ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.12, 10797–10805
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.12.63

ON SEMI WEAKLY (l, m)-HYPONORMAL WEIGHTED SHIFTS

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ABSTRACT. In this paper, semi-weak (l, m)–hyponormality is defined and a characterization for semi-weakly (l, m)–hyponormal weighted shift, is obtained. Stability of semi-weakly (l, m)–hyponormal weighted shift under small perturbation of the weight sequence is also discussed. In addition, we obtain the condition under which 3-hyponormality of the weighted shift is stable under small perturbation of second weight of the weight sequence.

1. INTRODUCTION

Let H be a separable complex Hilbert space and T be a bounded linear operators on H. $[A, B] := AB - BA$ denote the commutator of two operators A and B. An operator T is hyponormal if $[T^*, T] \geq 0$. An operator T is polynomially hyponormal if $p(T)$ is hyponormal for all (complex) polynomials p. An operator T is weakly m-hyponormal if $p(T)$ is hyponormal for any polynomial p with degree $\leq m$. An operator T is (strongly) m-hyponormal if the operator matrix $([T^{*j}, T^i])_{i,j=1}^m$ is positive.

The classes of (weakly) m-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([2–6]). The study of this gap has been only partially successful. For weighted shifts, positive results appear, although no concrete example of a weighted shift which

2020 *Mathematics Subject Classification.* 47B20, 47B37.
Key words and phrases. Weighted shift operator, m-hyponormal, Weakly m-hyponormal, Semi weakly m-hyponormal.

is polynomially hyponormal and not subnormal has yet been found (the existence of such weights was established in [6]). Let α be a weight sequence, $\alpha : \alpha_0, \alpha_1, \alpha_2, \ldots$ where it is without loss of generality to assume these are all positive. The weighted shift W_α acting on $\ell^2(\mathbb{N}_0)$, with standard basis $e_0, e_1, \ldots$ is defined by $W_\alpha(e_j) = \alpha_j e_{j+1}$ for all $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

In [2] finite rank perturbations of unilateral weighted shifts were established and in [4] a characterization for weakly m-hyponormal weighted shift W_α with weight sequence α was obtained, mainly to study such gaps. In [3] Semi-weak m-hyponormality was defined and studied using the notion of positive determinant partition to illustrates the gaps between various weak subnormalities. However, it is still unknown that whether the polynomial hyponormality of the weighted shift is stable under small perturbations of the weight sequence?

As a continuation of this study, in this present work, semi-weak (l, m)– hyponormality is defined and a characterization for semi-weakly (l, m)–hyponormal weighted shift W_α with weight sequence α, is obtained. Also it is shown that semi-weakly (l, m)–hyponormal weighted shift for ($l \geq 2$ and $m > l$) is stable under the perturbation of first and second weight of the weight sequence. In addition, we obtain the condition under which 3-hyponormality of the weighted shift is also stable under small perturbation of second weight of the weight sequence.

2. SEMI WEAKLY (l, m)–HYPONORMAL

Definition 2.1. *[3] A weighted shift W_α is semi weakly $m-hyponormal$ if $W_\alpha + sW_\alpha^m$ is hyponormal for any $s \in \mathbb{C}$.*

Definition 2.2. *A weighted shift W_α is semi weakly (l, m)-hyponormal if $W_\alpha^l + sW_\alpha^m$ $(1 \leq l < m)$ and $m \geq 2$ is hyponormal for any $s \in \mathbb{C}$.*

Clearly, a weakly m-hyponormal weighted shift is semi weakly (l, m)-hyponormal. Semi weakly m-hyponomal weighted shifts and semi weakly $(1, m)$-hyponormal weighted shifts are equivalent.

Let $P[x]$ be the set of polynomials with one variable x.

Theorem 2.1. *[4] Let W_α be a contractive weighted shift with weight sequence $\alpha := \{\alpha_i\}_{i=0}^{\infty}$. Then W_α is semi weakly (l, m)-hyponormal if and only if there exists a linear functional $\bar{\Lambda} : P[x] \to \mathbb{C}$ such that*

(1) *for all* $i \in \mathbb{N}_0, \bar{\Lambda}(x^i) = \gamma_i$, *where* $\gamma_0 := 1$ *and* $\gamma_i := \alpha_0^2 \dots \alpha_{i-1}^2 (i \geq 1)$,

(2) *for all finite sets* $\{p_i\}_{i\geq 0}$ *in* $\mathbb{C}$,

$$\bar{\Lambda}\left(\sum_{i\geq 0} |p_i|^2 x^i - \sum_{i\geq 0} |p_i|^2 x^{i+1}\right) \geq 0,$$

(3) *for all finite sets* $\{p_i\}_{i\geq 0}, \{q_i\}_{i\geq 0}$ *and* $\{\phi_i\}_{i=l,m}$ *in* $\mathbb{C}$,

$$\bar{\Lambda}\left(\sum_{k\geq 0} x^k \left| q_k + \sum_{i=l,m} p_{i+k}\phi_i x^i \right|^2 + \sum_{k>0} x^k \left| \sum_{i\geq 0} \phi_{k+i} p_i x^i \right|^2\right) \geq 0,$$

where ϕ_i *is set to* $0\,(i \neq l,m)$.

Theorem 2.2. *Let* W_α *be a contractive weighted shift with weight sequence* $\alpha := \{\alpha_i\}_{i=0}^\infty$. *Then* W_α *is semi weakly* (l,m)*-hyponormal if and only if*

$$\begin{aligned}
\Delta_m^\alpha := \sum_{i=l+1}^{m} |\phi_m p_{m-i}|^2 \gamma_{2m-i} &+ \left\langle \begin{pmatrix} \gamma_{2l-1} & \gamma_{m+l-1} \\ \gamma_{m+l-1} & \gamma_{2m-1} \end{pmatrix} \begin{pmatrix} \phi_l p_{l-1} \\ \phi_m p_{m-1} \end{pmatrix}, \begin{pmatrix} \phi_l p_{l-1} \\ \phi_m p_{m-1} \end{pmatrix} \right\rangle \\
&+ \left\langle \begin{pmatrix} \gamma_{2l-2} & \gamma_{m+l-2} \\ \gamma_{m+l-2} & \gamma_{2m-2} \end{pmatrix} \begin{pmatrix} \phi_l p_{l-2} \\ \phi_m p_{m-2} \end{pmatrix}, \begin{pmatrix} \phi_l p_{l-2} \\ \phi_m p_{m-2} \end{pmatrix} \right\rangle + \dots \\
&+ \left\langle \begin{pmatrix} \gamma_l & \gamma_m \\ \gamma_m & \gamma_{2m-l} \end{pmatrix} \begin{pmatrix} \phi_l p_0 \\ \phi_m p_{m-l} \end{pmatrix}, \begin{pmatrix} \phi_l p_0 \\ \phi_m p_{m-l} \end{pmatrix} \right\rangle \\
&+ \sum_{k\geq 0} \left\langle \begin{pmatrix} \gamma_k & \gamma_{k+l} & \gamma_{k+m} \\ \gamma_{k+l} & \gamma_{k+2l} & \gamma_{k+m+l} \\ \gamma_{k+m} & \gamma_{k+m+l} & \gamma_{k+2m} \end{pmatrix} \begin{pmatrix} q_k \\ \phi_l p_{k+l} \\ \phi_m p_{k+m} \end{pmatrix}, \begin{pmatrix} q_k \\ \phi_l p_{k+l} \\ \phi_m p_{k+m} \end{pmatrix} \right\rangle \geq 0.
\end{aligned}$$

Proof. Clearly, for any finite sets $\{p_i\}_{i\geq 0}$ in $\mathbb{C}$,

$$\bar{\Lambda}\left(\sum_{i\geq 0} |p_i|^2 x^i - \sum_{i\geq 0} |p_i|^2 x^{i+1}\right) = \sum_{i\geq 0} |p_i|^2 \gamma_i - \sum_{i\geq 0} |p_i|^2 \gamma_{i+1} \geq 0,$$

For all $k \geq 0$,

$$\begin{aligned}
&\bar{\Lambda}\left(x^k\left|q_k+\sum_{i=l,m}p_{i+k}\phi_i x^i\right|^2\right)\\
&=\bar{\Lambda}\left(x^k\left(q_k+\sum_{i=l,m}p_{i+k}\phi_i x^i\right)\left(\bar{q}_k+\sum_{i=l,m}\bar{p}_{i+k}\bar{\phi}_i x^i\right)\right)\\
&=\bar{\Lambda}(x^k)|q_k|^2+q_k\sum_{i=l,m}\bar{p}_{k+i}\bar{\phi}_i\bar{\Lambda}(x^{k+i})+\bar{q}_k\sum_{i=l,m}p_{k+i}\phi_i\bar{\Lambda}(x^{k+i})\\
&\quad+\sum_{i,j=l,m}p_{k+i}\bar{p}_{k+j}\phi_i\bar{\phi}_j\bar{\Lambda}(x^{k+i+j})\\
&=\gamma_k|q_k|^2+q_k\sum_{i=l,m}\bar{p}_{k+i}\bar{\phi}_i\gamma_{k+i}+\bar{q}_k\sum_{i=l,m}p_{k+i}\phi_i\gamma_{k+i}+\sum_{i,j=l,m}p_{k+i}\bar{p}_{k+j}\phi_i\bar{\phi}_j\gamma_{k+i+j}\\
&=\left\langle\begin{pmatrix}\gamma_k & \gamma_{k+l} & \gamma_{k+m}\\ \gamma_{k+l} & \gamma_{k+2l} & \gamma_{k+m+l}\\ \gamma_{k+m} & \gamma_{k+m+l} & \gamma_{k+2m}\end{pmatrix}\begin{pmatrix}q_k\\ \phi_l p_{k+l}\\ \phi_m p_{k+m}\end{pmatrix},\begin{pmatrix}q_k\\ \phi_l p_{k+l}\\ \phi_m p_{k+m}\end{pmatrix}\right\rangle.
\end{aligned}$$

Also,

$$\begin{aligned}
&\sum_{k>0}\bar{\Lambda}\left(x^k\left|\sum_{i\geq 0}\phi_{k+i}p_i x^i\right|^2\right)=\sum_{k<0}\sum_{i=l,m}\phi_j\bar{p}_{k+j}\bar{\phi}_i p_{k+i}\bar{\Lambda}(x^{k+i+j})\\
&=\sum_{k<0}\sum_{i=l,m}\phi_j\bar{p}_{k+j}\bar{\phi}_i p_{k+i}\gamma_{k+i+j}\\
&=\sum_{i=l,m}\left(\phi_j\bar{p}_{j-1}\bar{\phi}_i p_{i-1}\gamma_{i+j-1}+\phi_j\bar{p}_{j-2}\bar{\phi}_i p_{i-2}\gamma_{i+j-2}+\cdots+\phi_j\bar{p}_{j-l}\bar{\phi}_i p_{i-l}\gamma_{i+j-l}\right)\\
&\quad+\sum_{i=l,m}\left(\phi_j\bar{p}_{j-(l+1)}\bar{\phi}_i p_{i-(l+1)}\gamma_{i+j-(l+1)}+\cdots+\phi_j\bar{p}_{j-m}\bar{\phi}_i p_{i-m}\gamma_{i+j-m}\right)\\
&=\left\langle\begin{pmatrix}\gamma_{2l-1} & \gamma_{m+l-1}\\ \gamma_{m+l-1} & \gamma_{2m-1}\end{pmatrix}\begin{pmatrix}\phi_l p_{l-1}\\ \phi_m p_{m-1}\end{pmatrix},\begin{pmatrix}\phi_l p_{l-1}\\ \phi_m p_{m-1}\end{pmatrix}\right\rangle\\
&\quad+\left\langle\begin{pmatrix}\gamma_{2l-2} & \gamma_{m+l-2}\\ \gamma_{m+l-2} & \gamma_{2m-2}\end{pmatrix}\begin{pmatrix}\phi_l p_{l-2}\\ \phi_m p_{m-2}\end{pmatrix},\begin{pmatrix}\phi_l p_{l-2}\\ \phi_m p_{m-2}\end{pmatrix}\right\rangle+\ldots\\
&\quad+\left\langle\begin{pmatrix}\gamma_l & \gamma_m\\ \gamma_m & \gamma_{2m-l}\end{pmatrix}\begin{pmatrix}\phi_l p_0\\ \phi_m p_{m-l}\end{pmatrix},\begin{pmatrix}\phi_l p_0\\ \phi_m p_{m-l}\end{pmatrix}\right\rangle+|\phi_m|^2|p_{m-(l+1)}|^2\gamma_{2m-(l+1)}\\
&\quad+\cdots+|\phi_m|^2|p_0|^2\gamma_m
\end{aligned}$$

Since the sets $\{p_i\}$ and $\{q_i\}$ were arbitrary, therefore Theorem 2.1 and the above two equalities prove this theorem. $\square$

Lemma 2.1. *Let* $\alpha : \alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots$ *and let* $\alpha(\delta,\epsilon) : \delta\alpha_0, \epsilon\alpha_1, \alpha_2, \alpha_3, \ldots$ *with* $0 < \delta, \epsilon \leq 1$. *Then for* $l \geq 2$ *and* $m > l$, $\Delta_m^{\alpha(\delta,\epsilon)} = \delta^2\epsilon^2\Delta_m^{\alpha} + (1-\delta^2)|q_0|^2 + \delta^2(1-\epsilon^2)\gamma_1|q_1|^2$, *where* γ_j *are the moments of the sequence* α.

Proof. Here $\gamma'_0 = \gamma_0 = 1, \gamma'_1 = \delta^2\gamma_1$, and $\gamma'_j = \delta^2\epsilon^2\gamma_j$, $(j \geq 2)$, where γ'_j are the moments of the sequence $\alpha(\delta,\epsilon)$. Then

$$\begin{aligned}
\Delta_m^{\alpha(\delta,\epsilon)} := \sum_{i=l+1}^{m} |\phi_m p_{m-i}|^2 \gamma'_{2m-i} \\
&+ \left\langle \begin{pmatrix} \gamma'_{2l-1} & \gamma'_{m+l-1} \\ \gamma'_{m+l-1} & \gamma'_{2m-1} \end{pmatrix} \begin{pmatrix} \phi_l p_{l-1} \\ \phi_m p_{m-1} \end{pmatrix}, \begin{pmatrix} \phi_l p_{l-1} \\ \phi_m p_{m-1} \end{pmatrix} \right\rangle \\
&+ \left\langle \begin{pmatrix} \gamma'_{2l-2} & \gamma'_{m+l-2} \\ \gamma'_{m+l-2} & \gamma'_{2m-2} \end{pmatrix} \begin{pmatrix} \phi_l p_{l-2} \\ \phi_m p_{m-2} \end{pmatrix}, \begin{pmatrix} \phi_l p_{l-2} \\ \phi_m p_{m-2} \end{pmatrix} \right\rangle + \cdots \\
&+ \left\langle \begin{pmatrix} \gamma'_{l} & \gamma'_{m} \\ \gamma'_{m} & \gamma'_{2m-l} \end{pmatrix} \begin{pmatrix} \phi_l p_{0} \\ \phi_m p_{m-l} \end{pmatrix}, \begin{pmatrix} \phi_l p_{0} \\ \phi_m p_{m-l} \end{pmatrix} \right\rangle + \Omega_0 + \Omega_1 \\
&+ \sum_{k\geq 2} \left\langle \begin{pmatrix} \gamma'_k & \gamma'_{k+l} & \gamma'_{k+m} \\ \gamma'_{k+l} & \gamma'_{k+2l} & \gamma'_{k+m+l} \\ \gamma'_{k+m} & \gamma'_{k+m+l} & \gamma'_{k+2m} \end{pmatrix} \begin{pmatrix} q_k \\ \phi_l p_{k+l} \\ \phi_m p_{k+m} \end{pmatrix}, \begin{pmatrix} q_k \\ \phi_l p_{k+l} \\ \phi_m p_{k+m} \end{pmatrix} \right\rangle,
\end{aligned}$$

where

$$\Omega_0 := \left\langle \begin{pmatrix} \gamma'_0 & \gamma'_l & \gamma'_m \\ \gamma'_l & \gamma'_{2l} & \gamma'_{m+l} \\ \gamma'_m & \gamma'_{m+l} & \gamma'_{2m} \end{pmatrix} \begin{pmatrix} q_0 \\ \phi_l p_l \\ \phi_m p_m \end{pmatrix}, \begin{pmatrix} q_0 \\ \phi_l p_l \\ \phi_m p_m \end{pmatrix} \right\rangle$$

and

$$\Omega_1 := \left\langle \begin{pmatrix} \gamma'_1 & \gamma'_{l+1} & \gamma'_{m+1} \\ \gamma'_{l+1} & \gamma'_{2l+1} & \gamma'_{m+l+1} \\ \gamma'_{m+1} & \gamma'_{m+l+1} & \gamma'_{2+1m} \end{pmatrix} \begin{pmatrix} q_1 \\ \phi_l p_{l+1} \\ \phi_m p_{m+1} \end{pmatrix}, \begin{pmatrix} q_k \\ \phi_l p_{l+1} \\ \phi_m p_{m+1} \end{pmatrix} \right\rangle.$$

By direct computation, we have

$$\Omega_0 := \delta^2\epsilon^2 \left\langle \begin{pmatrix} \gamma_0 & \gamma_l & \gamma_m \\ \gamma_l & \gamma_{2l} & \gamma_{m+l} \\ \gamma_m & \gamma_{m+l} & \gamma_{2m} \end{pmatrix} \begin{pmatrix} q_0 \\ \phi_l p_l \\ \phi_m p_m \end{pmatrix}, \begin{pmatrix} q_0 \\ \phi_l p_l \\ \phi_m p_m \end{pmatrix} \right\rangle + (1-\delta^2)|q_0|^2$$

and

$$\Omega_1 := \delta^2\epsilon^2 \left\langle \begin{pmatrix} \gamma_1 & \gamma_{l+1} & \gamma_{m+1} \\ \gamma_{l+1} & \gamma_{2l+1} & \gamma_{m+l+1} \\ \gamma_{m+1} & \gamma_{m+l+1} & \gamma_{2+1m} \end{pmatrix} \begin{pmatrix} q_1 \\ \phi_l p_{l+1} \\ \phi_m p_{m+1} \end{pmatrix}, \begin{pmatrix} q_k \\ \phi_l p_{l+1} \\ \phi_m p_{m+1} \end{pmatrix} \right\rangle + \delta^2(1-\epsilon^2)\gamma_1|q_1|^2.$$

Thus, $\Delta_m^{\alpha(\delta,\epsilon)} = \delta^2\epsilon^2\Delta_m^\alpha + (1-\delta^2)|q_0|^2 + \delta^2(1-\epsilon^2)\gamma_1|q_1|^2$. □

Lemma 2.2. *Let* $\alpha : \alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots$ *and let* $\alpha(\delta,\epsilon) : \delta\alpha_0, \epsilon\alpha_1, \alpha_2, \alpha_3, \ldots$ *with* $0 < \delta, \epsilon \leq 1$. *Then for* $l = 1$ *and a fix* $m > l$, $\Delta_m^{\alpha(\delta,\epsilon)} = \delta^2\epsilon^2\Delta_m^\alpha + \delta^2(1-\epsilon^2)\gamma_1(|\phi_1 p_0|^2 + |q_1|^2) + (1-\delta^2\epsilon^2)|q_0|^2 + 2\delta^2(1-\epsilon^2)\gamma_1 Re(\phi_1 p_1 \bar{q}_0)$, *where* γ_j *are the moments of the sequence* α.

Proof. For $l = 1$ and a fix $m > l$,

$$\Delta_m^{\alpha(\delta,\epsilon)} := \sum_{i=2}^{m} |\phi_m p_{m-i}|^2\gamma'_{2m-i} + \left\langle \begin{pmatrix} \gamma'_1 & \gamma'_m \\ \gamma'_m & \gamma'_{2m-1} \end{pmatrix} \begin{pmatrix} \phi_1 p_0 \\ \phi_m p_{m-1} \end{pmatrix}, \begin{pmatrix} \phi_1 p_0 \\ \phi_m p_{m-1} \end{pmatrix} \right\rangle + \sum_{k\geq 0} \left\langle \begin{pmatrix} \gamma'_k & \gamma'_{k+1} & \gamma'_{k+m} \\ \gamma'_{k+1} & \gamma'_{k+2} & \gamma'_{k+m+1} \\ \gamma'_{k+m} & \gamma'_{k+m+1} & \gamma'_{k+2m} \end{pmatrix} \begin{pmatrix} q_k \\ \phi_1 p_{k+1} \\ \phi_m p_{k+m} \end{pmatrix}, \begin{pmatrix} q_k \\ \phi_1 p_{k+1} \\ \phi_m p_{k+m} \end{pmatrix} \right\rangle.$$

By direct computations, we have

$$\left\langle \begin{pmatrix} \gamma'_1 & \gamma'_m \\ \gamma'_m & \gamma'_{2m-1} \end{pmatrix} \begin{pmatrix} \phi_1 p_0 \\ \phi_m p_{m-1} \end{pmatrix}, \begin{pmatrix} \phi_1 p_0 \\ \phi_m p_{m-1} \end{pmatrix} \right\rangle = \delta^2(1-\epsilon^2)\gamma_1|\phi_1 p_0|^2 + \delta^2\epsilon^2 \left\langle \begin{pmatrix} \gamma_1 & \gamma_m \\ \gamma_m & \gamma_{2m-1} \end{pmatrix} \begin{pmatrix} \phi_1 p_0 \\ \phi_m p_{m-1} \end{pmatrix}, \begin{pmatrix} \phi_1 p_0 \\ \phi_m p_{m-1} \end{pmatrix} \right\rangle.$$

For $k = 0$,

$$\left\langle \begin{pmatrix} \gamma'_0 & \gamma'_1 & \gamma'_m \\ \gamma'_1 & \gamma'_2 & \gamma'_{m+1} \\ \gamma'_m & \gamma'_{m+1} & \gamma'_{2m} \end{pmatrix} \begin{pmatrix} q_0 \\ \phi_1 p_1 \\ \phi_m p_m \end{pmatrix}, \begin{pmatrix} q_0 \\ \phi_1 p_1 \\ \phi_m p_m \end{pmatrix} \right\rangle = (1-\delta^2\epsilon^2)|q_0|^2 + 2\delta^2(1-\epsilon^2)\gamma_1 Re(\phi_1 p_1 \bar{q}_0) + \delta^2\epsilon^2 \left\langle \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_m \\ \gamma_1 & \gamma_2 & \gamma_{m+1} \\ \gamma_m & \gamma_{m+1} & \gamma_{2m} \end{pmatrix} \begin{pmatrix} q_0 \\ \phi_1 p_1 \\ \phi_m p_m \end{pmatrix}, \begin{pmatrix} q_0 \\ \phi_1 p_1 \\ \phi_m p_m \end{pmatrix} \right\rangle.$$

For $k=1$,

$$\left\langle \begin{pmatrix} \gamma_1' & \gamma_2' & \gamma_{m+1}' \\ \gamma_2' & \gamma_3' & \gamma_{m+2}' \\ \gamma_{m+1}' & \gamma_{m+2}' & \gamma_{2m+1}' \end{pmatrix} \begin{pmatrix} q_1 \\ \phi_1 p_2 \\ \phi_m p_{m+1} \end{pmatrix}, \begin{pmatrix} q_1 \\ \phi_1 p_2 \\ \phi_m p_{m+1} \end{pmatrix} \right\rangle = \delta^2(1-\epsilon^2)\gamma_1|q_1|^2$$

$$+\delta^2\epsilon^2 \left\langle \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_{m+1} \\ \gamma_2 & \gamma_3 & \gamma_{m+2} \\ \gamma_{m+1} & \gamma_{m+2} & \gamma_{2m+1} \end{pmatrix} \begin{pmatrix} q_1 \\ \phi_1 p_2 \\ \phi_m p_{m+1} \end{pmatrix}, \begin{pmatrix} q_1 \\ \phi_1 p_2 \\ \phi_m p_{m+1} \end{pmatrix} \right\rangle.$$

Thus, $\Delta_m^{\alpha(\delta,\epsilon)} = \delta^2\epsilon^2\Delta_m^\alpha + \delta^2(1-\epsilon^2)\gamma_1(|\phi_1 p_0|^2 + |q_1|^2) + (1-\delta^2\epsilon^2)|q_0|^2 + 2\delta^2(1-\epsilon^2)\gamma_1 Re(\phi_1 p_1 \bar{q}_0)$. □

Theorem 2.3. *Let $\alpha : \alpha_0, \alpha_1, \alpha_2, \alpha_3, \dots$ be a weight sequence and let $\alpha(\delta) : \delta\alpha_0, \alpha_1, \alpha_2, \alpha_3, \dots$ be a weight sequence with $0 < \delta \leq 1$. If W_α is a semi weakly (1,m)–hyponormal, then $W_{\alpha(\delta)}$ is also semi weakly (1,m)-hyponormal.*

Proof. If W_α is a semi weakly (1,m)–hyponormal, then by Lemma 2.2, $\Delta_m^{\alpha(\delta)} = \delta^2\Delta_m^\alpha + (1-\delta^2)|q_0|^2 \geq 0$, for any $0 < \delta \leq 1$. Thus by Theorem 2.2, $W_{\alpha(\delta)}$ is semi weakly (1,m)-hyponormal. □

Theorem 2.4. *Let $\alpha : \alpha_0, \alpha_1, \alpha_2, \alpha_3, \dots$ be a weight sequence and let $\alpha(\delta,\epsilon) : \delta\alpha_0, \epsilon\alpha_1, \alpha_2, \alpha_3, \dots$ be a weight sequence with $0 < \delta, \epsilon \leq 1$. If W_α is a semi weakly (l,m)-hyponormal for $l \geq 2$ and $m > l$, then $W_{\alpha(\delta,\epsilon)}$ is also semi weakly (l,m)-hyponormal.*

Proof. Without loss of generality, we may assume that $\|W_\alpha\| = 1$. Since by Lemma 2.1, $\Delta_m^{\alpha(\delta,\epsilon)} = \delta^2\epsilon^2\Delta_m^\alpha + (1-\delta^2)|q_0|^2 + \delta^2(1-\epsilon^2)\gamma_1|q_1|^2 \geq 0$, for any $\delta, \epsilon \in (0,1]$, therefore by Theorem 2.2, $W_{\alpha(\delta,\epsilon)}$ is semi weakly (l,m)–hyponormal. □

3. 3-HYPONORMALITY

Lemma 3.1. *[1] If a and b are non-negative real numbers and c is a non-zero complex number, then $|z|^2 a + b + 2Re(zc) \geq 0$ for all complex number z if and only if $|c|^2 \leq ab$.*

A weighted shift W_α is m-hyponormal if and only if the Hankel matrix

$$A(n;m) = \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+m} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n+m} & \gamma_{n+m+1} & \cdots & \gamma_{n+2m} \end{pmatrix}$$

is positive for all $n \geq 0$, where $\gamma_0 = 1, \gamma_n = \alpha_0^2\alpha_1^2 \dots \alpha_{n-1}^2$ (called moments of W_α).

Proposition 3.1. *Let $\alpha : \alpha_0, \alpha_1, \alpha_2, \dots$ be a weight sequence of the weighted shift W_α and $\alpha(\epsilon) : \alpha_0, \epsilon\alpha_1, \alpha_2, \dots$ be the perturbed weight sequence, where $0 < \epsilon \leq 1$. If W_α is 3-hyponormal, then $W_{\alpha(\epsilon)}$ is also 3-hyponormal if and only if $1 \geq \epsilon \geq \frac{\gamma_1}{\sqrt{\gamma_1^2+r}}$.*

Proof. Here

$$\gamma_i' = \begin{cases} \gamma_i, & \text{for } i < 2 \\ \epsilon^2\gamma_i, & \text{for } i \geq 2 \end{cases} .$$

To show $W_{\alpha(\epsilon)}$ is 3-hyponormal, it is sufficient to check the positivity of the matrices $A'(0;3)$ and $A'(1;3)$.

Positivity of $A'(1;3)$: For any $z = (z_0, z_1, z_2, z_3)^T$,

$$\langle A'(1;3)z, z\rangle = \epsilon^2 \langle A(1;3)z, z\rangle + (1-\epsilon^2)\gamma_1|z_0|^2 \geq 0.$$

Positivity of $A'(0;3)$:

$$\langle A'(0;3)z, z\rangle = \epsilon^2 \langle A(0;3)z, z\rangle + (1-\epsilon^2)\gamma_0|z_0|^2 + (1-\epsilon^2)\gamma_1(z_1\bar{z}_0 + z_0\bar{z}_1).$$

If $z_1 = 0$, then $\langle A'(0;3)z, z\rangle \geq 0$. Consider the case $z_1 \in \mathbb{C} \setminus \{0\}$.

$$W_{\alpha(\epsilon)} \text{ is 3- hyponormal} \Leftrightarrow \langle A'(0;3)z, z\rangle \geq 0 \Leftrightarrow \frac{1}{|z_1|^2} \langle A'(0;3)z, z\rangle \geq 0$$

$$\Leftrightarrow \langle A'(0;3)z', z'\rangle \geq 0 \text{ where } z' = \frac{1}{z_1}z$$

$$\Leftrightarrow \epsilon^2 \langle A(0;3)z', z'\rangle + (1-\epsilon^2)|z_0'|^2 + 2(1-\epsilon^2)\gamma_1 Re(z_0') \geq 0$$

$$\Leftrightarrow \epsilon^2 r + (1-\epsilon^2)|z_0'|^2 + 2Re((1-\epsilon^2)\gamma_1 z_0') \geq 0, \text{ where } r = \langle A(0;3)z', z'\rangle$$

$$\Leftrightarrow (1-\epsilon^2)^2\gamma_1^2 \leq \epsilon^2(1-\epsilon^2)r, \text{ (using Lemma 3.1)}$$

$$\Leftrightarrow \epsilon \geq \frac{\gamma_1}{\sqrt{\gamma_1^2 + r}}$$

□

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