 points from the right side (9 smaller and one greater) strictly increasing in positional aspect as its left degree is 9 and The right degree is 10. 384^{th} periodic points will see 8 periodic points (7 points smaller and one greater) strictly increasing in positional aspect from the left side and 8 periodic points (7 points smaller and one greater) strictly increasing in positional aspect from right side. This can also be concluded that the whole set

of periodic points can be partitioned into several groups and every element of a particular partition (except one) observes in its locality the same number of points in same orientation.

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