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# HOMOTOPY PERTURBATION AND ADOMIAN DECOMPOSITION METHODS ON 12th-ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, Homotopy Perturbation Method (HPM) and Adomian Decomposition Method (ADM) are implemented on 12th-order boundary value problems (BVP) in finite domains. The HPM is based on the traditional perturbation and on homotopy while the ADM is based on modified multi-stage ADM. Two test problems were considered to validate and demonstrate our findings with the results compared with the analytical solutions. HPM gave numerical solutions whose accuracy reduced as it approaches the domain boundaries, while ADM gave exact solutions in form of Taylor's series expansion of the closed form solutions which were absolutely convergent. And, all the results were graphically represented in plotted graphs vis-a-vis the analytical solution.

## 1. INTRODUCTION

For over a decade now BVP of higher order (up to 24th-order) are being investigated due to their mathematical importance, and also due to their potential for application in diversified applied and engineering sciences. These class of problems arise from mathematical studies of systems in astrophysics, hydrodynamics and hydro-magnetic stability. For instance, modelling of an infinite horizontal layer of fluid heated from below is done with the assumption that a uniform magnetic field is subject to action of rotation. In the process instability do set in. When the

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instability sets in as ordinary convection, the system is modelled with 10th-order BVP. But, when the instability set in as over stability, the system is modelled with 12th-order BVP. See [13, 17] and the references there in.

On the whole, most mathematical models and mathematical scientific problems and phenomena in different fields of science and engineering occur nonlinearly. Except for a limited number of these problems, we encounter difficulties in finding their exact analytical solutions.

Consider the 12th-Order BVP of the form

$$y^{(xii)}(t) + \vartheta(t)y(t) = \eta(t), \qquad a \leqslant t \leqslant b, \tag{1.1}$$

with the boundary conditions

$$y^{2j}(a) = \delta_{2j},$$

$$y^{2j}(b) = \xi_{2j},$$

where $j = 0, 1, 2, 3, 4$ and $5$. Functions $y(t)$, $\vartheta(t)$ and $\eta(t)$ are continuous and defined on $[a, b]$, and $\delta$ and $\xi$ are finite real constant. Existence and uniqueness of solutions to 12th-order BVP are contain, without details, as theorems in Agarwal [1].

Also, different numerical methods have been proposed by various authors. For instance, the authors in [12] implemented the HPM to obtain approximate numerical solutions to 12th-order BVP. The method used was based on coupling of the traditional perturbation and homotopy methods. In the two examples considered, the results were compared with the exact analytical solutions and were found to be approximate. In [14] the authors applied HPM on 9th, 10th and 12th-orders BVP to obtain numerical solutions. In the three problems considered as examples, the overall results were found to also be approximate. In [11] the authors applied the Galerkin weighted residue technique with Berstein polynomial as a basis function on 10th and 12th orders BVP. The numerical result obtained in the example considered gave sizeable absolute errors when compared to the exact solutions. In [9] the authors implemented Chebychev method on 12th-order BVP to obtain numerical solutions. The method was based on Chebychev polynomial and the result compared with that obtained from Differential Transform method which were found to be approximate but relatively better. [17] presented a numerical algorithm to approximate 9th, 10th and 12th-order BVP. The technique was based on modified ADM. The results provided reliable solutions with sizeable errors in the numerical examples considered. [16] Applied polynomial spline on linear twelfth-order BVP to obtain approximate solutions. [13] presented HPM on 9th, 10th and 12th-order BVP which arise in the study of astrophysics, hydrodynamic and hydromagnetic stability. In the paper, numerical examples were considered which were correct to at most seven significant figures. [15] Implemented Variational iteration method to solve twelfth-order BVP with minimal computational process to obtain approximate solutions.

The main motivation in this work is to show that ADM, for the first time, can be used to obtain exact analytical solution of 12th-order BVP. The ADM is based on the modified multi-stage ADM, see [2,5–7] and the references there in. The overall results were correlated with the approximate solution obtained by implementing HPM.

## 2. Basic idea of HPM and ADM

2.1. **Theory of HPM.** He [10] proposed and developed the HPM in 1997. Since then the method has systematically improved and is still evolving, see [12,14] and the references there in. The method expresses equation (1.1) as

$$L(y) = 0, \tag{2.1}$$

where $L$ is a differential operator which can further be divided. The possible homotopy formulation is defined as

$$H(y,p) = (1-p)F(y) + pL(y), \tag{2.2}$$

where $F(y)$ is a functional operator with known initial solution $y_0$ and $p$ is the homotopy parameter $p \in (0,1]$. For

$$H(y,p) = 0, \tag{2.3}$$

we have $H(y,0) = F(y)$, $H(y,1) = L(y)$. And the solution of equation (2.1) is

$$y = \sum_{i=0}^{\infty} p^i y_i. \tag{2.4}$$

As $p \to 1$, equation (2.4) corresponds to equation (2.2) and becomes an approximate solution of the form

$$y = \lim_{p \to 1} y = \sum_{i=0}^{\infty} y_i. \tag{2.5}$$

The HPM is heavily dependent on a careful selection of the initial solution $y_0$. With that taken care of, equation (2.5) is convergent and the rate of convergence depend on $L(y)$. The method is applied without any discretization, restrictive assumption or transformation.

2.2. **Theory of ADM.** The ADM by G. Adomian has been widely reported in $[3, 4, 7, 8]$. Theoretically, ADM start by splitting equation (1.1) into linear and nonlinear parts. Then, inverting the highest order derivative operator contained in the linear operator on both sides of the equation. The Adomian polynomials is then calculated. See [2,3,8], and the references there in, and finally the successive terms of the series solution are then found by a recurrent relation.

Traditionally, ADM expresses equation (1.1) as

$$Ly(t) + Ry(t) + Ny(t) = f(t). \tag{2.6}$$

Here, $L$ is a 12th-order differential operator in this paper, $R$ is the remaining linear operator, which in this case is $\vartheta(t)y(t)$, and, $N$ is a nonlinear differential operator, which in this case is zero. So,

$$f(t) = \eta(t). \tag{2.7}$$

Suppose $L^{-1}$ exist, the solution of equation (1.1) is given as

$$y(t) = \sum_{k=0}^{\infty} y_k(t). \tag{2.8}$$

We modify $f(t)$ by applying Taylors' series expansion on it as contain in [2]. That is

$$f(t) = \sum_{k=0}^{\infty} \eta_k(t). \tag{2.9}$$

Substituting equations (2.7), (2.8) and (2.9) in equation (2.6), we have

$$\sum_{k=0}^{\infty} y_k(t) = \Psi(t) + L^{-1}\left[\sum_{k=0}^{\infty} \eta_k(t) - \vartheta(t)\sum_{k=0}^{\infty} y_k(t)\right], \tag{2.10}$$

where $L^{-1}$ is a 12-fold integral in this case. By the principle of ADM, it becomes trivial to see that

$$y_0(t) = \Psi(t), \tag{2.11}$$

where $\Psi(t)$ is the term arising from the source term that may or may not include the initial/boundary conditions. Subsequently,

$$y_1(t) = L^{-1}\left[\eta_0(t) - \vartheta(t)y_0(t)\right],$$
$$y_2(t) = L^{-1}\left[\eta_1(t) - \vartheta(t)y_1(t)\right],$$
$$\vdots$$
$$y_{k+1}(t) = L^{-1}\left[\eta_k(t) - \vartheta(t)y_k(t)\right].$$

**Theorem 2.1.** *$y(t) = \sum_{k=0}^{\infty} y_k(t)$ is a power series that converges absolutely $\forall t$ with $|t| < |\phi|$.*

*Proof.* Since $y(t)$ is a convergent power series $\sum_{k=0}^{\infty} \beta_k t^k$, $\beta_k\phi^k \longrightarrow 0$ as $k \longrightarrow \infty$ and $\beta_k\phi^k$ is bounded. $\beta_k$ is independent on $t^k$. Choose $\alpha \ni \forall k \geqslant 0\ |\beta_k\phi^k| \leqslant \alpha$. Then for $|t| < |\phi|$ and $n \geqslant 1$, $|\beta_k t^k| = |\beta_k\phi^k||\frac{t}{\phi}|^n \leqslant \alpha|\frac{t}{\phi}|^n$. Since $|\frac{t}{\phi}|$ is independent on $k$ and $|\frac{t}{\phi}| < 1$, $\sum_{n=1}^{\infty} |\frac{t}{\phi}|^n$ converges. And, by comparison, $\sum_{k=0}^{\infty} \beta_k t^k$ converges absolutely □

## 3. Numerically Computed Examples

In this section we take two numerically computed examples to justify our claim.

**Example 1.** *In relation to equation* (1.1)*, consider the 12th-order BVP which is also found in [9, 12, 16]*

$$\vartheta(t) = -1, \qquad \eta(t) = -12(2\cos t + 11\sin t), \qquad a = -1 \quad \textit{and} \quad b = 1,$$

*with the boundary conditions*

$$y(-1) = y(1) = 0,$$
$$y^{'}(-1) = y^{'}(1) = 2\sin 1,$$
$$y^{''}(-1) = -y^{''}(1) = -4\cos 1 - 2\sin 1,$$
$$y^{'''}(-1) = y^{'''}(1) = 6\cos 1 - 6\sin 1,$$
$$y^{(iv)}(-1) = -y^{(iv)}(1) = 8\cos 1 + 12\sin 1,$$
$$y^{(v)}(-1) = y^{(v)}(1) = -20\cos 1 + 10\sin 1.$$

The exact solution is

$$y = \sin t(t^2 - 1)$$

and the Taylor's series equivalent is

$$y = -t + \frac{7}{6}t^3 - \frac{7}{40}t^5 + \frac{43}{5040}t^7 - \frac{73}{362880}t^9 + \frac{37}{13305600}t^{11} - \frac{157}{6227020800}t^{13} + \frac{211}{1307674368000}t^{15} - \cdots .$$

Applying the HPM by using equations (2.1)-(2.5), we have the result as shown in Table 1. See [12] for details.

TABLE 1. Absolute errors obtained using HPM on Example 1

| t | Exact solution | Absolute Error of HPM [12] |
|---|---|---|
| -1.0 | 0.0000000000 | 1.6E-09 |
| -0.8 | 0.2582481927 | 2.0E-10 |
| -0.6 | 0.3613711830 | 1.0E-09 |
| -0.4 | 0.3271114075 | 3.0E-09 |
| -0.2 | 0.1907225576 | 3.9E-19 |
| 0.0 | 0.0000000000 | 0 |
| 0.2 | -0.1907225576 | 3.3E-09 |
| 0.4 | -0.3271114075 | 2.9E-09 |
| 0.6 | -0.3613711830 | 1.0E-08 |
| 0.8 | -0.2582481927 | 5.0E-10 |
| 1.0 | 0.0000000000 | 1.6E-09 |

Applying equations (2.6)-(2.11) of ADM theory, we have

$$\sum_{k=0}^{\infty} y_k = \sum_{j=0}^{n-1} \frac{t^j}{j!}a_j + L^{-1}(\eta(t) + y),$$

where $n = 12$, k and j are non-negative integers, $a_j$'s are $y(0)$, $y^{'}(0)$, $y^{''}(0)$, $\ldots$, $y^{(xi)}(0)$ respectively. The $a_j$'s, $j = 0, 1, \ldots, 11$ are not given in the boundary conditions so we solve the system of equations using the multi-stage ADM in [5]. We easily obtain the $a_j$'s, as $0, -1, 0, 7, 0, -21, 0, 43, 0, -73, 0$ and $111$ respectively. This automatically produces

$$y_0(t) = -t + \frac{7}{6}t^3 - \frac{7}{40}t^5 + \frac{43}{5040}t^7 - \frac{73}{362880}t^9 + \frac{37}{1330560}t^{11}$$

Subsequently, using the modification of $\eta(t)$ as found in [2], we have

$$\begin{aligned} y_1 &= L^{-1}[\eta_0(t) - \vartheta(t)y_0(t)] \\ &= L^{-1}[-156t + y_0] \\ &= -\frac{157}{6227020800}t^{13} + \frac{1}{186810624000}t^{15} - \frac{1}{16937496576000}t^{17} + \cdots, \\ y_2 &= L^{-1}[\eta_1(t) - \vartheta(t)y_1(t)] \\ &= L^{-1}[34t^3 + y_1] \\ &= \frac{17}{10897284000}t^{15} - \frac{157}{15511210004333098598400000}t^{25} - \cdots. \end{aligned}$$

Continuing in this order, we obtain

$$y = \sum_{k=0}^{\infty} y_k = \sin t(t^2 - 1),$$

which is the exact analytical solution of Example 1. The graphical representation of the results in the two method is as shown in figure 1. To demonstrate a clear behaviour of the plotted graph, we deliberately used $y = \sum_{k=0}^{2} y_k$ of the solution by ADM.

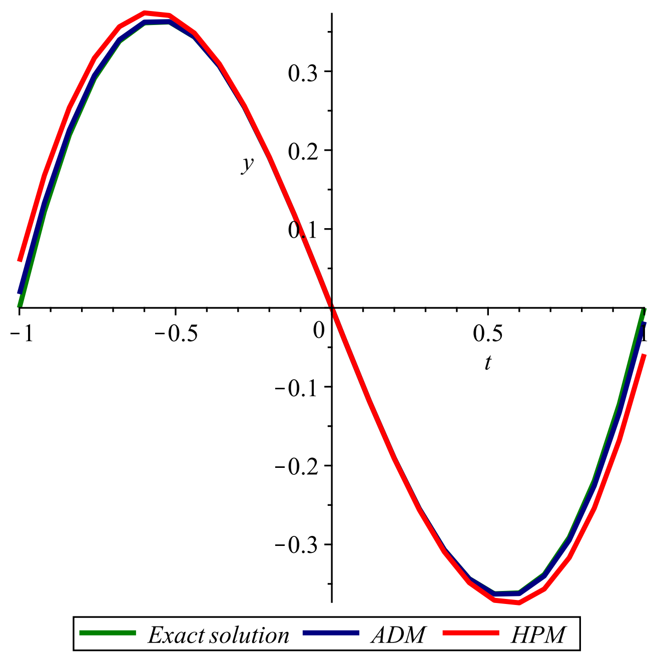


FIGURE 1. Graphical results of Example 1

**Example 2.** *Similarly, in relation to equation* (1.1)*, consider the 12th-order BVP which is also found in [9, 11, 12, 16]:*

$$\vartheta(t) = t, \qquad \eta(t) = -(120 + 23t + t^3)e^t, \qquad a = 0 \quad \textit{and} \quad b = 1,$$

*with the boundary conditions*

$$\begin{aligned} y(0) &= 0, y(1) = 0, \\ y^{'}(0) &= 1, y^{'}(1) = -e, \\ y^{''}(0) &= 0, y^{''}(1) = -4e, \\ y^{'''}(0) &= -3, y^{'''}(1) = -9e, \\ y^{(iv)}(0) &= -8, y^{(iv)}(1) = -16e, \\ y^{(v)}(0) &= -15, y^{(v)}(1) = -25e. \end{aligned}$$

*The exact solution is*

$$y = te^t(1 - t),$$

*and the Taylor's series form is expressed as*

$$\begin{aligned} y = t &- \frac{t^3}{2} - \frac{t^4}{3} - \frac{t^5}{8} - \frac{t^6}{30} - \frac{t^7}{144} - \frac{t^8}{840} - \frac{t^9}{5760} - \frac{t^{10}}{45360} \\ &- \frac{t^{11}}{403200} - \frac{t^{12}}{3991680} - \frac{t^{13}}{43545600} - \frac{t^{14}}{518918400} - \dots \end{aligned}$$

*Applying the HPM by using equations* (2.1)-(2.5)*, we have the result as shown in table 2. See [12] for details.*

*Also, applying equations* (2.6)-(2.11) *of ADM theory, we have*

$$\sum_{k=0}^{\infty} y_k = \sum_{j=0}^{n-1} \frac{t^j}{j!} a_j + L^{-1}(\eta(t) - ty)$$

*Where* $n = 12$*, k and j are non-negative integers,* $a_j$*'s are* $y(0)$*,* $y^{'}(0)$*,* $y^{''}(0)$*, ...,* $y^{(xi)}(0)$ *respectively. The* $a_j$*'s,* $j = 0, 1, \dots, 5$ *are given in the boundary conditions. Using the Multi-stage ADM in [5] we easily obtain the* $a_j$*'s,* $j = 6, 7, \dots, 11$ *as* $-24, -35, -48, -63, -80$ *and* $-99$ *respectively. This automatically produces*

$$y_0 = t - \frac{t^3}{2} - \frac{t^4}{3} - \frac{t^5}{5} - \frac{t^6}{30} - \frac{t^7}{144} - \frac{t^8}{840} - \frac{t^9}{5760} - \frac{t^{10}}{45360} - \frac{t^{11}}{403200}.$$

TABLE 2. Absolute errors obtained using HPM on Example 2

| *t* | *Exact solution* | *Absolute Error of HPM [12]* |
|---|---|---|
| *0.0* | *0.0000000000* | *0.000E-00* |
| *0.1* | *0.0994653826* | *3.000E-11* |
| *0.2* | *0.1954244413* | *0.000E-00* |
| *0.3* | *0.2834703497* | *-1.000E-10* |
| *0.4* | *0.3580379275* | *2.000E-10* |
| *0.5* | *0.4121803178* | *1.100E-09* |
| *0.6* | *0.4373085120* | *4.400E-09* |
| *0.7* | *0.4228880685* | *1.350E-08* |
| *0.8* | *0.3560865484* | *3.680E-08* |
| *0.9* | *0.2213642800* | *9.010E-08* |
| *1.0* | *0.0000000000* | *2.027E-07* |

*Subsequently, following similar steps as in example 1 and using the modification of* $\eta(t)$ *as contain in [2] we have*

$$y_1 = -\frac{t^{12}}{3991680} - \frac{t^{14}}{43589145600} + \frac{t^{16}}{1743565824000} - \cdots,$$
$$y_2 = -\frac{t^{13}}{43545600} + \frac{t^{25}}{994308336110960} + \cdots,$$

*and*

$$y = \sum_{k=0}^{\infty} y_k = te^t(1-t),$$

*which is the exact solution of example 2. Similarly, the graphical representation of the results in the two method is as shown in figure 2. And, to demonstrate a clear behaviour of the plotted graph, we deliberately used* $y = \sum_{k=0}^{4} y_k$ *of the solution by ADM.*

## 4. CONCLUSIONS

We have, in this work, for the first time been able to show that ADM can also be used to obtain exact solutions of 12th-order boundary value problems in a finite domain. The HPM on the same class of equations gave approximate solutions that gradually deviated from normal as $t \to a/b$. And, the ADM expressed the solutions

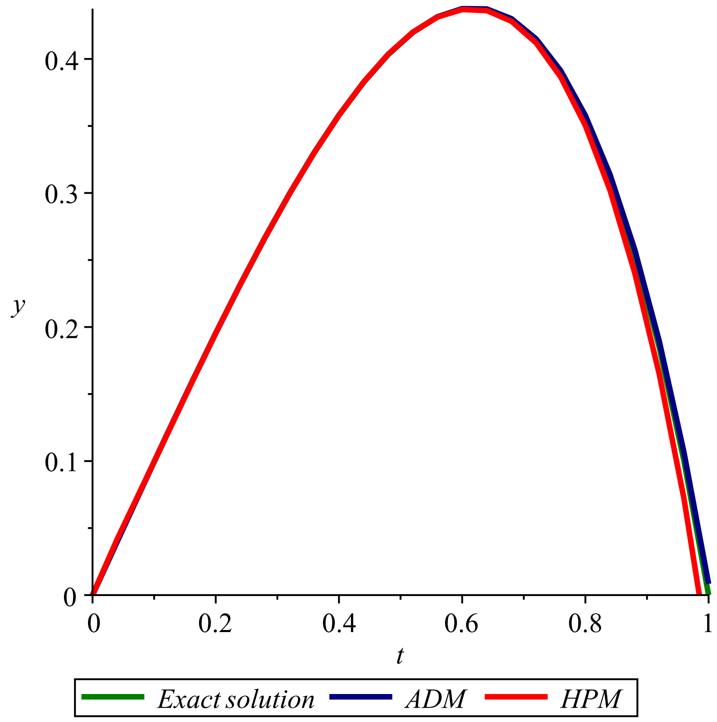


FIGURE 2. Graphical results of Example 2

in rapidly converging series which were the same as the Taylor's series expansion of the exact analytical solutions.

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