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SUFFICIENT CONDITIONS FOR STARLIKENESS USING SUBORDINATION METHOD

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ABSTRACT. Let f be analytic in the unit disk and normalized by $f(0) = f'(0)-1 = 0$. In this paper using a method from the theory of first order differential subordination we investigate the sufficient conditions over the differential subordination

$$p(z) + zp'(z) \prec \frac{1 + Az(2 + Bz)}{(1 + Bz)^2}$$

that implies $p(z) \prec \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, and further use it for obtaining inequalities over the function f.

1. INTRODUCTION AND PRELIMINARIES

Analytic function f defined in the domain D is univalent if it is injective. Let $\mathcal{A}$ denotes the class of functions f that are analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$, i.e., such that $f(z) = z + a_2z^2 + \cdots$.

A function $f \in \mathcal{A}$ is said to be *starlike* if, and only if

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0, \quad z \in \mathbb{D}.$$

We denote by $\mathcal{S}^*$ the class of all such functions which are at the same time univalent. Their geometrical characterisations is the following: f is starlike if, and only

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if, $t\omega \in f(\mathbb{D})$ for all $\omega \in f(\mathbb{D})$ and all $t \in [0,1]$, i.e., for all $z \in \mathbb{D}$, $f(z)$ is visible from the origin. For details see [1, 7].

A special subclass of $\mathcal{S}^*$ is the class of *starlike function of order* α with $0 \le \alpha < 1$, given by

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \;\; z \in \mathbb{D} \right\}.$$

Further, a function f is said to be subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if there exists a function w analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. If F is univalent, then $f \prec F$ if, and only if, $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$. For details see [2].

Using subordination, another generalisation is defined by

$$\mathcal{S}^*[A,B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \;\; z \in \mathbb{D} \right\},$$

$-1 \le B < A \le 1$. Geometrically, this means that the image of $\mathbb{D}$ by $zf'(z)/f(z)$ is inside the open disk centered on the real axis with diameter endpoints $(1-A)/(1-B)$ and $(1+A)/(1+B)$. In [5] it is given that special selections of A and B lead us to the following:

- $\mathcal{S}^*[1,-1] \equiv \mathcal{S}^*$;
- $\mathcal{S}^*[1-2\alpha,-1] \equiv \mathcal{S}^*(\alpha)$, $0 \le \alpha < 1$.

Next, we denote by $\mathcal{K}$ the class of *convex functions*, i.e., the class of function $f(z) \in \mathcal{A}$ for which

$$\operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0, \quad z \in \mathbb{D}.$$

and its generalization, the class of *convex functions of order* α, with $0 \le \alpha < 1$, given by

$$\mathcal{K}(\beta) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha, \, z \in \mathbb{D} \right\}.$$

Both these classes ($\mathcal{S}^*$ and $\mathcal{K}$) are subclasses of univalent function in $\mathbb{D}$ and even more $\mathcal{K} \subset \mathcal{S}^*$. For details see [1, 7].

In this paper we study the differential subordination of the form

$$p(z) + zp'(z) \prec \frac{1 + Az(2+Bz)}{(1+Bz)^2}, \quad -1 \le B < A \le 1,$$

and conditions when it implies the subordination $p(z) \prec (1+Az)/(1+Bz)$, where $p(z)$ is analytic function and $p(0)=1$. For special selection of the function $p(z)$, for example for $p(z) = zf'(z)/f(z)$, $p(z) = f(z)/z$ and $p(z) = f'(z)$ the left hand side of this subordination will give special cases that will imply results over inequalities involving the function f.

For that purpose we will use a method from the theory of first order differential subordinations. If $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ is analytic in the domain D, if $h(z)$ is univalent in $\mathbb{D}$, and if $p(z)$ is analytic in $\mathbb{D}$ with $(p(z), zp'(z)) \in D$ when $z \in \mathbb{D}$, then we say tha $p(z)$ satisfies the *(first-order) differential subordination*

$$\psi\left(p(z), zp'(z)\right) \prec h(z). \tag{1.1}$$

The function $p(z)$ is called the *solution of differential subordination* (1.1). The univalent function $q(z)$ is called *dominant* of the solution of differential equation (1.1) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.1). The dominant $\widetilde{q}(x)$ satisfies $\widetilde{q}(x) \prec q(z)$ for all dominants $q(z)$ of (1.1) is said to be *the best dominant* of (1.1).

From this theory we will make use of the following lemma due to Miller and Mocanu [2].

Lemma 1.1. *[2] Let q be univalent in the unit disk $\mathbb{D}$, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(\mathbb{D})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that:*

(i) Q is starlike in the unit disk $\mathbb{D}$,
(ii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} = \operatorname{Re}\left[\dfrac{\theta'(q(z))}{\phi(q(z))} + \dfrac{zQ'(z)}{Q(z)}\right] > 0$, $z \in \mathbb{D}$.

If p is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z) \tag{1.2}$$

then $p(z) \prec q(z)$, and q is the best dominant of (1.2).

2. Main results and consequences

First we will prove a lemma that will later lead to the main result.

Lemma 2.1. *Let $p(z)$ be analytic in the unit disk $\mathbb{D}$, $p(0) = 1$, $0 \notin p(\mathbb{D})$. Also, let A, B be a real number with $-1 \leq B < A \leq -1$. If*

$$p(z) + zp'(z) \prec \frac{1 + Az(2 + Bz)}{(1 + Bz)^2}, \tag{2.1}$$

then $p(z) \prec q(z) = \frac{1+Az}{1+Bz}$ and $q(z)$ is the best dominant of (2.1).

Proof. In Lemma 1.1 we choose $\theta(\omega) = \omega$ and $\phi(\omega) = 1$, which are analytic in domain $D = \mathbb{C}$. Then $q(z)$ is univalent in $\mathbb{D}$ and $\phi(\omega)$ and $\theta(\omega)$ are analytic in domain $D = \mathbb{C}$ containing $q(z) = \frac{1+Az}{1+Bz}$ with $\phi(\omega) \neq 0$ when $\omega \in q(\mathbb{D})$. Further, set

$$Q(z) = zq'(z)\phi(q(z)) = \frac{(A - B)z}{(1 + Bz)^2},$$

which is starlike because

$$\frac{zQ'(z)}{Q(z)} = \frac{1 - Bz}{1 + Bz}$$

and for $z = e^{i\theta}$, $\theta \in [-\pi, \pi]$,

$$\mathrm{Re}\left\{\frac{zQ'(z)}{Q(z)}\right\} = \frac{1 - B^2}{|1 + Be^{i\theta}|^2} \geq 0.$$

Next,

$$h(z) = \theta(q(z)) + Q(z) = \frac{1 + Az(2 + Bz)}{(1 + Bz)^2}$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{2}{1 + Bz}.$$

For $z = e^{i\theta}$, $\theta \in [-\pi, \pi]$ we have

$$\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \frac{2 + 2B\cos\theta}{1 + 2B\cos\theta + B^2} = \frac{2 + 2B\cos\theta}{|1 + Be^{i\theta}|^2} \geq 0.$$

So, from $p(0) = q(0) = 1$ and from (1.2) we receive that $p(z) \prec q(z)$ and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant od (2.1). □

Putting $p(z) = \frac{zf'(z)}{f(z)}$ in Lemma 1 we obtain the main result.

Theorem 2.1. *Let $f \in \mathcal{A}$,and let A, B be a real numbers, $-1 \leq B < A \leq -1$. If*

$$\frac{zf'(z)}{f(z)}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right) + \left(1 - \frac{zf'(z)}{f(z)}\right)\right] \prec \frac{1 + Az(2 + Bz)}{(1 + Bz)^2} \equiv h(z) \tag{2.2}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}. \tag{2.3}$$

The right hand side of (2.3) *is the best dominant of* (2.2).

Proof. Let $p(z) = \frac{zf'(z)}{f(z)}$ and $q(z) = \frac{1+Az}{1+Bz}$. Then,

$$p(z) + zp'(z) = \frac{zf'(z)}{f(z)}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right) + \left(1 - \frac{zf'(z)}{f(z)}\right)\right],$$

$$q(z) + zq'(z) = \frac{1 + Az(2 + Bz)}{(1 + Bz)^2}$$

and

$$p(z) + zp'(z) \prec q(z) + zq'(z).$$

Since, $p(0) = q(0) = 1$ from Lemma 1.1 we have $p(z) \prec q(z)$, i.e.,

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},$$

where $q(z)$ is best dominant. □

Corollary 2.1. *Let* $f \in \mathcal{A}$.

(i) *If* $-1 \leq B < A \leq 1$ *and*

$$\left|\frac{zf'(z)}{f(z)}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right) + \left(1 - \frac{zf'(z)}{f(z)}\right)\right] - 1\right| < (A - B)\frac{2 + |B|}{(1 + |B|)^2},$$

$z \in \mathbb{D}$, *then*

$$\left|\frac{zf'(z)}{f(x)} - 1\right| < \frac{A - B}{1 - |B|}, \quad z \in \mathbb{D}.$$

(ii) *If* $B = 0$ *and* $0 < A \leq 1$, *then*

$$\left|\frac{zf'(z)}{f(z)}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right) + \left(1 - \frac{zf'(z)}{f(z)}\right)\right] - 1\right| \leq 2A, \quad z \in \mathbb{D},$$

implies

$$\left|\frac{zf'(z)}{f(x)} - 1\right| < A, \quad z \in \mathbb{D}.$$

(iii) *If* $B = -1$ *and* $A = 1 - 2\alpha$, *for* $\alpha \in [0, 1)$, *then:*

$$\left|\frac{zf'(z)}{f(z)}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right) + \left(1 - \frac{zf'(z)}{f(z)}\right)\right] - 1\right| < \frac{3}{2}(1 - \alpha), \quad z \in \mathbb{D},$$

implies that $f \in \mathcal{S}^*(\alpha)$.

Proof.
(i) By the definition of subordination, we have that

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$$

is equivalent to

$$\frac{zf'(z)}{f(z)} - 1 \prec \frac{(A-B)z}{1+Bz}$$

and implies that

$$\sup_{z\in\mathbb{D}} \left|\frac{zf'(z)}{f(z)} - 1\right| = \inf_{|z|=1} \frac{(A-B)z}{1+Bz} = \frac{A-B}{1-|B|},$$

i.e.,

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{A-B}{1-|B|} \quad (z \in \mathbb{D}).$$

So, the conclusion in (i) will follow from Theorem 2.1 if we show that

$$\min_{\theta\in[0,2\pi]} |h(e^{i\theta}) - 1)| = (A-B)\frac{2+|B|}{(1+|B|)^2},$$

where h is defined in Theorem 2.1,

$$h(z) = \frac{1+Az(2+Bz)}{(1+Bz)^2}.$$

From

$$|h(e^{i\theta}) - 1|^2 = \frac{(A-B)^2(4+B^2+4B\cos t)}{(1+B^2+2B\cos t)},$$

using $x = \cos t$, $-1 \le x \le 1$, we have

$$\varphi(x) = \frac{(A-B)^2(4+B^2+4Bx)}{(1+B^2+2Bx)}$$

and

$$\varphi'(x) = -\frac{4(A-B)^2B(3+2Bx)}{(1+B^2+2Bx)}.$$

The equation $\varphi'(x) = 0$ is equivalent to $-\frac{4(A-B)^2B(3+2Bx)}{(1+B^2+2Bx)} = 0$ with solution $x = -\frac{3}{2B}$. But, since $-1 \le B \le 1$, we have $-\frac{3}{2B} \notin [-1,1]$, meaning that the extreme values occur for $x=-1$ and $x=1$ (at the endpoints). Next,

$$\varphi(\pm 1) = (A-B)^2\frac{(2\pm B)^2}{(1\pm B)^4} = (A-B)^2\left[\frac{2\pm B}{(1\pm B)^2}\right]^2$$

and

$$\varphi(1) > \varphi(-1) \quad \Leftrightarrow \quad B < 0.$$

This means that

$$B < 0 \quad \Rightarrow \quad \min_{t\in[0,2\pi]} |h(e^{i\theta}) - 1)|^2 = \varphi(-1)$$

and

$$B > 0 \quad \Rightarrow \quad \min_{t\in[0,2\pi]} |h(e^{i\theta}) - 1)|^2 = \varphi(1),$$

i.e., that

$$\min_{t\in[0,2\pi]} |h(e^{i\theta}) - 1)|^2 = (A-B)^2 \left(\frac{2+|B|}{(1+|B|)^2}\right)^2$$

and

$$\min_{t\in[0,2\pi]} |h(e^{i\theta}) - 1)| = (A-B)\frac{2+|B|}{(1+|B|)^2}.$$

This completes the proof of (i).

(ii) Follows from (i) for $B = 0$.

(iii) When $B = -1$ and $A = 1 - 2\alpha$, for $\alpha \in [0, 1)$, from

$$\min_{t\in[0,2\pi]} |h(e^{i\theta}) - 1)| = (A-B)\frac{2+|B|}{(1+|B|)^2} = \frac{3}{2}(1-\alpha)$$

and

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} = \frac{1+(1-2\alpha)z}{1-z},$$

we have that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{D},$$

or $f \in \mathcal{S}^*(\alpha)$. □

Putting $p(z) = \frac{f(z)}{z}$ in Lemma 2.1 we obtain the following theorem.

Theorem 2.2. *Let $f \in \mathcal{A}$,and let A, B be a real numbers such that $-1 \le B < A \le -1$. If*

$$f'(z) \prec \frac{1+Az(2+Bz)}{(1+Bz)^2} \tag{2.4}$$

then

$$\frac{f(z)}{z} \prec \frac{1+Az}{1+Bz}. \tag{2.5}$$

And the right hand side of (2.5) *is the best dominant of* (2.4).

In a similar way as we obtained Corollary 2.1 from Theorem 2.1, Theorem 2.2 implies the following result.

Corollary 2.2. *Let* $f \in \mathcal{A}$.

(i) *If* $-1 \le B < A \le 1$*, then*

$$|f'(z) - 1| \le (A - B)\frac{2+|B|}{(1+|B|)^2}, \quad z \in \mathbb{D},$$

implies

$$\left|\frac{f(z)}{z} - 1\right| < \frac{A-B}{1-|B|}, \quad z \in \mathbb{D}.$$

(ii) *If* $B = 0$*, and* $0 < A \le 1$*, then*

$$|f'(z) - 1| < 2A, \quad z \in \mathbb{D}.$$

implies

$$\left|\frac{f(z)}{z} - 1\right| < A, \quad z \in \mathbb{D}.$$

(iii) *If* $B = -1$ *and* $A = 1 - 2\alpha, \alpha \in [0,1), (-1 < A \le 1)$ *then*

$$|f'(z) - 1| < \frac{3}{2}(1-\alpha), \quad z \in \mathbb{D},$$

implies

$$\operatorname{Re}\left(\frac{f(z)}{z}\right) > \alpha, \quad z \in \mathbb{D}.$$

By specifying values for A and B in Corollary 2.2 we receive the following

Example 1.

(i) *For* $A = 1$ *and* $B = \frac{-3+\sqrt{17}}{4} = 0.28\ldots$ *we have that*

$$|f'(z) - 1| < 1, \quad z \in \mathbb{D},$$

implies

$$\left|\frac{f(z)}{z} - 1\right| < 1, \quad z \in \mathbb{D}.$$

(ii) *For* $A = 1$ *and* $B = 0$ *we have that*

$$|f'(z) - 1| < 2, \quad z \in \mathbb{D},$$

implies

$$\left| \frac{f(z)}{z} - 1 \right| < 1, \quad z \in \mathbb{D}.$$

(iii) *For* $B = -1$ *and* $\alpha = \frac{1}{2}$ *we have that*

$$|f'(z) - 1| < \frac{3}{4}, \quad z \in \mathbb{D}$$

implies

$$\operatorname{Re} \frac{f(z)}{z} > \frac{1}{2}, \quad z \in \mathbb{D}.$$

Putting $p(z) = f'(z)$ in Lemma 2.1 we obtain the following theorem followed by a corollary in a analogue way as before.

Theorem 2.3. *Let* $f \in \mathcal{A}$*, and let* A, B *be a real numbers such that* $-1 \le B < A \le -1$*. If*

$$f'(z) + zf''(z) \prec \frac{1 + Az(2 + Bz)}{(1 + Bz)^2} \tag{2.6}$$

then

$$f'(z) \prec \frac{1 + Az}{1 + Bz}. \tag{2.7}$$

The right hand side of (2.7) *is the best dominant of* (2.6).

Corollary 2.3. *Let* $f \in \mathcal{A}$.

(i) *If* $-1 \le B < A \le 1$*, then*

$$|f'(z) + zf''(z) - 1| < (A - B)\frac{2 + |B|}{(1 + |B|)^2}, \quad z \in \mathbb{D},$$

implies

$$|f'(z) - 1| < \frac{A - B}{1 - |B|}, \quad z \in \mathbb{D}.$$

(ii) *If* $B = 0$*, and* $0 < A \le 1$*, then*

$$|f'(z) + zf''(z) - 1| < 2A, \quad z \in \mathbb{D},$$

implies

$$|f'(z) - 1| < A, \quad z \in \mathbb{D}.$$

(iii) *If* $B=-1$ *and* $A=1-2\alpha, \alpha\in[0,1)$, $(-1<A\le 1)$ *then*

$$|f'(z)+zf''(z)-1|<\frac{3}{2}(1-\alpha),\quad z\in\mathbb{D},$$

implies

$$\operatorname{Re} f'(z)>\alpha,\quad z\in\mathbb{D}.$$

Specifying values for A, B and α in Corollary 2.1 and Corollary 2.3, in a similar way as in Example 1, we can obtain other examples.

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