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EXTENSION OF BM-ALGEBRA INTO AN ALGEBRAIC STRUCTURE WITH TWO BINARY OPERATIONS

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ABSTRACT. In this article, we combine the notions of BM-algebra and semigroup to introduce a ring like structure with two binary operations, which is named as a "BM-semigroup". The concepts of left and right ideals of a BM-semigroup are also introduced and related properties are studied. Some results on homomorphism defined on a BM-semigroup are also given.

1. INTRODUCTION

Several researchers have considered different types of logical algebras (see [1]) and studied their characterizations. Here, we consider the algebraic structure - BM-algebra - introduced by Kim and Kim [2]. Megalai and Tamilarasi [3] renamed it as TM-algebra. Later on, many authors have considered different characterizations of BM-algebra ([6], [9], [10], [11]) and made fuzzifications to BM/TM-algebra ([4], [5], [7], [8]). In this article, we try to add one more binary operation to the BM-algebra and study its properties. For this, first let us see the basic definition of BM-algebra.

Definition 1.1. *A BM-algebra is a triple $(X, *, \theta)$ where $X(\neq \phi)$ is a set with a fixed element θ and a binary operation $*$, satisfying the two conditions: (i) $x * \theta = x$ and (ii) $(x * y) * (x * z) = z * y$ for all $x, y, z \in X$.*

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2. BM-SEMIGROUP

Definition 2.1. *A (right) BM-semigroup* $(X, *, \cdot, \theta)$ *is a non-empty set together with two binary operations* $*$ *and* $\cdot$*, defined on* X *such that the following axioms are satisfied:*

(i) $(X, *, \theta)$ *is a BM-algebra*
(ii) $(X, \cdot)$ *is a semigroup*
(iii) *The operation* $\cdot$ *is distributive on right over the operation* $*$ *: for all* x, y, z *in* X*, the right distributive law* $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$ *hold.*

Similarly, it is possible to define a left BM-semigroup by replacing the right distributive law (iii) by the corresponding left distributive law, $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ *for all* x, y, z *in* X*. If the operation* $\cdot$ *is distributive on both sides (left as well as right) over the operation* $*$*, then we call* X *as a BM-semigroup.*

Example 1. *Let* $X = \{\theta, a, b, c\}$*. Define* $*$ *and* $\cdot$ *by the following Table 1. Then*

TABLE 1

$*$	θ	a	b	c
θ	θ	c	b	a
a	a	θ	c	b
b	b	a	θ	c
c	c	b	a	θ

$\cdot$	θ	a	b	c
θ	θ	θ	θ	θ
a	θ	b	θ	b
b	θ	θ	θ	θ
c	θ	b	θ	b

$(X, *, \cdot, \theta)$ *is a BM-semigroup.*

Example 2. *Let* $Y = \{\theta', x, y, z\}$*. Define* $*'$ *and* $\cdot'$ *by the following Table 2. Then*

TABLE 2

$*'$	θ'	x	y	z
θ'	θ'	x	y	z
x	x	θ'	z	y
y	y	z	θ'	x
z	z	y	x	θ'

$\cdot'$	θ'	x	y	z
θ'	θ'	θ'	θ'	θ'
x	θ'	x	y	z
y	θ'	x	y	z
z	θ'	θ'	θ'	θ'

$(Y, *', \cdot', \theta')$ *is a BM-semigroup.*

Definition 2.2. *Assume* $(X, *, \cdot, \theta)$ *and* $(Y, *_1, \cdot_1, \theta_1)$ *be two BM-semigroups. A mapping* $\Phi : X \to Y$ *is said to be a BM-semigroup homomorphism if* $\Phi(a * b) = \Phi(a) *_1 \Phi(b)$ *and* $\Phi(a \cdot b) = \Phi(a) \cdot_1 \Phi(b)$ *for all* $a, b \in X$.

Example 3. *Let* $X = \{\theta, a, b, c\}$ *and* $Y = \{\theta', x, y, z\}$ *be sets with cayley tables given in Example 1 and 2 respectively. Thus* $(X, *, \cdot, \theta)$ *and* $(Y, *', \cdot', \theta')$ *are BM-semigroups. Define a map* $\Phi : X \to Y$ *by*

$$\Phi(p) = \begin{cases} \theta', \ if \ p = \theta, b \\ z, \ if \ p = a, c. \end{cases}$$

Then Φ *is a BM-semigroup homomorphism. Define* $\rho : X \to Y$ *by*

$$\rho(p) = \begin{cases} \theta', \ if \ p = \theta, b \\ x, \ if \ p = a, c \end{cases}$$

is a BM-homomorphism, but not a BM-semigroup homomorphism since $\rho(a \cdot c) = \rho(b) = \theta$ *but* $\rho(a) \cdot \rho(c) = x \cdot x = x$.

Proposition 2.1. *Let* $(X, *, \cdot, \theta)$ *be a BM-semigroup. Then* $x \cdot \theta = \theta \cdot x = \theta, \forall x \in X$.

Proof. Let $x \in X$. Then $x \cdot \theta = x \cdot (\theta * \theta) = (x \cdot \theta) * (x \cdot \theta) = \theta$ and $\theta \cdot x = (\theta * \theta) \cdot x = (\theta \cdot x) * (\theta \cdot x) = \theta$. □

Definition 2.3. *An element* $x \neq \theta$ *in a BM-semigroup* $(X, *, \cdot, \theta)$ *is said to be a left (resp., right) unit divisor if* $\exists y \neq \theta \in X$ *such that* $x \cdot y = \theta$ *(resp.* $y \cdot x = \theta$*). A unit divisor is an element of X which is both a left and a right unit divisors.*

Proposition 2.2. *Let* $(X, *, \cdot, \theta)$ *be a BM-semigroup. Then* X *satisfies the left (resp., right) cancellation law for the operation* $\cdot$ *if and only if* X *contains no left (resp., right) unit divisors.*

Proof. Let X satisfies the left cancellation law for the operation $\cdot$ and suppose $x \cdot y = \theta$ where $x \neq \theta$. Then $x \cdot y = \theta = x \cdot \theta$ which implies $y = \theta$. Similarly it holds for the right case also. Thus X does not contain any left (resp., right) unit divisors. Conversely, suppose X contains no left unit divisors and let $x \cdot y = x \cdot z$ for some $x \neq \theta$. Then $x \cdot (y * z) = (x \cdot y) * (x \cdot z) = \theta$. Since X has no left unit divisor, we get $y * z = \theta$ which implies $y = z$. Hence left cancellation law holds. Similarly, suppose X contains no right unit divisors and let $y \cdot x = z \cdot x$ for some

$x \neq \theta$. Then $(x * y) \cdot z = (x \cdot z) * (y \cdot z) = \theta$. Since X has no right unit divisor, we get $y * z = \theta$ which implies $y = z$. Hence X satisfies the right cancellation law. $\square$

Definition 2.4. *A subset $Y \neq \phi$ of a BM-semigroup $(X, *, \cdot, \theta)$ is said to be a BMS-subalgebra if $a, b \in Y$ implies $a * b \in Y$ and $a \cdot b \in Y$.*

Proposition 2.3. *Let $(X, *, \cdot, \theta)$ and $(Y, *_1, \cdot_1, \theta_1)$ be two BM-semigroups and let $\Phi : X \to Y$ be a BM-semigroup homomorphism. Then $K = ker(\Phi)$ is a BMS-subalgebra of X.*

Proof. Let $a, b \in K$. Then $\Phi(a) = \Phi(b) = \theta_1$. So $\Phi(a \cdot b) = \Phi(a) \cdot \Phi(b) = \theta_1 \cdot \theta_1 = \theta_1$. Hence $a \cdot b \in K$. Similarly, $\Phi(a * b) = \Phi(a) * \Phi(b) = \theta_1 * \theta_1 = \theta_1$. Hence $a * b \in K$. $\square$

Remark 2.1. *As BM-algebras, $(\mathbb{Z}, -, 0)$ and $(2\mathbb{Z}, -, 0)$ are isomorphic under the map $\Phi : \mathbb{Z} \to 2\mathbb{Z}$ with $\Phi(x) = 2x$ for $x \in \mathbb{Z}$. But Φ is not a BM-semigroup homomorphism, for $\Phi(x \cdot y) = 2xy$, while $\Phi(x) \cdot \Phi(y) = 2x2y = 4xy$.*

Theorem 2.1. *The set $End(X)$ of all endomorphisms of a BM-algebra X forms a BM-semigroup under homomorphism addition and homomorphism multiplication.*

Proof. Since the composition $\circ$ of two homomorphisms of X into itself is again such a homomorphism and since $\circ$ is associative, we can define function composition $\circ$ as the multiplication on $End(X)$. Now, define a binary operation $+$ on $End(X)$ as $(\varphi + \psi)(x) = \varphi(x) * \psi(x)$ for any $\varphi, \psi \in End(X)$ and for each $x \in X$. Now let $\varphi, \psi, \xi \in End(X)$. Since

$$\begin{aligned}(\varphi + \psi)(x * y) = \varphi(x * y) * \psi(x * y) &= [\varphi(x) * \varphi(y)] * [\psi(x) * \psi(y)] \\ &= [\varphi(x) * \psi(x)] * [\varphi(y) * \psi(y)] \\ &= (\varphi + \psi)(x) * (\varphi + \psi)(y) \text{ for all } x, y \in X,\end{aligned}$$

we get $\varphi + \psi \in End(X)$. Define an endomorphism ϑ on X as $\vartheta(x) = \theta$ for all $x \in X$. Now, $(\varphi + \vartheta)(x) = \varphi(x) * \vartheta(x) = \varphi(x) * \vartheta = \varphi(x)$ for all $x \in X$ and

$$\begin{aligned}((\varphi + \psi) + (\varphi + \xi))(x) &= (\varphi + \psi)(x) * (\varphi + \xi)(x) \\ &= [\varphi(x) * \psi(x)] * [\varphi(x) + \xi(x)] \\ &= \xi(x) * \psi(x) = (\xi + \psi)(x) \text{ for all } x \in X.\end{aligned}$$

Hence $(End(X), +, \vartheta)$ is a BM-algebra. Now for $\varphi, \psi, \xi \in End(X)$ and $x \in X$,

$$\begin{aligned}[\varphi \circ (\psi + \xi)](x) &= \varphi[(\psi + \xi)(x)] \\ &= \varphi[\psi(x) * \xi(x)] \\ &= \varphi(\psi(x)) * \varphi(\xi(x)) \text{ (since } \varphi \text{ is a BM-homomorphism)} \\ &= (\varphi \circ \psi)(x) * (\varphi \circ \xi)(x) \\ &= [(\varphi \circ \psi) + (\varphi \circ \xi)](x),\end{aligned}$$

and

$$\begin{aligned}[(\varphi + \psi) \circ \xi](x) &= (\varphi + \psi)(\xi(x)) \\ &= \varphi(\xi(x)) * \psi(\xi(x)) \\ &= (\varphi \circ \xi)(x) * (\psi \circ \xi)(x) \\ &= [(\varphi \circ \xi) + (\psi \circ \xi)](x).\end{aligned}$$

Thus the operation $\circ$ is distributive on left as well as right over the operation $+$. Hence $(End(X), +, \circ, \vartheta)$ is a BM-semigroup. □

Definition 2.5. *A non-empty subset I of a BM-semigroup $(X, *, \cdot, \theta)$ is called a left (resp. right) BMS-ideal of X if it satisfies*

(i) *$x \cdot a \in I$ (resp. $a \cdot x \in I$) whenever $x \in X$ and $a \in I$;*
(ii) *for any $x, y \in X$, $x * y \in I$ and $y \in I$ imply that $x \in I$.*

I is said to be a BMS-ideal of BM-semigroup X if it is both left and right BMS-ideal of X.

Remark 2.2. *For any BM-semigroup X, the subset $\{\theta\}$ and X are always BMS-ideals of X. Also, if I is a left (resp. right) BMS-ideal of BM-semigroup X, then $\theta \in I$. Hence I is a BM-ideal of the underlying BM-algebra X.*

Theorem 2.2. *Let a mapping $\Phi : X \to Y$ be an epimorphism of BM-semigroups. Then we have the following:*

(i) *$ker(\Phi)$ is a BMS-ideal of X;*
(ii) *If I is a left (resp. right) BMS-ideal of X, then $\Phi(I)$ is a left (resp. right) BMS-ideal of Y;*
(iii) *If I' is a left (resp. right) BMS-ideal of Y, then $\Phi^{-1}(I')$ is a left (resp. right) BMS-ideal of X.*

Proof. It is easily seen as similar the proofs of the ring theory. □

Remark 2.3. *Let $(X, *, \cdot, \theta)$ be a BM-semigroup and I be a BMS-ideal of X. For any $x, y \in X$, define a binary relation $\sim_I$ on X as follows:*

$$x \sim_I y \text{ if and only if } x * y \in I \text{ and } y * x \in I.$$

Then $\sim_I$ is an equivalence relation on X. Denote the equivalence class containing x by I_x and the set of equivalence classes in X by X/I. Then $I_\theta = I$.

Theorem 2.3. *If I is a BMS-ideal of a BM-semigroup $(X, *, \cdot, \theta)$, then $(X/I, *_\sim, \cdot_\sim, I_\theta)$ is a BM-semigroup under the operations $I_x *_\sim I_y = I_{x*y}$ and $I_x \cdot_\sim I_y = I_{x\cdot y}$ for all $I_x, I_y \in X/I$.*

Proof. Clearly, $(X/I, *_\sim, I_\theta)$ is a BM-algebra. Also, $*_\sim$ and $\cdot_\sim$ are well defined since I is a BMS-ideal and hence $(X/I, \cdot_\sim)$ is a semigroup. Now, for any $I_x, I_y, I_z \in X/I$, we get $(I_x *_\sim I_y) \cdot_\sim I_z = I_{x*y} \cdot_\sim I_z = I_{(x*y)\cdot z} = I_{(x\cdot z)*(y\cdot z)} = I_{(x\cdot z)} *_\sim I_{(y\cdot z)} = (I_x \cdot_\sim I_z) *_\sim (I_y \cdot_\sim I_z)$. Similarly, $I_x \cdot_\sim (I_y *_\sim I_z) = (I_x \cdot_\sim I_y) *_\sim (I_x \cdot_\sim I_z)$. Thus $(X/I, *_\sim, \cdot_\sim, I_\theta)$ is a BM-semigroup. □

Proposition 2.4. *If I and J are BMS-ideals of a BM-semigroup X and $I \subset J$, then J/I is a BMS-ideal of X/I.*

Theorem 2.4. *Let $\Phi : X \to Y$ be a BM-homomorphism of BM-semigroups with $ker(\Phi) = K$. Then for any BMS-ideal I of X, we have $I/(K \cap I) \cong \Phi(I)$.*

Proof. Clearly, $J = K \cap I$ is a BMS-ideal of I. Define a map $\rho : I/J \to \Phi(I)$ by $\rho(J_x) = \Phi(x)$ for all $x \in I$. Then for any $J_x, J_y \in I/J$,

$$\begin{aligned} J_x = J_y &\iff x * y, y * x \in J, \\ &\iff \Phi(x * y) = \theta, \Phi(y * x) = \theta, \\ &\iff \Phi(x) *_\sim \Phi(y) = \theta, \\ &\iff \Phi(x) = \Phi(y), \\ &\iff \rho(J_x) = \rho(J_y) \end{aligned}$$

Hence ρ is well defined and one-to-one. For all $J_x, J_y \in I/J$, we have

$$\rho(J_x *_\sim J_y) = \rho(J_{x*y}) = \Phi(x * y) = \Phi(x) * \Phi(y) = \rho(J_x) * \rho(J_y),$$

and

$$\rho(J_x \cdot_\sim J_y) = \rho(J_{x\cdot y}) = \Phi(x \cdot y) = \Phi(x) \cdot \Phi(y) = \rho(J_x) \cdot \rho(J_y).$$

Hence ρ is a BM-homomorphism of BM-semigroups. Also, $Im(\rho) = \{\rho(J_x) | x \in I\} = \{\Phi(x) | x \in I\} = \Phi(I)$. Hence ρ is onto. Hence the proof. □

Corollary 2.1. *Let $\Phi : X \to Y$ be a BM-epimorphism of BM-semigroups with $ker(\Phi) = K$. Then $X/K \cong Y$.*

Theorem 2.5. *If $\Phi : X \to Y$ be a BM-epimorphism of BM-semigroups and X is Noetherian, then so is Y.*

Proof. Let $K = ker(\Phi)$. By Corollary 2.1, $X/K \cong Y$. By Proposition 2.4, every BMS-ideal of X/K is of the form I/K, where I is a BMS-ideal of X with $K \subseteq I$. Let $I_1/K \subseteq I_2/K \subseteq \cdots$ be any ascending chain of BMS-ideals in Y. Then $K \subseteq I_1 \subseteq I_2 \subseteq \cdots$ is an ascending chain of BMS-ideals of X. Since X is Noetherian, we have $I_n = I_{n+1} = \cdots$ for some natural number n. Hence we obtain $I_n/K = I_{n+1}/K = \cdots$. Hence Y is Noetherian. $\square$

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