ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.12, 10717–10728
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.12.56

ON THE EXTREMAL FUNCTIONS ASSOCIATED TO *m*-POSITIVE CLOSED CURRENT

JAWHAR HBIL

ABSTRACT. The aim of this paper is to give a partial answer to a question posed by Elkhadhra and Dhouib [3] about the existence of the extremal function for capacity relative to m-positive closed currents. We give a sufficient condition on a given m-positive closed current T defined on an open subset Ω of $\mathbb{C}^n$ to ensure the existence of the associated $(m, T)-$Extremal function.

1. INTRODUCTION

In this paper we will denote by Ω a domain of $\mathbb{C}^n$, $PSH(\Omega)$ (resp. $SH_m(\Omega)$) the set of plurisubharmonic (resp. m-subharmonic) functions on Ω and T an m-positive closed current of bidimension $(n-p, n-p)$ defined on Ω. The notion of capacity is a very useful tool in the field of complex analysis and multi potential theory since it is in connection with various problem of analytic theory. One of the most useful capacity is the Bedford Taylor's one introduced using the Monge-Ampere operator $(dd^c.)^n$ and defined as follows:

$$C_{BT}(K) := \sup \left\{ \int_K (dd^c v)^n, v \in PSH(\Omega), -1 \le v \le 0 \right\},$$

where K is a compact subset of Ω.

The extremal function, if it exists, is the function v where the capacity attains its supremum. The extremal function is used to give the connection between

2020 *Mathematics Subject Classification.* 32U40, 32U05, 32U20.
Key words and phrases. m-positive closed current, m-subharmonic function, Capacity, Hessian operator.

the capacity and the pluripolar sets. In 1982 Bedford and Taylor [1] proved the existence of the extremal function, more precisely they proved that $C_{BT}(K) = \int_K (dd^c u_K^\star)^n$ where $u_K^\star$ is the regularized extremal function associated to K (see [1]). In this paper we study the same problem but with respect to the following capacity

$$\begin{aligned} &Cap_{m,T}(K,\Omega) = Cap_{m,T}(K) \\ :=& \sup\{\int_K (dd^c v)^{m-p} \wedge \beta^{n-m} \wedge T, v \in SH_m(\Omega), -1 \leq v \leq 0\}, \end{aligned}$$

introduced by Dhouib and ElKhadhra in [4]. We will give a sufficient condition on the current T so that the capacity $Cap_{m,T}$ attains its supremum on a function u. Such function will be called the $(m,T)-$Extremal function. This work generalizes Bedford and Taylor's one [1], it suffices to take $m = n$ and $T = 1$ to recover it and the work of ElKhadhra [5] when $m = n$. Note that the proof of Bedford and Taylor used essentially the crucial fact that the equation $(dd^c.)^n = 0$ on $\Omega \setminus K$ has a solution which is $u_K^\star$. This argument can not be extended to our case (the case of m-positive current) since the problem of existence of a local solution to the equation $T \wedge \beta^{n-m} \wedge (dd^c.)^{m-p} = 0$ still open. So a condition on T is needed to solve the problem. This is our main result which is given by the following theorem.

Theorem 1.1. *Let K be a compact subset of a bounded domain Ω and T be an m-positive closed current of bidimension $(n-p, n-p)$ on Ω. Let $(u_j)_j \subset SH(\Omega, [-1,0])$ such that $\lim_{j\to+\infty} \int_K (dd^c u_j)^{m-p} \wedge \beta^{n-m} \wedge T = Cap_{m,T}(K,\Omega)$. Assume that:*

(1) *Every m-pluripolar set is $(m,T)-$pluripolar.*
(2) *For every $A \subset\subset \Omega$, one has that*

$$\lim_{j\to+\infty} ||(dd^c u_{j+1})^{m-p} \wedge \beta^{n-m} \wedge T - (dd^c u_j)^{m-p} \wedge \beta^{n-m} \wedge T||_A = 0.$$

Then there exists $u_{m,K,T} \in SH(\Omega, [-1,0])$ (that depends on m, K and T) such that

$$Cap_{m,T}(K,\Omega) = \int_K (dd^c u_{m,K,T})^{m-p} \wedge \beta^{n-m} \wedge T.$$

We end this paper by giving some consequences of the above theorem as well as some open problems.

2. PRELIMINARIES

The notion of m-positivity of forms was introduced by Blocki in [2] and then generalized by Lu [7] and Dhouib and Elkhadhra [4] to the case of currents. We cite below those definitions and theirs properties which will be used throughout this paper.

Definition 2.1. *Let Ω be an open subset of $\mathbb{C}^n$, $\beta := dd^c|z|^2$ the standard Khaler form of $\mathbb{C}^n$ and m an integer such that $1 \leq p \leq m \leq n$.*

(1) *A real form α of bidegree $(1,1)$ in a domain Ω of $\mathbb{C}^n$ is said to be m-positive if at every point of Ω one has*

$$\alpha^j \wedge \beta^{n-j} \geq 0, \quad \forall j = 1, \dots, m.$$

(2) *A (p,p)–form φ is said to be m-positive on Ω if at every point of Ω one has:*

$$\varphi \wedge \beta^{m-n} \wedge \alpha_1 \wedge \cdots \wedge \alpha_{m-p} \geq 0$$

for every m-positive $(1,1)$-forms $\alpha_1, \dots, \alpha_{m-p}$.

(3) *A current T of bidimension $(n-p, n-p)$ ($p \leq m \leq n$) is called m-positive if*

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{m-p} \wedge \beta^{n-m} \wedge T$$

for every $\alpha_1, \cdots, \alpha_{m-p}$ m-positive (1,1)-$forms$.

Remark 2.1. *If we take $m = n$ in the above definitions we get the standard well-known definitions of positivity for forms and currents.*

Recall also the notion of m-subharmonicity introduced by Blocki [2]

Definition 2.2. *A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called m-subharmonic if it is subharmonic and*

$$dd^c u \wedge \beta^{n-m} \wedge \alpha_1 \wedge \cdots \wedge \alpha_{m-1} \geq 0$$

for all m-positive forms $\alpha_1, \dots, \alpha_{m-1}$. We denote by $SH_m(\Omega, I)$ the set of all m-subharmonic functions defined on Ω with value in $I \subset \mathbb{R} \cup \{-\infty\}$ and when $I = \mathbb{R} \cup \{-\infty\}$ it will be denoted $SH_m(\Omega)$.

An m-subharmonic function is in particular subharmonic. So The set $SH_m(\Omega, I)$ has a basics properties as $SH(\Omega, I)$. We summarize below those properties. For more details one can refer to [2], [9] and [4].

Proposition 2.1.

(1) *If* $u, v \in SH_m(\Omega)$ *then* $au + bv \in SH_m(\Omega)$ *for every* $a, b > 0$.
(2) *If* $u \in SH_m(\Omega)$ *then the current* $dd^c u$ *is* m*-positive.*
(3) $PSH(\Omega) = SH_n(\Omega) \subseteq \cdots \subseteq SH_m(\Omega) \subseteq \cdots \subseteq SH_1(\Omega) = SH(\Omega)$.
(4) *If* u *is* m*-subharmonic on* Ω *then the standard regularization* $u * \chi_\varepsilon$ *is also* m*-subharmonic on* $\Omega_\varepsilon := \{x \in \Omega \ / \ d(x, \partial\Omega) > \varepsilon\}$.
(5) *If* $(u_i)_j$ *is a decreasing sequence of* m*-subharmonic functions then* $u := \lim u_j$ *is either* m*-subharmonic or identically equal to* $-\infty$.
(6) $(u_j)_j \in SH_m(\Omega)$ *a locally uniformly sequence bounded from above, then* $(sup u_j)^\star \in SH_m(\Omega)$ *where* $f^\star$ *is the upper semicontinuous regularization of* f.

Definition 2.3.

(1) *Let* $\mathcal{U}$ *be a family of* m*-subharmonic functions locally bounded from above. If* $u(z) = \sup\{v(z), v \in \mathcal{U}\}$ *then the set*
$$\mathcal{N} := \{u(z) < u^\star(z)\}$$
as well as its subsets will be called m*-negilgible set.*
(2) *A subset* $E \subset \Omega$ *is said to be* m*-pluripolar if for every* $z \in E$ *there exist a neighborhood* V *of* z *and* $v \in SH_m(V)$ *such that* $E \cap V \subset \{v = -\infty\}$.
(3) *Let* μ *be a measure defined on* Ω*, the total variation of the measure* μ *will be denoted as* $||\mu||$.

In [8], Lu gave a relationship between m-pluripolar sets and m-negligible sets. He proved the following result:

Lemma 2.1. *(see [8]) Every* m*-negligeable set is* m*-pluripolar.*

3. THE $(m, T)-$EXTREMAL FUNCTION

Throughout this section Ω will be a bounded domain of $\mathbb{C}^n$ and T an m-positive closed current of bidimension $(n-p, n-p)$ defined on Ω for $1 \le p \le m \le n$. We will study the following problem given by Elkhadhra and Dhouib [4]:

$(\mathcal{P})$: Let K be a compact subset of Ω. Is there a function $u \in SH_m(\Omega, [-1, 0])$ such that $Cap_{m,T}(K, \Omega) = \int_K (dd^c u)^{m-p} \wedge \beta^{n-m} \wedge T$.

To give an answer to the cited problem we need first to recall the notion of m-capacity of a subset E in Ω with a respect to T which is defined as follows:

Definition 3.1. *For every compact K of Ω the m-capacity of K relatively to an m-positive current T denoted by $Cap_{m,T}(K)$ is defined by*

$$\begin{aligned} &Cap_{m,T}(K,\Omega) = Cap_{m,T}(K) \\ := &\sup\{\int_K (dd^c v)^{m-p} \wedge \beta^{n-m} \wedge T, v \in SH_m(\Omega), -1 \leq v \leq 0\}, \end{aligned}$$

and for every $E \subset \Omega$, $Cap_{m,T}(E,\Omega) = \sup\{Cap_{m,T}(K), K \text{ compact of } \Omega\}$. If $Cap_{m,T}(A,\Omega) = 0$ then A is called an $(m,T)-$pluripolar set.

Remark 3.1.

(1) *If $T = 1$ and $m = n$ then the above capacity is exactly the Bedford and Taylors's capacity C_{BT} and in this case it was proved that $C_{BT}(A) = 0$ if and only if A is a pluripolar set.*
(2) *If $T = 1$ then the above capacity is the capacity introduced by Lu in [7].*

The Capacity defined above have similar properties as the Bedford-Taylor's one introduced in [1]. We cite those properties in the following proposition and such properties will be used frequently throughout this paper.

Proposition 3.1.

(1) *If $A \subset B$ then $Cap_{m,T}(A) \leq Cap_{m,T}(B)$.*
(2) *If $E = \bigcup_j E_j$ then $Cap_{m,T}(E) \leq \sum_j Cap_{m,T}(E_j)$.*
(3) *If the sequence of subsets $(E_j)_j$ is increasing to E then $Cap_{m,T}(E) = \lim_{j\to+\infty} Cap_{m,T}(E_j)$.*

The notion of convergence in capacity $Cap_{m,T}$ was introduced by [4] as follows:

Definition 3.2. *Let T be an m-positive closed current of bidimension $(n-p, n-p)$, $p \leq m \leq n$ on an open subset Ω of $\mathbb{C}^n$ and $E \subset \Omega$. A sequence of functions $(u_j)_j$ defined on Ω is said to be convergent with respect to $cap_{m,T}$ to u on E if for all $t > 0$, one has:*

$$\lim_{j\to+\infty} Cap_{m,T}(E \cap \{|u - u_j| > t\}) = 0.$$

Using Bedford and Taylor's technics and lemma 2.1, one can prove the following result.

Proposition 3.2. *Let $(u_j)_j$ be a sequence of m-subharmonic functions such that $\limsup u_j \neq -\infty$. Then there exist an m-subharmonic function u such that the following set*

$$\{\limsup u_j \neq u\}$$

is m-pluripolar.

The following lemma is a direct generalization of Corollary 1.3.15 of Lu [8] and Xing [10]. It was recently proved by Alaini and Elkhadhra [6].

Lemma 3.1. *:*
Let Ω a bounded open subset of $\mathbb{C}^n$ and T an m-positive closed current of bidimension $(n-p, n-p)$ ($p \leq m \leq n$) defined on Ω. Let $u,\ v \in SH_m(\Omega) \cap L^\infty(\Omega)$ such that

$$\lim \sup_{\xi \to \partial\Omega,\ \xi \in SuppT} \mid u(\xi) - v(\xi) \mid = 0.$$

Then for all $\delta > 0$ and $0 < k < 1$, one has:

$$Cap_{m,T}(\{\mid u - v \mid \geq \delta\}) \leq \frac{[(m-p)!]^2}{(1-k)^{m-p}\delta^{m-p}} \mid\mid (dd^c u)^{m-p} \wedge \beta^{n-m} \wedge T - (dd^c v)^{m-p} \wedge \beta^{n-m} \wedge T \mid\mid_{\{|u-v|>k\delta\}} .$$

Lemma 3.2. *[4] Let Ω be a bounded subset of $\mathbb{C}^n$, $u \in \mathcal{S}H_m(\Omega) \cap L^\infty_{loc}(\Omega)$ and T an m-positive closed current on Ω of bidimension $(n-p, n-p)$, $p \leq m \leq n$. Then for all $\varepsilon > 0$, there exist an open set $\mathcal{O}_\varepsilon$ of Ω such that $Cap_{m,T}(\mathcal{O}_\varepsilon, \Omega) < \varepsilon$ and u is continuous $\Omega \backslash \mathcal{O}_\varepsilon$.*

Before giving the main theorem we will establish the following lemma which gives the connection between convergence in Capacity and weak convergence in terms of currents

Theorem 3.1. *Let Ω be an open subset of $\mathbb{C}^n$, T an m-positive colsed current on Ω of bidimension $(n-p, n-p)$ and $(u_j)_j$ a sequence of locally uniformly bounded m-subharmonic functions and $u \in \mathcal{S}H_m(\Omega) \cap L^\infty_{loc}(\Omega)$. If u_j converges to u in Capacity $cap_{m,T}$ on every $E \subset\subset \Omega$, then the sequence of currents $(dd^c u_j)^{m-p} \wedge \beta^{n-m} \wedge T$ converges weakly to $(dd^c u)^{m-p} \wedge \beta^{n-m} \wedge T$.*

Proof. 1) We proceed by induction on $m-p$. The case $m-p=1$ will be proved if we show that $u_j\beta^{n-m} \wedge T$ converges to $u\beta^{n-m} \wedge T$.

Let φ be a smooth form with compact support in Ω ($\varphi \in \mathcal{D}_{m-p,m-p}(\Omega)$), $supp\varphi \subset \Omega_1 \subset\subset \Omega$, then:

$$\begin{aligned}
&\left|\int_\Omega (u_jT - uT)\varphi \wedge \beta^{n-m} \wedge T\right| \\
\leq\ & C\int_{\Omega_1} |u_j - u|\beta^{n-p} \wedge T \\
=\ & C\int_{\{|u_j-u|\leq\delta\}\cap\Omega_1} |u_j - u|\beta^{n-p} \wedge T + C\int_{\{|u_j-u|>\delta\}\cap\Omega_1} |u_j - u|\beta^{n-p} \wedge T \\
\leq\ & C\delta||\beta^{n-m} \wedge T||_{\Omega_1} + C||u_j - u||_{L^\infty(\Omega_1)}\int_{\{|u_j-u|\geq\delta\}\cap\Omega_1} \beta^{n-p} \wedge T \\
\leq\ & C\delta||\beta^{n-m} \wedge T||_{\Omega_1} + MCap_{m,T}(\{z \in \Omega_1; |u_j(z) - u(z)| > \delta\}).
\end{aligned}$$

This proves the case $m - p = 1$ since δ is arbitrary, u_j converges to u in capacity $cap_{m,T}$ and M is independent on j.

Assume now by induction that $(dd^cu_j)^s \wedge \beta^{n-m} \wedge T$ converges weakly to $(dd^cu)^s \wedge \beta^{n-m} \wedge T$ for $s < m - p$. It suffices to prove that $u_j(dd^cu_j)^s \wedge \beta^{n-m} \wedge T$ converges weakly to $u(dd^cu)^s \wedge \beta^{n-m} \wedge T$. By lemma 3.2, for all $\varepsilon > 0$ there exists an open subset O_ε such that $cap_{m,T}(O_\varepsilon) < \varepsilon$ and $u = \varphi + \psi$ where φ is continuous on Ω and $\psi = 0$ on $\Omega \setminus O_\varepsilon$. Note that

$$\begin{aligned}
&u_j(dd^cu_j)^s \wedge \beta^{n-m} \wedge T - u(dd^cu)^s \wedge \beta^{n-m} \wedge T \\
=&(u_j - u)(dd^cu_j)^s \wedge \beta^{n-m} \wedge T \\
&+ \psi((dd^cu_j)^s \wedge \beta^{n-m} \wedge T - (dd^cu)^s \wedge \beta^{n-m} \wedge T) \\
&+ \varphi((dd^cu_j)^s \wedge \beta^{n-m} \wedge T - (dd^cu)^s \wedge \beta^{n-m} \wedge T) \\
=&(1) + (2) + (3).
\end{aligned}$$

Since φ is continuous on Ω and using induction's hypothesis, we get that (3) tends weakly to 0 when $j \to \infty$.

For (1), let $\varphi \in \mathcal{D}_{m-p-s,m-p-s}(\Omega)$ such that $supp\ \varphi \subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$. Then

$$\begin{aligned}
&\left|\int_\Omega (u_j - u)(dd^cu_j)^s \wedge \beta^{n-m} \wedge T \wedge \varphi\right| \\
\leq& C\int_{\Omega_1} |u_j - u|(dd^cu_j)^s \wedge \beta^{n-m} \wedge T \wedge (dd^c|z|^2)^{m-p-s}
\end{aligned}$$

$$
\begin{aligned}
\leq & C \int_{\Omega_1} |u_j - u| (dd^c (u_j + |z|^2))^{m-p} \wedge \beta^{n-m} \wedge T \\
\leq & C \int_{\{|u_j - u| > \delta\} \cap \Omega_1} |u_j - u| dd^c (u_j + |z|^2)^{m-p} \wedge \beta^{n-m} \wedge T \\
& + C \int_{\{|u_j - u| \leq \delta\} \cap \Omega_1} |u_j - u| dd^c (u_j + |z|^2)^{m-p} \wedge \beta^{n-m} \wedge T \\
\leq & C_1 Cap_{m,T} (z \in \Omega_1; |u_j(z) - u(z)| > \delta) \\
& + \delta M ||\beta^{n-m} \wedge T||_{\Omega_2}.
\end{aligned}
$$

Since the sequence $(u_j)_j$ is uniformly bounded, M and C_1 does not depend on j and $u_j \to u$ in capacity $cap_{m,T}$, we get that (1) tends to 0.

The same reason for (2), gives

$$
\begin{aligned}
& \left| \int_{\Omega_1 \cap O_\varepsilon} \psi (dd^c u_j)^s \wedge \beta^{n-m} \wedge T \wedge \varphi \right| \\
\leq \; & A \int_{\Omega_1 \cap O_\varepsilon} (dd^c (u_j + |z^2|))^{m-p} \wedge \beta^{n-m} \wedge T \\
\leq \; & B_1 cap_{m,T} (O_\varepsilon) \\
\leq \; & \varepsilon B_1
\end{aligned}
$$

Using the same reason as above, one can obtain that:

$$
\left| \int_{\Omega_1 \cap O_\varepsilon} \psi (dd^c u)^s \wedge \beta^{n-m} \wedge T \wedge \varphi \right| \leq \varepsilon B_2
$$

□

In the following theorem, we give an answer to the problem $(\mathcal{P})$. We give a sufficient condition on T so that the $(m, T)-$Extremal function exists. It should be noted that the technics of Bedford and Taylor [1] cannot be extended to the case of the m-positive currents since the problem of existence of a local solution to the equation $T \wedge \beta^{n-m} \wedge (dd^c.)^{m-p} = 0$ still open. So the proof will be completely different.

Theorem 3.2. *(Main Theorem): Let K be a compact subset of a abounded domain Ω and T be an m-positive closed current of bidimension $(n-p, n-p)$ on Ω. Let $(u_j)_j \subset SH(\Omega, [-1, 0])$ such that $\lim_{j \to +\infty} \int_K (dd^c u_j)^{m-p} \wedge \beta^{n-m} \wedge T = Cap_{m,T}(K, \Omega)$. Assume that:*

(1) *Every m-pluripolar set is $(m,T)-$pluripolar.*
(2) *For every $A\subset\Omega$, one has that*

$$\lim_{j\to+\infty}||(dd^cu_{j+1})^{m-p}\wedge\beta^{n-m}\wedge T-(dd^cu_j)^{m-p}\wedge\beta^{n-m}\wedge T||_A=0.$$

Then there exists $u_{m,K,T}\in SH(\Omega,[-1,0])$ (that depends on m,K and T) such that

$$Cap_{m,T}(K,\Omega)=\int_K(dd^cu_{m,K,T})^{m-p}\wedge\beta^{n-m}\wedge T.$$

Proof. By Proposition 3.2 there exists $v\in SH(\Omega,[-1,0]$ such that the set $\{v\neq\limsup u_j\}$ is m-pluripolar on Ω. Using the function v, we will proof the theorem in two steps.

First step. We will prove that $T\wedge\beta^{n-m}\wedge(dd^cu_j)^{m-p}$ converges weakly to $T\wedge\beta^{n-m}\wedge(dd^cv)^{m-p}$.

Thanks to Theorem 3.1, it suffices to prove the convergence of u_j to v with respect to the capacity $Cap_{m,T}$. Let $v_j=sup_{k\geq j}u_k$ and $v_j^\star$ the associated upper semicontinuous regularization of v_j. The set $L:=\bigcup_j\{v_j<v_j^\star\}$, is m-pluripolar and then is $(m,T)-$pluripolar using the first hypothesis. Moreover, one has that for all j, $v_j=v_j^\star$ outside L and $v_j^\star$ decreases to v. Using the properties of capacities and a simple computation one can obtain that for every $\varepsilon>0$ and for any compact subset M of Ω

$$\begin{aligned}Cap_{m,T}\left(M\cap\{|u_j-v|\geq\varepsilon\}\right)\leq\ & Cap_{m,T}\left(\{|u_j-v_j|\geq\frac{\varepsilon}{2}\}\right)\\&+Cap_{m,T}\left(M\cap\{|v_j-v|\geq\frac{\varepsilon}{2}\}\right).\end{aligned}\tag{3.1}$$

We will estimate the right hand side of the previous inequality. For the first term we can assume, since the problem is local and without loss of generality, that all functions u_j coincide near the boundary of Ω and for $0<k<1$, we consider $E\subset\Omega$ such that $\bigcup_j\{|u_{j+1}-u_j|\geq k\varepsilon\}\subset E$. By applying lemma 3.1 for every j, we obtain:

$$
\begin{aligned}
Cap_{m,T}\left(\{|u_{j+1}-u_j|\geq\varepsilon\}\right) \leq & \frac{[(m-p)!]^2}{(1-k)^{m-p}\varepsilon^{m-p}} \mid\mid (dd^c u_{j+1})^{m-p}\wedge\beta^{n-m}\wedge T \\
& - (dd^c u_j)^{m-p}\wedge\beta^{n-m}\wedge T \mid\mid_{\{|u_{j+1}-u_j|>k\varepsilon\}} \\
\leq & \frac{[(m-p)!]^2}{(1-k)^{m-p}\varepsilon^{m-p}} \mid\mid (dd^c u_{j+1})^{m-p}\wedge\beta^{n-m}\wedge T \\
& - (dd^c u_j)^{m-p}\wedge\beta^{n-m}\wedge T \mid\mid_{E} \\
\leq & \frac{2}{2^{(m-p+1)j}}
\end{aligned}
\tag{3.2}
$$

Note that the last inequality is deduced by hypothesis 2. Now it is easy to check that

$$\{|u_j-v_j|\geq\frac{\varepsilon}{2}\}\subset\bigcup_{k=1}^{\infty}\{|u_{j+k+1}-u_{j+k}|\geq\frac{\varepsilon}{2^{k+j+2}}\}.$$

Using equality (3.2) and the subadditivity of Capacity we obtain

$$
\begin{aligned}
Cap_{m,T}\left(\{|u_j-v_j\geq\frac{\varepsilon}{2}|\}\right) &\leq \sum_{k=1}^{+\infty} Cap_{m,T}\{|u_{j+k+1}-u_{j+k}|\geq\frac{\varepsilon}{2^{k+j+2}}\} \\
&\leq 2\sum_{k=1}^{+\infty}\frac{2^{(p(k+j+2)}}{2^{(p+1)(k+j}}=\frac{p+1}{2^j}.
\end{aligned}
$$

We conclude that the desired term tends to zero when j goes to infinity.

For the second term in the right hand side of equation (3.1) we observe that using the Lemma 3.2 that: $\forall\varepsilon>0,\exists V_j,V$ such that $Cap_{m,T}(V_j)\leq\frac{\varepsilon}{2^{j+1}}$, $Cap_{m,T}(V)\leq\frac{\varepsilon}{2}$, $v_j^\star$ is continuous on $M\setminus V_j$ and v is continuous on $M\setminus V$. If we take $G:=V_j\cup V$ then it is clear that $v_j^\star$ and v are continous on G and $Cap_{m,T}(G)\leq\varepsilon$. By Dini's theorem, the sequence $v_j^\star$ is uniformly decreasing to v outside G. Moreover since the set A is $(m,T)-$pluripolar then the second term in the right hand side of equation (3.1) is less than ε when $j\to+\infty$.

Second step. The construction of the desired function $u_{m,K,T}$.

Let $(\varphi_s)_s$ a sequence of smooth compactly supported functions in Ω such that $0\leq\varphi_s\leq 1$ for every s and φ_s decreases to $\mathbf{1}_K$. Using the first step, we obtain

$$\begin{aligned} Cap_{m,T}(K) &= \lim_{j\to+\infty}\int_K (dd^c u_j)^{m-p}\wedge\beta^{n-m}\wedge T \\ &\leq \lim_{j\to+\infty}\int_\Omega \varphi_s(dd^c u_j)^{m-p}\wedge\beta^{n-m}\wedge T \\ &= \int_\Omega \varphi_s(dd^c v)^{m-p}\wedge\beta^{n-m}\wedge T. \end{aligned}$$

When s goes to ∞, we get $Cap_{m,T}(K)\leq\int_K T\wedge\beta^{n-m}\wedge(dd^c v)^{m-p}$. As the converse inequality is obvious we take $w=u_{m,K,T}$ to finish the proof. □

As a consequence of the main result we get the following corollary

Corollary 3.1. *Let T and $u_{m,K,T}$ be as in Theorem 3.2. If $u_{m,K,T}>\alpha>0$ for some $\alpha>0$ on K then K is $(m,T)-$pluripolar.*

Proof. Assume by contradiction that $Cap_{m,T}(K)>0$. If we take $v:=\frac{u_{m,K,T}}{1-\alpha}+1$ then it is easy to check that $v\in SH_m(\Omega,[-1,0])$. Let $\varepsilon>0$ and $v_\varepsilon:=\max(v,-\varepsilon)+\varepsilon$, then v_ε is m-subharmonic $v_\varepsilon\equiv v$ on K and $0\leq\frac{v_\varepsilon}{1+\varepsilon}\leq 1$ on Ω. Using Theorem 3.2 one has

$$\begin{aligned} Cap_{m,T}(K) &\geq \frac{1}{(1+\varepsilon)^p}\int_K (dd^c v_\varepsilon)^{m-p}\wedge\beta^{n-m}\wedge T \\ &= \frac{1}{(1+\varepsilon)^p}\int_K (dd^c v_\varepsilon)^{m-p}\wedge\beta^{n-m}\wedge T \\ &= \frac{1}{(1+\varepsilon)^p}(\frac{1}{1-\alpha})^p\int_K (dd^c u_{m,K,T}^{m-p}\wedge\beta^{n-m}\wedge T \\ &= \frac{1}{(1+\varepsilon)^p}(\frac{1}{1-\alpha})^p Cap_{m,T}(K). \end{aligned}$$

When ε goes to zero we get contradiction. □

Open Problems:

P1: " Can we characterize the $(m,T)-$Pluripolar sets for $T\neq 1$?" This problem still open even in the case $m=n$. However, in the particular case $T=1$ and $m=n$, pluripolar sets are exactly negligible sets.

P2: In the classic case, the extremal function is an essential tool to solve the Dirichlet problem. Using the main result in this paper, can we solve the following Hessian equation

$$(dd^c.)^{m-p}\wedge\beta^{n-m}\wedge T=\mu$$

for a suitable measure μ.

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Mathematics Departement, College of Science, Jouf University
P.O. Box:2014, Sakaka, Saudi Arabia
Irescomath Laboratory, Gabes University, 6072 Zrig Gabes, Tunisia
Email address: jmhbil@ju.edu.sa