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A STUDY ON BIPOLAR FUZZY BCI-IMPLICATIVE IDEALS

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ABSTRACT. This paper studies bipolar fuzzy BCI-implicative ideals considering bipolar fuzzy (translation, multiplication, extension, magnified translation) and relations between them obtaining several theorems.

1. INTRODUCTION

The notions of translation, multiplication and extension applied in different aspects on different algebraic structures. In [7], fuzzy translation, fuzzy extension and fuzzy multiplication applied on fuzzy subalgebras in BCK/BCI-algebras. The same notions were applied to fuzzy H-ideals in BCK/BCI-algebras [12]. In [11], intuitionistic fuzzy translation is applied to intuitionistic fuzzy subalgebras and ideals in B-algebras. In [4], fuzzy translation and fuzzy multiplication studied on BF-algebras. Other properties have been studied on anti fuzzy implicative ideals of BCK-algebras [2]. Zhang [13] in 1998 was the first to introduce the notion of bipolar fuzzy sets as an extension of fuzzy sets. Lots of studies have been done later on the notion, see for example [3, 5, 10].

Liu et al. in [9] introduced the notion BCI-implicative ideals and investigated the connection with other ideals in BCK/BCI-algebras. Then the authors introduced fuzzy BCI-implicative ideals and fuzzy BCI-positive implicative ideals [8]. Recently, the notion of bipolar fuzzy BCI-implicative ideals of BCI-algebras were introduced, characterizations were given and some properties were studied [1].

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This paper studies more properties of the bipolar fuzzy BCI-implicative ideal of BCI-algebras defined in [1]. For a bipolar fuzzy set of BCI-algebras, say μ, we give conditions for when a bipolar fuzzy translation, multiplication, extension, magnified translation) of μ become a bipolar fuzzy BCI-implicative ideal and relations between them are investigated. The negative and positive cut of a bipolar fuzzy set is defined and some properties are studied.

2. Preliminaries

Definition 2.1. *An algebra $(W;*,0)$ of type $(2,0)$ is called a $BCI-algebra$ if for all $w,s,q \in W$ it satisfies:*

(1) $((w*s)*(w*q))*(q*s)=0$,
(2) $(w*(w*s))*s=0$,
(3) $w*w=0$,
(4) $w*s=0$ *and* $s*w=0 \Rightarrow w=s$.

We denote a bipolar fuzzy set in a BCI-algebra W by the triple $\mu=(W;\mu_n,\mu_p)$, where μ_n,μ_p are the maps from W to $[-1,0]$ and from W to $[0,1]$, respectively.

For any $\mu=(W;\mu_n,\mu_p)$ in W and for any $w\in W$, let $\mathrm{B}=-1-inf\{\mu_n(w)\}$ and $\mathrm{T}=1-sup\{\mu_p(w)\}$.

Let $\{r_j \mid j\in J\}$, be a set of real numbers where J is finite. Define $\vee$ and $\wedge$ as: $\vee\{r_j \mid j\in J\}:=max\{r_j \mid j\in J\}$ and $\wedge\{r_j \mid j\in J\}:=min\{r_j \mid j\in J\}$.

A nonempty subset A of W is called an $ideal$ of W if it satisfies:

(1) $0\in A$,
(2) $\forall w,s\in W, w*s\in A, s\in A\Rightarrow w\in A$.

A nonempty subset A of W is called a $BCI-implicative\ ideal$ of W if $\forall w,s,q\in W$ it satisfies:

(1) $0\in A$,
(2) $(((w*s)*s)*(0*s))*q\in A, q\in A\Rightarrow w*((s*(s*w))*(0*(0*(w*s))))\in A$.

Definition 2.2. *[6] A bipolar fuzzy set $\mu=(W;\mu_n,\mu_p)$ in W is called a $bipolar\ fuzzy\ ideal$ of W if it satisfies the following assertions:*

(1) $(\forall w\in W)\begin{pmatrix}\mu_n(0)\le\mu_n(w),\\ \mu_p(0)\ge\mu_p(w)\end{pmatrix}$,

(2) $(\forall w,s\in W)\begin{pmatrix}\mu_n(w)\le\mu_n(w*s)\vee\mu_n(s),\\ \mu_p(w)\ge\mu_p(w*s)\wedge\mu_p(s)\end{pmatrix}$.

Definition 2.3. *[1] A bipolar fuzzy set $\mu = (W; \mu_n, \mu_p)$ in a BCI-algebra W is said to be a $bipolar\ fuzzy\ BCI - implicative\ ideal$ of W if $\forall w, s, q \in W$ it satisfies:*

(1) $\mu_n(0) \leq \mu_n(w),\ \mu_p(0) \geq \mu_p(w)$,

(2) $\mu_n(w*((s*(s*w))*(0*(0*(w*s))))) \leq \mu_n((((w*s)*s)*(0*s))*q) \vee \mu_n(q)$,
$\mu_p(w*((s*(s*w))*(0*(0*(w*s))))) \geq \mu_p((((w*s)*s)*(0*s))*q) \wedge \mu_p(q)$.

Definition 2.4. *[5] Let $\mu = (W; \mu_n, \mu_p)$ be a bipolar fuzzy set in W and let $\iota \in [\mathrm{B}, 0]$ and $\kappa \in [0, \mathrm{T}]$. Then a bipolar fuzzy (ι, κ)-translation of $\mu = (W; \mu_n, \mu_p)$, denoted by $\mu^{(T,\iota,\kappa)} = (W; \mu_n^{(T,\iota)}, \mu_p^{(T,\kappa)})$ is a bipolar fuzzy set where, $\mu_n^{(T,\iota)}$ and $\mu_p^{(T,\kappa)}$ are maps from W to $[-1, 0]$ and from W to $[0, 1]$ given by $w \mapsto \mu_n(w) + \iota$ and $w \mapsto \mu_p(w) + \kappa$, respectively.*

Definition 2.5. *[5] Let $\mu_1 = (W; \mu_{n_1}, \mu_{p_1})$ and $\mu_2 = (W; \mu_{n_2}, \mu_{p_2})$ be bipolar fuzzy sets in W. Then it said that $\mu_2 = (W; \mu_{n_2}, \mu_{p_2})$ is a bipolar fuzzy extension of $\mu_1 = (W; \mu_{n_1}, \mu_{p_1})$ if $\mu_{n_1}(w) \geq \mu_{n_2}(w)$ and $\mu_{p_1}(w) \leq \mu_{p_2}(w)$, for all $w \in W$.*

Definition 2.6. *[3] Let $\mu = (W; \mu_n, \mu_p)$ be a bipolar fuzzy set in W and let $0 < \beta, \gamma \leq 1$. We define the bipolar fuzzy (β, γ)-multiplication of $\mu = (W; \mu_n, \mu_p)$ as the bipolar fuzzy set $\mu^{(M,\beta,\gamma)} = (W; \mu_n^{(M,\beta)}, \mu_p^{(M,\gamma)})$ where $\mu_n^{(M,\beta)}$ and $\mu_p^{(M,\gamma)}$ are maps from W to $[-1, 0]$ and from W to $[0, -1]$ given by $\mu_n.\beta$ and $\mu_p.\gamma$, respectively.*

3. RESULTS

Throughout the next parts we will use W to denote a BCI-algebra and BF-BCI-implicative ideals to denote bipolar fuzzy BCI-implicative ideals.

Theorem 3.1. *Let $\mu = (W; \mu_n, \mu_p)$ be a bipolar fuzzy set of W. Then for $(\iota, \kappa) \in [\mathrm{B}, 0] \times [0, \mathrm{T}]$, the bipolar fuzzy (ι, κ)-translation of $\mu = (W; \mu_n, \mu_p)$, $\mu^{(T,\iota,\kappa)} = (W; \mu_n^{(T,\iota)}, \mu_p^{(T,\kappa)})$ is a BF-BCI-implicative ideal of W if and only if $\mu = (W; \mu_n, \mu_p)$ is a BF-BCI-implicative ideal of W.*

Proof. Let $\mu^{(T,\iota,\kappa)} = (W; \mu_n^{(T,\iota)}, \mu_p^{(T,\kappa)})$ be a BF-BCI-implicative ideal of W. Then, for all $w, s, q \in W$,

$$\begin{aligned} &\mu_n^{(T,\iota)}(w*((s*(s*w))*(0*(0*(w*s))))) \\ &\leq \mu_n^{(T,\iota)}((((w*s)*s)*(0*s))*q) \vee \mu_n^{(T,\iota)}(q), \end{aligned}$$

$$\begin{aligned} &\mu_p^{(T,\kappa)}(w*((s*(s*w))*(0*(0*(w*s))))) \\ &\geq \mu_p^{(T,\kappa)}((((w*s)*s)*(0*s))*q) \wedge \mu_p^{(T,\kappa)}(q). \end{aligned}$$

Then,

$$\mu_n(w*((s*(s*w))*(0*(0*(w*s)))))+\iota$$
$$\leq[\mu_n((((w*s)*s)*(0*s))*q)\vee\mu_n(q)]+\iota,$$
$$\mu_p(w*((s*(s*w))*(0*(0*(w*s)))))+\kappa$$
$$\geq[\mu_p((((w*s)*s)*(0*s))*q)\wedge\mu_p(q)]+\kappa.$$

Thus,

$$\mu_n(w*((s*(s*w))*(0*(0*(w*s)))))$$
$$\leq\mu_n((((w*s)*s)*(0*s))*q)\vee\mu_n(q),$$
$$\mu_p(w*((s*(s*w))*(0*(0*(w*s)))))$$
$$\geq\mu_p((((w*s)*s)*(0*s))*q)\wedge\mu_p(q).$$

Also we have,

$$\mu_n(0)+\iota=\mu_n^{(T,\iota)}(0)\leq\mu_n^{(T,\iota)}(w)=\mu_n(w)+\iota$$

and

$$\mu_p(0)+\kappa=\mu_p^{(T,\kappa)}(0)\geq\mu_p^{(T,\kappa)}(w)=\mu_p(w)+\kappa.$$

Hence, $\mu_n(0)\leq\mu_n(w)$ and $\mu_p(0)\geq\mu_p(w)$. Therefore, $\mu=(W;\mu_n,\mu_p)$ is a BF-BCI-implicative ideal of W.

Conversely, let $\mu=(W;\mu_n,\mu_p)$ be a BF-BCI-implicative ideal of W. Then for all $w,s,q\in W$ we have,

$$\mu_n(w*((s*(s*w))*(0*(0*(w*s)))))+\iota$$
$$\leq\mu_n[((((w*s)*s)*(0*s))*q)\vee\mu_n(q)]+\iota,$$
$$\mu_p(w*((s*(s*w))*(0*(0*(w*s)))))+\kappa$$
$$\geq\mu_p[((((w*s)*s)*(0*s))*q)\wedge\mu_p(q)]+\kappa.$$

Hence,

$$\mu_n^{(T,\iota)}(w*((s*(s*w))*(0*(0*(w*s)))))$$
$$\leq[\mu_n((((w*s)*s)*(0*s))*q)+\iota]\vee[\mu_n(q)+\iota]$$
$$=\mu_n^{(T,\iota)}((((w*s)*s)*(0*s))*q)\vee\mu_n^{(T,\iota)}(q),$$

$$\begin{aligned}
&\mu_p^{(T,\kappa)}(w*((s*(s*w))*(0*(0*(w*s)))))\\
\geq&[\mu_p((((w*s)*s)*(0*s))*q)+\kappa]\wedge[\mu_p(q)+\kappa\\
=&\mu_p^{(T,\kappa)}((((w*s)*s)*(0*s))*q)\wedge\mu_p^{(T,\kappa)}(q).
\end{aligned}$$

Also, it is obvious that $\mu_n^{(T,\iota)}(0)\leq\mu_n^{(T,\iota)}(w)$ and $\mu_p^{(T,\kappa)}(0)\geq\mu_p^{(T,\kappa)}(w)$. Hence, $\mu^{(T,\iota,\kappa)}=(W;\mu_n^{(T,\iota)},\mu_p^{(T,\kappa)})$ is a BF-BCI-implicative ideal of W. □

Theorem 3.2. *Let $\mu=(W;\mu_n,\mu_p)$ be a bipolar fuzzy set of W. For $\beta,\gamma\in[0,1]$, the bipolar fuzzy (β,γ)-multiplication $\mu^{(M,\beta,\gamma)}=(W;\mu_n^{(M,\beta)},\mu_p^{(M,\gamma)})$ of $\mu=(W;\mu_n,\mu_p)$, is a BF-BCI-implicative ideal in W if and only if $\mu=(W;\mu_n,\mu_p)$ is a BF-BCI-implicative ideal in W.*

Proof. Let (β,γ)-multiplication, $\mu^{(M,\beta,\gamma)}=(W;\mu_n^{(M,\beta)},\mu_p^{(M,\gamma)})$ be a BF-BCI-implicative ideal in W. Then

$$\mu_n^{(M,\beta)}(0)\leq\mu_n^{(M,\beta)}(w),\ \mu_p^{(M,\gamma)}(0)\geq\mu_p^{(M,\gamma)}(w)$$

and

$$\begin{aligned}
&\mu_n^{(M,\beta)}(w*((s*(s*w))*(0*(0*(w*s)))))\\
&\leq\mu_n^{(M,\beta)}((((w*s)*s)*(0*s))*q)\vee\mu_n^{(M,\beta)}(q),
\end{aligned}$$

and

$$\begin{aligned}
&\mu_p^{(M,\gamma)}(w*((s*(s*w))*(0*(0*(w*s)))))\\
&\geq\mu_p^{(M,\gamma)}((((w*s)*s)*(0*s))*q)\wedge\mu_p^{(M,\gamma)}(q),
\end{aligned}$$

and so,

$$\begin{aligned}
&\mu_n(w*((s*(s*w))*(0*(0*(w*s))))).\beta\\
&\leq\mu_n((((w*s)*s)*(0*s))*q).\beta\vee\mu_n(q).\beta,
\end{aligned}$$

and

$$\begin{aligned}
&\mu_p(w*((s*(s*w))*(0*(0*(w*s))))).\gamma\\
&\geq\mu_p[((((w*s)*s)*(0*s))*q).\gamma\wedge\mu_p(q).\gamma.
\end{aligned}$$

Hence,

$$\mu_n(0).\beta\leq\mu_n(w).\beta,\ \mu_p(0).\gamma\geq\mu_p(w).\gamma$$

and

$$\mu_n(w*((s*(s*w))*(0*(0*(w*s))))).\beta$$
$$\leq [\mu_n((((w*s)*s)*(0*s))*q)\vee \mu_n(q)].\beta,$$

and

$$\mu_p(w*((s*(s*w))*(0*(0*(w*s))))).\gamma$$
$$\geq \mu_p[((((w*s)*s)*(0*s))*q)\wedge \mu_p(q)].\gamma.$$

It follows that,

$$\mu_n(0)\leq \mu_n(w),\ \mu_p(0)\geq \mu_p(w)$$

and

$$\mu_n(w*((s*(s*w))*(0*(0*(w*s)))))$$
$$\leq \mu_n((((w*s)*s)*(0*s))*q)\vee \mu_n(q),$$

and

$$\mu_p(w*((s*(s*w))*(0*(0*(w*s)))))$$
$$\geq \mu_p((((w*s)*s)*(0*s))*q)\wedge \mu_p(q).$$

Hence, $\mu=(W;\mu_n,\mu_p)$ is a BF-BCI-implicative ideal in W.

Conversely, Let $\mu=(W;\mu_n,\mu_p)$ be a BF-BCI-implicative ideal in W. Then we have,

$$\mu_n(0)\leq \mu_n(w),\ \mu_p(0)\geq \mu_p(w)$$

and

$$\mu_n(w*((s*(s*w))*(0*(0*(w*s)))))$$
$$\leq \mu_n((((w*s)*s)*(0*s))*q)\vee \mu_n(q),$$

and

$$\mu_p(w*((s*(s*w))*(0*(0*(w*s)))))$$
$$\geq \mu_p((((w*s)*s)*(0*s))*q)\wedge \mu_p(q).$$

Thus, having $0\leq \beta,\gamma\leq 1$ implies

$$\mu_n(0).\beta\leq \mu_n(w).\beta,\ \mu_p(0).\gamma\geq \mu_p(w).\gamma$$

and

$$\mu_n(w*((s*(s*w))*(0*(0*(w*s))))).\beta$$
$$\leq [\mu_n((((w*s)*s)*(0*s))*q)\vee \mu_n(q)].\beta,$$

and

$$\mu_p(w*((s*(s*w))*(0*(0*(w*s))))).\gamma$$
$$\geq \mu_p[((((w*s)*s)*(0*s))*q) \wedge \mu_p(q)].\gamma.$$

From the two latter inequalities we have,

$$\mu_n(w*((s*(s*w))*(0*(0*(w*s))))).\beta$$
$$\leq \mu_n((((w*s)*s)*(0*s))*q).\beta \vee \mu_n(q).\beta,$$

and

$$\mu_p(w*((s*(s*w))*(0*(0*(w*s))))).\gamma$$
$$\geq \mu_p[((((w*s)*s)*(0*s))*q).\gamma \wedge \mu_p(q).\gamma.$$

That is,

$$\mu_n^{(M,\beta)}(0) \leq \mu_n^{(M,\beta)}(w),\ \mu_p^{(M,\gamma)}(0) \geq \mu_p^{(M,\gamma)}(w)$$

and

$$\mu_n^{(M,\beta)}(w*((s*(s*w))*(0*(0*(w*s)))))$$
$$\leq \mu_n^{(M,\beta)}((((w*s)*s)*(0*s))*q) \vee \mu_n^{(M,\beta)}(q),$$

and

$$\mu_p^{(M,\gamma)}(w*((s*(s*w))*(0*(0*(w*s)))))$$
$$\geq \mu_p^{(M,\gamma)}((((w*s)*s)*(0*s))*q) \wedge \mu_p^{(M,\gamma)}(q).$$

Hence, $\mu^{(M,\beta,\gamma)} = (W; \mu_n^{(M,\beta)}, \mu_p^{(M,\gamma)})$ is a BF-BCI-implicative ideal in W. □

Definition 3.1. *Let $\mu = (W; \mu_n, \mu_p)$ be a bipolar fuzzy set in W and let $\beta, \gamma \in [0,1]$, $(\iota, \kappa) \in [\mathrm{B}, 0] \times [0, \mathrm{T}]$. Then a bipolar fuzzy $(M^{(\beta,\gamma)}, T^{(\iota,\kappa)})$-magnified translation of $\mu = (W; \mu_n, \mu_p)$, denoted by $\mu^{M^{(\beta,\gamma)}T^{(\iota,\kappa)}} = (W; \mu_n^{(MT,\beta,\iota)}, \mu_p^{(MT,\gamma,\kappa)})$ is a bipolar fuzzy set where, $\mu_n^{(MT,\beta,\iota)}$ and $\mu_p^{(MT,\gamma,\kappa)}$ are maps from W to $[-1,0]$ and from W to $[0,1]$ given by $w \mapsto \mu_n(w).\beta + \iota$ and $w \mapsto \mu_n(w).\gamma + \kappa$,respectively.*

Using the same manner used in Theorem 3.2 and Theorem 3.1 we have the following theorem.

Theorem 3.3. *Let $\mu = (W; \mu_n, \mu_p)$ be a BF-BCI-implicative ideal of W and let $\beta, \gamma \in [0,1]$, $(\iota, \kappa) \in [\mathrm{B}, 0] \times [0, \mathrm{T}]$. Then the bipolar fuzzy $(M^{(\beta,\gamma)}, T^{(\iota,\kappa)})$-magnified translation of $\mu = (W; \mu_n, \mu_p)$, $\mu^{M^{(\beta,\gamma)}T^{(\iota,\kappa)}} = (W; \mu_n^{(MT,\beta,\iota)}, \mu_p^{(MT,\gamma,\kappa)})$ is a BF-BCI-implicative ideal of W if and only if $\mu = (W; \mu_n, \mu_p)$ is a BF-BCI-implicative ideal of W.*

Definition 3.2. *Let $\mu = (W; \mu_n, \mu_p)$ be a bipolar fuzzy set in W. We define , the negative $(t^- - \iota)$-cut and the positive $(t^+ - \kappa)$-cut of $\mu = (W; \mu_n, \mu_p)$ as follows:*

$\mathcal{N}^{(t^- - \iota, \mu)} := \{w \in W | \mu_n(w) \leq t^- - \iota\}$ *where* $\iota \in [\mathrm{B}, 0]$ *and* $t^- \in [-1, 0]$ *with* $t^- \leq \iota$;

$\mathcal{P}^{(t^+ - \kappa, \mu)} := \{w \in W | \mu_p(w) \geq t^+ - \kappa\}$ *where* $\kappa \in [0, \mathrm{T}]$ *and* $t^+ \in [0, 1]$ *with* $t^+ \geq \kappa$, *respectively.*

Corollary 3.1. *For a bipolar fuzzy set $\mu = (W; \mu_n, \mu_p)$ in W, if $\mu = (W; \mu_n, \mu_p)$ is not BF-BCI-implicative ideal of W, then $\mathcal{N}^{(t^- - \iota, \mu)}$ is not BF-BCI-implicative ideal of W nor $\mathcal{P}^{(t^+ - \kappa, \mu)}$ as will be shown in the next example.*

Example 1. *Let $W = \{0, g, h, f\}$ and $*$ is given by the Cayley Table 1.*

TABLE 1. Cayley Table of the binary operation $*$

$*$	0	g	h	f
0	0	0	0	0
g	g	0	0	g
h	h	g	0	h
f	f	f	f	0

Then $(W; *, 0)$ is a BCI-algebra.

Let $\mu = (W; \mu_n, \mu_p)$ be a bipolar fuzzy set in W represented by Table 2.

TABLE 2. Table representation of μ

W	0	g	h	f
μ_n	-0.5	-0.5	-0.3	-0.4
μ_p	0.8	0.8	0.5	0.7

Then we have, $\mu_n(h * ((f * (f * h)) * (0 * (0 * (h * f))))) = \mu_n(h) = -0.3 \not\leq -0.5 = \mu_n((((h*f)*f)*(0*f))*g) \vee \mu_n(g) = \mu_n(g)$, and so $\mu = (W; \mu_n, \mu_p)$ is not BF-BCI-implicative ideal of W. Then $\mathrm{B} = -1-(-0.5) = -0.5$. Take $\iota = -0.2$ and $t^- = -0.4$. Then $\mathcal{N}^{(-0.2, \mu)} := \{w \in W | \mu_n(w) \leq -0.2\} = \{0, g, h, f\}$. Also $\mathrm{T} = 1 - 0.8 = 0.2$. Take $\kappa = 0.1$ and $t^+ = 0.5$, then $\mathcal{P}^{(0.4, \mu)} := \{w \in W | \mu_p(w) \geq 0.4\} = \{0, g, h, f\}$. It is obvious that both $\mathcal{N}^{(-0.2, \mu)}$ and $\mathcal{P}^{(0.4, \mu)}$ are not BF-BCI-implicative ideals of W.

If $\mu = (W; \mu_n, \mu_p)$ is a BF-BCI-implicative ideal of W, then we get the following.

Theorem 3.4. *For a bipolar fuzzy set $\mu = (W; \mu_n, \mu_p)$ in W, let $\mu = (W; \mu_n, \mu_p)$ be a BF-BCI-implicative ideal of W. Then $\mathcal{N}^{(t^- - \iota,\mu)}$ and $\mathcal{P}^{(t^+ - \kappa,\mu)}$ are BF-BCI-implicative ideals of W.*

Proof. Direct. ☐

Theorem 3.5. *Let $\mu = (W; \mu_n, \mu_p)$ be a bipolar fuzzy set in W. Then the following are equivalent, for $(\iota, \kappa) \in [\mathrm{B}, 0] \times [0, \mathrm{T}]$ and $(t^-, t^+) \in [-1, \iota] \times [\kappa, 1]$:*

(1) $\mu^{(T,\iota,\kappa)} = (W; \mu_n^{(T,\iota)}, \mu_p^{(T,\kappa)})$ of $\mu = (W; \mu_n, \mu_p)$ is a BF-BCI-implicative ideal of W.

(2) Both $\mathcal{N}^{(t^- - \iota,\mu)}$ and $\mathcal{P}^{(t^+ - \kappa,\mu)}$ are BCI-implicative ideals of W.

Proof.

$(1 \Rightarrow 2)$. Let $\mu^{(T,\iota,\kappa)} = (W; \mu_n^{(T,\iota)}, \mu_p^{(T,\kappa)})$ of $\mu = (W; \mu_n, \mu_p)$ be a BF-BCI-implicative ideal of W.Then
$\mu_n(0) + \iota = \mu_n^{(T,\iota)}(0) \leq \mu_n^{(T,\iota)}(w) = \mu_n(w) + \iota \leq t^-$ as $t^- \leq \iota$ and so $\mu_n(0) \leq t^- - \iota$.

Also, $\mu_p(0) + \kappa = \mu_p^{(T,\kappa)}(0) \geq \mu_p^{(T,\kappa)}(w) = \mu_p(w) + \kappa \geq t^+$ as $t^+ \geq \kappa$ and so $\mu_p(0) \geq t^+ - \kappa$. Therefore, 0 is an element in $\mathcal{N}^{(t^- - \iota,\mu)}$ and $\mathcal{P}^{(t^+ - \kappa,\mu)}$.

Let

$$((((w * s) * s) * (0 * s)) * q) \in \mathcal{N}^{(t^- - \iota,\mu)}, \qquad q \in \mathcal{N}^{(t^- - \iota,\mu)}$$

and

$$((((w * s) * s) * (0 * s)) * q) \in \mathcal{P}^{(t^+ - \kappa,\mu)}, \qquad q \in \mathcal{P}^{(t^+ - \kappa,\mu)}.$$

Then,

$$\mu_n((((w * s) * s) * (0 * s)) * q) \leq t^- - \iota, \quad \text{and} \quad \mu_n(q) \leq t^- - \iota,$$

$$\mu_p((((w * s) * s) * (0 * s)) * q) \geq t^+ - \kappa, \quad \text{and} \quad \mu_p(q) \geq t^+ - \kappa.$$

Hence,

$$\mu_n((((w * s) * s) * (0 * s)) * q) + \iota \leq t^-, \quad \text{and} \quad \mu_n(q) + \iota \leq t^-,$$

$$\mu_p((((w * s) * s) * (0 * s)) * q) + \kappa \geq t^+, \quad \text{and} \quad \mu_p(q) + \kappa \geq t^+.$$

Thus,

$$\mu_n^{(T,\iota)}((((w * s) * s) * (0 * s)) * q) \leq t^- \quad \text{and} \quad \mu_n^{(T,\iota)}(q) \leq t^-,$$
$$\mu_p^{(T,\kappa)}((((w * s) * s) * (0 * s)) * q) \geq t^+ \quad \text{and} \quad \mu_p^{(T,\kappa)}(q) \geq t^+.$$

Therefore,

$$\mu_n^{(T,\iota)}((((w * s) * s) * (0 * s)) * q) \vee \mu_n^{(T,\iota)}(q) \leq t^-,$$
$$\mu_p^{(T,\kappa)}((((w * s) * s) * (0 * s)) * q) \wedge \mu_p^{(T,\kappa)}(q) \geq t^+.$$

By assumption we have,

$$\mu_n^{(T,\iota)}((((w*s)*s)*(0*s))*q) \vee \mu_n^{(T,\iota)}(q)$$
$$\geq \mu_n^{(T,\iota)}(w*((s*(s*w))*(0*(0*(w*s))))),$$
$$\mu_p^{(T,\kappa)}((((w*s)*s)*(0*s))*q) \wedge \mu_p^{(T,\kappa)}(q)$$
$$\leq \mu_p^{(T,\kappa)}(w*((s*(s*w))*(0*(0*(w*s))))).$$

As

$$\mu_n^{(T,\iota)}((((w*s)*s)*(0*s))*q) \vee \mu_n^{(T,\iota)}(q) \leq t^-,$$
$$\mu_p^{(T,\kappa)}((((w*s)*s)*(0*s))*q) \wedge \mu_p^{(T,\kappa)}(q) \geq t^+,$$

we have,

$$\mu_n^{(T,\iota)}(w*((s*(s*w))*(0*(0*(w*s))))) \leq t^-,$$
$$\mu_p^{(T,\kappa)}(w*((s*(s*w))*(0*(0*(w*s))))) \geq t^+.$$

Thus,

$$\mu_n(w*((s*(s*w))*(0*(0*(w*s))))) + \iota \leq t^-,$$
$$\mu_p(w*((s*(s*w))*(0*(0*(w*s))))) + \kappa \geq t^+.$$

Then,

$$\mu_n(w*((s*(s*w))*(0*(0*(w*s))))) \leq t^- - \iota,$$
$$\mu_p(w*((s*(s*w))*(0*(0*(w*s))))) \geq t^+ - \kappa.$$

Therefore,

$$(w*((s*(s*w))*(0*(0*(w*s))))) \in \mathcal{N}^{(t^- - \iota,\mu)},$$
$$(w*((s*(s*w))*(0*(0*(w*s))))) \in \mathcal{P}^{(t^+ - \kappa,\mu)}.$$

Hence, $\mathcal{N}^{(t^- - \iota,\mu)}$ and $\mathcal{P}^{(t^+ - \kappa,\mu)}$ are BCI-implicative ideals.

$(2 \Rightarrow 1)$. Let $\mathcal{N}^{(t^- - \iota,\mu)}$ and $\mathcal{P}^{(t^+ - \kappa,\mu)}$ be BCI-implicative ideals of W. Let $w \in \mathcal{N}^{(t^- - \iota,\mu)}$ and $w \in \mathcal{P}^{(t^+ - \kappa,\mu)}$. Then $\mu_n(w) \leq t^- - \iota$ and $\mu_p(w) \geq t^+ - \kappa$.

We prove by contradiction. So assume that $\mu_n^{(T,\iota)}(0) \geq \mu_n^{(T,\iota)}(w)$ and $\mu_p^{(T,\kappa)}(0) \leq \mu_p^{(T,\kappa)}(w)$. Then for some $t^- \in [-1,\iota)$ and $t^+ \in (\kappa,1]$, $\mu_n^{(T,\iota)}(0) > t^- \geq \mu_n^{(T,\iota)}(w)$ and $\mu_p^{(T,\kappa)}(0) < t^+ \leq \mu_p^{(T,\kappa)}(w)$.

Thus, $\mu_n(0) + \iota > t^- \geq \mu_n(w) + \iota$ and $\mu_p(0) + \kappa < t^+ \leq \mu_p(w) + \kappa$. Then, $\mu_n(0) > t^- - \iota$, $\mu_n(w) \leq t^- - \iota$ and $\mu_p(0) < t^+ - \kappa$, $\mu_p(w) \geq s - \kappa$. Hence, $0 \notin \mathcal{N}^{(t^- - \iota,\mu)}$, $w \in \mathcal{N}^{(t^- - \iota,\mu)}$ and $0 \notin \mathcal{P}^{(t^+ - \kappa,\mu)}$, $w \in \mathcal{P}^{(t^+ - \kappa,\mu)}$ which is a contradiction to the assumption that $\mathcal{N}^{(t^- - \iota,\mu)}$ and $\mathcal{P}^{(t^+ - \kappa,\mu)}$ are BCI-implicative ideals of W. Therefore, $\mu_n^{(T,\iota)}(0) \leq \mu_n^{(T,\iota)}(w)$ and $\mu_p^{(T,\kappa)}(0) \geq \mu_p^{(T,\kappa)}(w)$.

Now suppose that

$$\mu_n^{(T,\iota)}(w*((s*(s*w))*(0*(0*(w*s)))))$$
$$\geq \mu_n^{(T,\iota)}((((w*s)*s)*(0*s))*q)\vee \mu_n^{(T,\iota)}(q),$$
$$\mu_p^{(T,\kappa)}(w*((s*(s*w))*(0*(0*(w*s)))))$$
$$\leq \mu_p^{(T,\kappa)}((((w*s)*s)*(0*s))*q)\wedge \mu_p^{(T,\kappa)}(q).$$

Then there exist $t^- \in [-1,\iota)$ and $t^+ \in (\kappa,1]$ such that

$$\mu_n^{(T,\iota)}(w*((s*(s*w))*(0*(0*(w*s))))) > t^-$$
$$\geq \mu_n^{(T,\iota)}((((w*s)*s)*(0*s))*q)\vee \mu_n^{(T,\iota)}(q),$$

$$\mu_p^{(T,\kappa)}(w*((s*(s*w))*(0*(0*(w*s))))) < t^+$$
$$\leq \mu_p^{(T,\kappa)}((((w*s)*s)*(0*s))*q)\wedge \mu_p^{(T,\kappa)}(q).$$

Thus,

$$\mu_n^{(T,\iota)}(w*((s*(s*w))*(0*(0*(w*s))))) > t^-,$$
$$\mu_n^{(T,\iota)}((((w*s)*s)*(0*s))*q) \leq t^- \vee \mu_n^{(T,\iota)}(q)\leq t^-$$

and

$$\mu_p^{(T,\kappa)}(w*((s*(s*w))*(0*(0*(w*s))))) < t^+,$$
$$\mu_p^{(T,\kappa)}((((w*s)*s)*(0*s))*q) \geq t^+ \wedge \mu_p^{(T,\kappa)}(q)\geq t^+.$$

Therefore,

$$\mu_n(w*((s*(s*w))*(0*(0*(w*s))))) > t^- - \iota,$$
$$\mu_n((((w*s)*s)*(0*s))*q) \leq t^- - \iota \vee \mu_n(q)\leq t^- - \iota$$

and

$$\mu_p(w*((s*(s*w))*(0*(0*(w*s))))) < t^+ - \kappa,$$
$$\mu_p((((w*s)*s)*(0*s))*q) \geq t^+ - \kappa \wedge \mu_p(q)\geq t^+ - \kappa.$$

Implies that

$$((((w*s)*s)*(0*s))*q) \in \mathcal{N}^{(t^- - \iota,\mu)} \quad \text{and} \quad q \in \mathcal{N}^{(t^- - \iota,\mu)}$$

but

$$(w*((s*(s*w))*(0*(0*(w*s))))) \notin \mathcal{N}^{(t^- - \iota,\mu)}.$$

Also

$$((((w*s)*s)*(0*s))*q) \in \mathcal{P}^{(t^+ - \kappa,\mu)} \quad \text{and} \quad q \in \mathcal{P}^{(t^+ - \kappa,\mu)}$$

while

$$(w*((s*(s*w))*(0*(0*(w*s))))) \notin \mathcal{P}^{(t^{+}-\kappa,\mu)}.$$

This gives a contradiction to the assumption that $\mathcal{N}^{(t^{-}-\iota,\mu)}$ and $\mathcal{P}^{(t^{+}-\kappa,\mu)}$ are BCI-implicative ideals of W. Thus we have

$$\begin{aligned}&\mu_n^{(T,\iota)}(w*((s*(s*w))*(0*(0*(w*s)))))\\ \leq&\mu_n^{(T,\iota)}((((w*s)*s)*(0*s))*q) \vee \mu_n^{(T,\iota)}(q),\end{aligned}$$

$$\begin{aligned}&\mu_p^{(T,\kappa)}(w*((s*(s*w))*(0*(0*(w*s)))))\\ \geq&\mu_p^{(T,\kappa)}((((w*s)*s)*(0*s))*q) \wedge \mu_p^{(T,\kappa)}(q).\end{aligned}$$

□

Definition 3.3. *Let $\mu_1 = (W;\mu_{n_1},\mu_{p_1})$ and $\mu_2 = (W;\mu_{n_2},\mu_{p_2})$ be bipolar fuzzy sets in W such that $\mu_{n_1}(w) \geq \mu_{n_2}(w)$ and $\mu_{p_1}(w) \leq \mu_{p_2}(w)$, for all $w \in W$. Then it said is that $\mu_2 = (W;\mu_{n_2},\mu_{p_2})$ is a bipolar fuzzy BCI-implicative ideal extension of $\mu_1 = (W;\mu_{n_1},\mu_{p_1})$ if whenever $\mu_1 = (W;\mu_{n_1},\mu_{p_1})$ is a bipolar fuzzy BCI-implicative ideal of W then $\mu_2 = (W;\mu_{n_2},\mu_{p_2})$ is a bipolar fuzzy BCI-implicative ideal of W.*

Obviously, if $\mu_2 = (W;\mu_{n_2},\mu_{p_2})$ is a BF-BCI-implicative ideal extension of $\mu_1 = (W;\mu_{n_1},\mu_{p_1})$ then $\mu_1 = (W;\mu_{n_1},\mu_{p_1})$ is a bipolar fuzzy BCI-implicative ideal of W and vice versa.

Theorem 3.6. *Let $\mu = (W;\mu_n,\mu_p)$ be a BF-BCI-implicative ideal of W. Then the bipolar fuzzy (ι,κ)-translation of $\mu = (W;\mu_n,\mu_p)$, $\mu^{(T,\iota,\kappa)} = (W;\mu_n^{(T,\iota)},\mu_p^{(T,\kappa)})$ is a BF-BCI-implicative ideal extension of $\mu = (W;\mu_n,\mu_p)$. The converse is not true in general.*

Proof. Let $\mu = (W;\mu_n,\mu_p)$ be a BF-BCI-implicative ideal of W and let $\mu^{(T,\iota,\kappa)} = (W;\mu_n^{(T,\iota)},\mu_p^{(T,\kappa)})$ be a bipolar fuzzy (ι,κ)-translation of $\mu = (W;\mu_n,\mu_p)$. Using Theorem 3.1 we have $\mu^{(T,\iota,\kappa)} = (W;\mu_n^{(T,\iota)},\mu_p^{(T,\kappa)})$ is a BF-BCI-implicative ideal of W. As $(\iota,\kappa) \in [\mathrm{B},0]\times[0,\mathrm{T}]$ then we know that $\mu_n(w) \geq \mu_n^{(T,\iota)}(w)$ and $\mu_p(w) \leq \mu_p^{(T,\kappa)}$. Therefore, $\mu^{(T,\iota,\kappa)} = (W;\mu_n^{(T,\iota)},\mu_p^{(T,\kappa)})$ is a BF-BCI-implicative ideal extension of $\mu = (W;\mu_n,\mu_p)$.

Conversely, consider a BCI-algebra $(W;*,0)$ where $W = \{0,g,h,f\}$ and $*$ is given by the Cayley Table 3.

Let $\mu = (W;\mu_n,\mu_p)$ be a bipolar fuzzy set in W represented by Table 4.

TABLE 3. Cayley Table of the binary operation $*$

$*$	0	g	h	f
0	0	g	h	f
g	g	0	f	h
h	h	f	0	g
f	f	h	g	0

TABLE 4. Table representation of μ

W	0	g	h	f
μ_n	-0.8	-0.8	-0.5	-0.5
μ_p	0.9	0.9	0.3	0.3

Then, by routine calculations, $\mu = (W; \mu_n, \mu_p)$ is a BF-BCI-implicative ideal of W. Let $\eta = (W; \eta_n, \eta_p)$ be a bipolar fuzzy set in W given by Table 5.

TABLE 5. Table representation of η

W	0	g	h	f
η_n	-0.85	-0.85	-0.51	-0.51
η_p	0.91	0.91	0.33	0.33

Then, $\eta = (W; \eta_n, \eta_p)$ is a BF-BCI-implicative ideal extension of $\mu = (W; \mu_n, \mu_p)$ and is not bipolar fuzzy (ι, κ)-translation of $\mu = (W; \mu_n, \mu_p)$. $\square$

Theorem 3.7. *Let $\mu = (W; \mu_n, \mu_p)$ be a BF-BCI-implicative ideal in W and let $\mu^{(T,\iota_1,\kappa_1)} = (W; \mu_n^{(T,\iota_1)}, \mu_p^{(T,\kappa_1)})$ be a bipolar fuzzy (ι_1, κ_1)-translation of $\mu = (W; \mu_n, \mu_p)$ and $\mu^{(T,\iota_2,\kappa_2)} = (W; \mu_n^{(T,\iota_2)}, \mu_p^{(T,\kappa_2)})$ be a bipolar fuzzy (ι_2, κ_2)-translation of $\mu = (W; \mu_n, \mu_p)$ where $\iota_1, \iota_2 \in [\mathrm{B}, 0]$ and $\kappa_1, \kappa_2 \in [0, \mathrm{T}]$. Then (ι_1, κ_1)-translation $\mu^{(T,\iota_1,\kappa_1)} = (W; \mu_n^{(T,\iota_1)}, \mu_p^{(T,\kappa_1)})$ of $\mu = (W; \mu_n, \mu_p)$ is a BF-BCI-implicative ideal extension of (ι_2, κ_2)-translation $\mu^{(T,\iota_2,\kappa_2)} = (W; \mu_n^{(T,\iota_2)}, \mu_p^{(T,\kappa_2)})$ of $\mu = (W; \mu_n, \mu_p)$ if $(\iota_1, \kappa_1) \geq (\iota_2, \kappa_2)$.*

Proof. Having $(\iota_1, \kappa_1) \geq (\iota_2, \kappa_2)$ implies $\iota_1 \leq \iota_2$ and $\kappa_1 \geq \kappa_2$ and so, using Theorem 3.1, (ι_1, κ_1)-translation $\mu^{(T,\iota_1,\kappa_1)} = (W; \mu_n^{(T,\iota_1)}, \mu_p^{(T,\kappa_1)})$ of $\mu = (W; \mu_n, \mu_p)$ is a BF-BCI-implicative ideal extension of (ι_2, κ_2)-translation $\mu^{(T,\iota_2,\kappa_2)} = (W; \mu_n^{(T,\iota_2)}, \mu_p^{(T,\kappa_2)})$ of $\mu = (W; \mu_n, \mu_p)$. $\square$

Theorem 3.8. *Let $\mu = (W; \mu_n, \mu_p)$ be a bipolar fuzzy set of W. Then the bipolar fuzzy (ι, κ)-translation of $\mu = (W; \mu_n, \mu_p)$, $\mu^{(T,\iota,\kappa)} = (W; \mu_n^{(T,\iota)}, \mu_p^{(T,\kappa)})$ is a BF-BCI-implicative ideal extension of the (β, γ)-multiplication $\mu^{(M,\beta,\gamma)} = (W; \mu_n^{(M,\beta)}, \mu_p^{(M,\gamma)})$ of $\mu = (W; \mu_n, \mu_p)$.*

Proof. Let $\mu^{(T,\iota,\kappa)} = (W; \mu_n^{(T,\iota)}, \mu_p^{(T,\kappa)})$ be a bipolar fuzzy (ι, κ)-translation of $\mu = (W; \mu_n, \mu_p)$. We show that $\mu^{(T,\iota,\kappa)} = (W; \mu_n^{(T,\iota)}, \mu_p^{(T,\kappa)})$ is a BF-BCI-implicative ideal extension of the (β, γ)-multiplication $\mu^{(M,\beta,\gamma)} = (W; \mu_n^{(M,\beta)}, \mu_p^{(M,\gamma)})$.

We have

$$\mu_n^{(T,\iota)}(w) = \mu_n(w) + \iota \leq \mu_n(w) \leq \mu_n(w).\beta = \mu_n^{(M,\beta)}(w)$$

and

$$\mu_p^{(T,\kappa)}(w) = \mu_p(w) + \kappa \geq \mu_p(w) \geq \mu_p(w).\gamma = \mu_p^{(M,\gamma)}(w).$$

Thus, $\mu_n^{(M,\beta)}(w) \geq \mu_n^{(T,\iota)}(w)$ and $\mu_p^{(M,\gamma)}(w) \leq \mu_p^{(T,\kappa)}(w)$. Let (β, γ)-multiplication $\mu^{(M,\beta,\gamma)} = (W; \mu_n^{(M,\beta)}, \mu_p^{(M,\gamma)})$ of $\mu = (W; \mu_n, \mu_p)$ be a BF-BCI-implicative ideal of W. Thus, $\mu = (W; \mu_n, \mu_p)$ is a BF-BCI-implicative ideal of W from Theorem 3.2 and so (ι, κ)-translation of $\mu = (W; \mu_n, \mu_p)$ is a BF-BCI-implicative ideal of W from Theorem 3.1. Hence, the bipolar fuzzy (ι, κ)-translation of $\mu = (W; \mu_n, \mu_p)$, $\mu^{(T,\iota,\kappa)} = (W; \mu_n^{(T,\iota)}, \mu_p^{(T,\kappa)})$ is a BF-BCI-implicative ideal extension of the (β, γ)-multiplication $\mu^{(M,\beta,\gamma)} = (W; \mu_n^{(M,\beta)}, \mu_p^{(M,\gamma)})$ of $\mu = (W; \mu_n, \mu_p)$. □

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