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# GRAPH WITH A GIVEN UPPER DOMINATION NUMBER

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ABSTRACT. In this paper we have given a method of constructing a graph with a given positive integer $r$ as its upper domination number using simple number theoretic concepts.

## 1. INTRODUCTION

All graphs considered here are finite, simple, undirected and loopless. A set $D$ of vertices in a graph $G = (V, E)$ is a dominating set of $G$ if every vertex in $V \setminus D$ is adjacent to some vertex in $D$. The concept of domination in graphs was first introduced by Konig [3], Berge [2] and Ore [4]. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of all dominating sets in $G$. The upper domination number is defined as the maximum cardinality of all minimal dominating sets in a graph $G$. The method of construction of a graph with 1. a given domination number is given by [5] and 2. a given maximal domination number is given by [6]. In this paper we have discussed the method of construction of a graph with a given positive integer as its upper domination number. In Section 2 we have given the definitions of the concepts used in this paper. In Section 3 we have established the existence of a graph $G$ with a given positive integer r as the upper domination number of $G$ and in Section 4 we have given a method of constructing such a graph and its application to communication network.

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## 2. Preliminaries

**Definition 2.1.** *The upper domination number of $G$ denoted by $\Gamma(G)$, is the maximum cardinality of all minimal dominating sets in $G$.*

**Definition 2.2.** *A dominating set $D$ of a graph $G$ is minimal if for each vertex $v \in D,\ D \setminus \{v\}$ is not a dominating set of $G$.*

**Definition 2.3.** *Let $m$ be a positive integer such that $m = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ where p's are primes and $'s > 0$. The arithmetic graph $V_m$ is defined as a graph with its vertex set as the set of all divisors of $m$ (excluding 1) and two distinct vertices $a, b$ are adjacent in this graph if $gcd(a,b) = p_i$ (where $p_i | m$).*

## 3. Existence of a graph with a given upper domination number

In this section, we show that given a positive integer $r$ there exists a connected graph $G$ whose upper domination number is $r$. We first prove the following two Lemmas.

**Lemma 3.1.** *Let $m$ be a positive integer such that $m = \prod_{i=1}^{n} p_i^{a_i}$ where $p_i' s$ are primes and $a_i' s$ are positive integers and $\mu(m) \neq 0$ where $\mu$ is the Mobius function[1]. Then the upper domination number of arithmetic graph $V_m$ is given by*

1. $\Gamma(V_m) = r,$ *if* $r < 4$.
2. $\Gamma(V_m) = r + 1,$ *if* $r \geq 4,$

*Proof.* The proof is trivial when $r = 1, 2$.

When $r = 3$, choosing $m = p_1 p_2 p_3$ where $p_1, p_2, p_3$ are any three distinct primes, construct the graph $V_m$ (see Figure-1). The vertex set $\{p_1, p_2, p_3\}$ is a minimal dominating set of maximum cardinality. Hence $\Gamma(V_m) = 3$.

When $r \geq 4$, choosing $m = p_1 p_2 p_3 \cdots p_r$, where $p_1, p_2, \cdots, p_r$ are r distinct primes, construct the graph $V_m$. Intuitively one can observe that the vertices of lower degree generaly constitute the minimal dominating set of maximum cardinality. Also $deg(p_1 p_2 \cdots p_t) > deg(p_1 p_2 \cdots p_{t+1})$ since $deg(p_1 p_2 \cdots p_t) = t + t(2^{r-t} - 1)$ and $deg(p_1 p_2 \cdots p_{t+1}) = (t+1) + (t+1)(2^{r-(t+1)} - 1)$. It follows that $D = \{m\} \cup$ {vertices which are formed of all $r - 1$ combinations of $r$ primes} is a minimal dominating set of maximum cardinality. Thus $\Gamma(V_m) = r + 1$. □

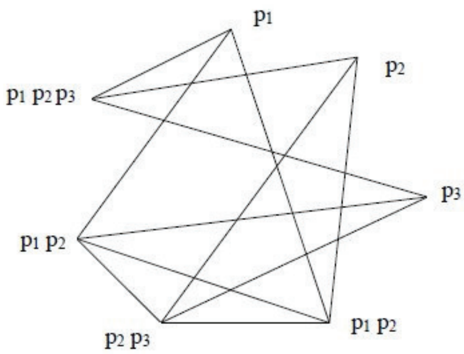


FIGURE 1

**Lemma 3.2.** *Let $m$ be a positive integer such that $m = p_1^2 p_2^2$ whee $p_1, p_2$ are any two distinct primes, then $\Gamma(V_m) = 4$.*

*Proof.* The minimal dominating set of maximum cardinality in $V_m$ is $\{p_1p_2, p_1^2p_2, p_1p_2^2, p_1^2p_2^2\}$ and thus $\Gamma(V_m) = 4$ □

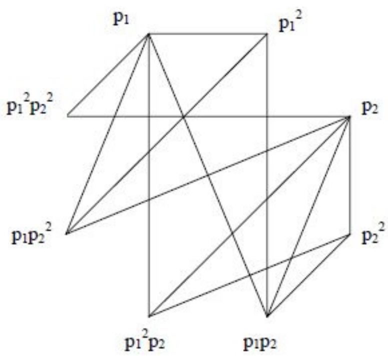


FIGURE 2

**Theorem 3.1.** *For a given positive integer $r$, there exists a graph $G$ with $r$ as its upper domination number.*

*Proof.* When $r = 1, 2, 3$ choosing m to be $p_1, p_1p_2, p_1p_2p_3$ (where $p_1, p_2, p_3$ are any $3$ distinct primes) respectively, we have from Lemma 3.1 that the $V_m$ graph will be a graph with the required upper domination number. When $r = 4$, choosing $m = p_1^2 p_2^2$ where $p_1, p_2$ are any two primes,then $V_m$ graph has the upper domination number $4$ from Lemma 3.2. When $r \geq 5$, choose $m = p_1 p_2 \cdots p_{r-1}$, where $p_i'$s are distinct primes, we obtain that $V_m$ graph has the upper domination number $r$ from Lemma 3.1. □

## 4. CONSTRUCTION

We now give an algorithm for the construction of a graph with given upper domination number using theorem-3.1.

**Algorithm:**

Input : Given positive integer $r$

Output : Graph with a given upper domination number $r$

Step 1 : Enter upper domination number $r$

Step 2 : If $r = 1$, enter the prime number $p_1$ and let $m = p_1$

Step 3 : If $r = 2$, enter the prime numbers $p_1, p_2$ and let $m = p_1p_2$

Step 4 : If $r = 3$, enter the prime numbers $p_1, p_2, p_3$ and let $m = p_1p_2p_3$.

Step 5 : If $r = 4$, enter the prime numbers $p_1, p_2$ and set $m = p_1^2p_2^2$.

Step 6 : If $r \geq 5$, enter the prime numbers $p_1, p_2, \cdots, p_{r-1}$ and set $m = p_1p_2p_3 \cdots p_{r-1}$.

Step 7 : The vertex set $v = \{d \mid d$ is a divisor of $m, d \neq 1\}$

Step 8 : Construct a $V_m$ graph with the vertex set $V$ in which two distinct vertices $a$ and $b$ are adjacent if $gcd(a, b) = p_i$ (where $p_i|m$)

**Illustrations.**

Graph with its upper domination number $5$. Now, since $\Gamma = 5$, enter the distinct primes $p_1, p_2, p_3, p_4$ and set $m = p_1p_2p_3p_4$, (see Figure-3). The vertext set of $V_m = \{d/d$ is a divisor of $m, d \neq 1\}$.

Adjacency : Two vertices $a$ and $b$ are adjacent if $gcd(a, b) = p_i$ (where $p_i|m$).

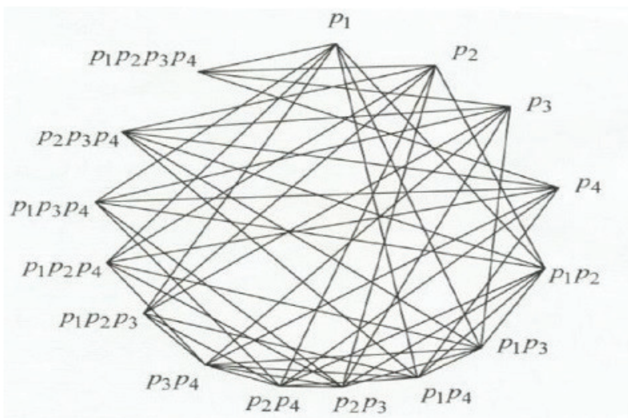


FIGURE 3

## 5. Applications

The construction of a graph with a given positive integer as its upper domination number has an interesting application. Consider a network in which each site of the network corresponds to a vertex of the graph. If two sites have direct communication link joining them the corresponding vertices of the graph are made adjacent. Suppose the number of transmitters in an organization is given. We have to construct a network using the capacity of the power of transmission of each transmitter completely. These transmitters have to be placed in the network at some sites such that every site in the network that does not have a transmitter is joined by a direct communication link to the site that has a transmitter. The utilization of all the transmitters to their complete capacity serves as a upper domination number and the network serves as the graph with this as the upper domination number.

## 6. Conclusions

In this article, the construction of a graph for any positive integer as its upper domination number has been effectively constituted by using the concepts of number theory. An elegant algorithm was employed for the construction of said graph. Also the application on communication Networks is flourished the significance of this article. It is also suggested that one may extend these ideas to construct a graph with a positive integer as other graphical parameter.

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