

A STUDY ON ASSET PRICE WITH STOCHASTIC VOLATILITY

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ABSTRACT. In this paper a detailed study on the situations wherein the environmental impacts such as the psychological pushups and pull-downs produce an upward or downward jump in the volatility and these jumps occur in an alternating way and in between any two successive jumps the asset price changes according to a diffusion process. The study is based on a type of finance market in a diffusion environment coupled with a jump-type stochastic volatility. The impact of the environment on the movement of the price is reflected in the volatility of the stock-price. To be specific, the problem of asset pricing in the context of a diffusion environment whose volatility switches between two constant values alternately at random time points is governed by a Poisson process.

1. INTRODUCTION

There is available a massive corpus of literature on jump-diffusion models in finance. To mention a few phenomenal works, books by Cont and Tankov [1], Kijima [2] top the list. To include 'volatility smile' in option pricing varied models are suggested. Stochastic volatility and ARCH models are seen in Hull and White [3]. We see a constant elasticity model (CEV) model by Cox and Ross [4].

Other relevant references are [5-11].

Jump-diffusion models lead to incomplete markets. Hence, there are many ways to choose the pricing measure. A few popular methods include mean-variance hedging, local mean-variance hedging, entropy methods, indifference pricing, etc.

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While appropriating these methods, partial integro-differential equations may be used but it involves difficulties related to barrier-crossing problems. Due to its convolutions and irregularities, direct application of *Itô* formula and Feymann - Kac formula may be complex. However but many a times we find that, Martingale arguments are handy in solving such problems. Laplace transforms are deployed for solving equations, which may lead to expressions which are easy and solvable and even quicker calculations are made possible.

Even though the problem is not just easy to find analytical solutions for option pricing, we attempt to give one in this paper. In this study we shall focus on the issue related to jump-diffusion models for asset pricing with Stochastic Volatility in financial engineering. Laplace transforms are employed to evaluate pricing options and evaluations with the partial integro-differential equations related to barrier-crossing problems are also given.

2. ASSET PRICE OF A FINANCIAL MARKET MODEL

The basic assumption is by considering a market which consists of a bond and a stock. Let B_t and S_t be the prices of the bond and stock at time t respectively. These two financial instruments are governed by the dynamical equations:

$$(2.1) \quad dB_t = r_t B_t dt$$

$$(2.2) \quad dS_t = a_t S_t dt + \sigma_t S_t dW_t$$

where $B_0 = 1$ and W_t is a standard Brownian motion starting at 0. Here S_0 may be a constant or a random variable independent of W_t , a_t and σ_t are respectively called the drift and the volatility of the stock market. For simplicity, it is assumed that the interest rate $r_t = r$ is a positive constant. From the equation (2.1) and the initial condition, it is evident that $B_t = e^{rt}$. Further assume that $a_t = a$, a positive constant and the volatility σ_t is a stochastic process. Suppose that the volatility σ_t alternates between two constant values σ_1 and σ_2 and the fluctuations occur at random time points. Then we note that volatility σ_t is a two state stochastic process with state space $\{\sigma_1, \sigma_2\}$. Assume that $\lim_{\Delta \rightarrow 0} \sigma_{-\Delta} = \sigma_2$ and $\lim_{\Delta \rightarrow 0} \sigma_{\Delta} = \sigma_1$. Let $T_1, T_2, T_3, \dots$ be the successive sojourn times of the volatility σ_t we observe that the sojourn time in the state σ_1 are $T_1, T_3, T_5, \dots$ and the sojourn time in the state σ_2 are $T_2, T_4, T_6, \dots$ and the sojourn times $T_1, T_3, T_5, \dots$ are independent and identically distributed with probability density function

$f(t)$. Similarly the sojourn times in the state σ_2 are independent and identically distributed with probability density function $g(t)$.

Since, that all the sojourn times are independent, the Counting process N_t is defined as

$$N_t = \max\{n/T_1 + T_2 \cdots + T_n \leq t, T_1 + T_2 \cdots + T_n + T_{n+1} > t\}, t > 0$$

and also N_t gives the total number of fluctuations of σ_t in the interval $(0, 1]$ and

$$(2.3) \quad \sigma_t = \left(\frac{\sigma_1 + \sigma_2}{2} \right) + \left(\frac{\sigma_1 - \sigma_2}{2} \right) (-1)^{N_t}$$

Let $\gamma = \frac{\sigma_1 + \sigma_2}{2}$, $\delta = \frac{\sigma_1 - \sigma_2}{2}$. Now the equation (2.2) becomes

$$(2.4) \quad dS_t = aS_t dt (\gamma + \delta(-1)^{N_t}) S_t dW_t$$

Since, the stock market is governed by the probability space (Ω, F, P) the equation (2.4) can be solved by applying the *Itô* - differentiation formula

$$\begin{aligned} d\log_e S_t &= \log_e \left\{ 1 + \frac{dS_t}{S_t} \right\} = \frac{dS_t}{S_t} - \frac{1}{2} \left\{ \frac{dS_t}{S_t} \right\}^2 + \frac{1}{3} \left\{ \frac{dS_t}{S_t} \right\}^3 - \cdots \\ &= a dt + \left\{ \gamma + \delta(-1)^{N_t} \right\} dW_t - \frac{1}{2} \left[a dt + \left\{ \gamma + \delta(-1)^{N_t} \right\} dW_t \right]^2 + \cdots \\ &= \left[a - \frac{1}{2} \left\{ \left(\frac{\sigma_1^2 + \sigma_2^2}{2} \right) + \left(\frac{\sigma_1^2 - \sigma_2^2}{2} \right) (-1)^{N_t} \right\} \right] dt + \left\{ \gamma + \delta(-1)^{N_t} \right\} dW_t. \end{aligned}$$

Hence,

$$\begin{aligned} S_t &= S_0 \exp \left[at - \frac{1}{2} \left\{ \left(\frac{\sigma_1^2 + \sigma_2^2}{2} \right) t + \left(\frac{\sigma_1^2 - \sigma_2^2}{2} \right) \int_0^t (-1)^{N_u} du \right\} \right] \\ &\quad \times \exp \left\{ \gamma W_t + \delta \int_0^t (-1)^{N_u} dW_u \right\} \\ (2.5) \quad &= S_0 \exp \left[\alpha t + \beta \int_0^t (-1)^{N_u} du \right] \times \exp \left\{ \gamma W_t + \delta \int_0^t (-1)^{N_u} dW_u \right\}, \end{aligned}$$

where $\alpha = a - \left(\frac{\sigma_1^2 + \sigma_2^2}{4} \right)$ and $\beta = \left(\frac{\sigma_1^2 - \sigma_2^2}{4} \right)$.

Setting $X_t = \alpha t + \beta \int_0^t (-1)^{N_u} du$ and $Y_t = \gamma W_t + \delta \int_0^t (-1)^{N_u} dW_u$, the equation (2.5) gives

$$(2.6) \quad S_t = S_0 \exp(X_t + Y_t)$$

Since N_t is an alternating renewal process, the probabilities $p_n(t) = P_r\{N_t = n\}$, $n = 1, 2, 3, \dots$ are given by

$$\begin{aligned} p_0(t) &= \overline{F}(t), \\ p_1(t) &= \int_0^t f(u) \overline{G}(t-u) du, \\ p_{2n}(t) &= \int_0^t (f * g)^{(n)}(u) \overline{F}(t-u) du, \quad n = 1, 2, 3, \dots \\ p_{2n+1}(t) &= \int_0^t \left((f * g)^{(n)} * f \right)(u) \overline{G}(t-u) du, \quad n = 1, 2, 3, \dots, \end{aligned}$$

where

$$(f * g)^{(1)}(t) = (f * g)(t) = \int_0^t f(u)g(t-u) du$$

and

$$(f * g)^{(n)}(t) = \int_0^t (f * g)^{(n-1)}(u) (f * g)(t-u) du, \quad n = 2, 3, \dots$$

The Laplace transforms $p_n(t)$, $n = 0, 1, 2, \dots$ are given by

$$\begin{aligned} \hat{p}_0(s) &= \frac{1 - \hat{f}(s)}{s}, \\ \hat{p}_1(s) &= \hat{f}(s) \left(\frac{1 - \hat{g}(s)}{s} \right), \\ \hat{p}_{2n}(s) &= \left\{ \hat{f}(s) \right\}^n \left\{ \hat{g}(s) \right\}^n \left(\frac{1 - \hat{f}(s)}{s} \right), \quad n = 1, 2, 3, \dots \\ \hat{p}_{2n+1}(s) &= \left\{ \hat{f}(s) \right\}^{n+1} \left\{ \hat{g}(s) \right\}^n \left(\frac{1 - \hat{g}(s)}{s} \right), \quad n = 1, 2, 3, \dots \end{aligned}$$

If $G(\theta, t) = \sum_{n=0}^{\infty} p_n(t)\theta^n$, then the Laplace transform of

$$G(\theta, t) = \hat{G}(\theta, s) = \sum_{n=0}^{\infty} \hat{p}_n(s)\theta^n = \frac{1 - \hat{f}(s) + \theta \hat{f}(s) \{1 - \hat{g}(s)\}}{s \{1 - \theta^2 \hat{f}(s) \hat{g}(s)\}}.$$

In the particular case $f(t) = \lambda e^{-\lambda t}$, $g(t) = \mu e^{-\mu t}$, we have $\hat{G}(\theta, s) = \frac{s + \lambda\theta + \mu}{(s + \lambda)(s + \mu) - \lambda\mu\theta^2}$. The roots of the equation $(s + \lambda)(s + \mu) - \lambda\mu\theta^2 = 0$ are real and distinct, since its discriminant is $(\lambda - \mu)^2 + 4\lambda\mu\theta^2$ which is > 0 . Let α and β be the roots, then $\alpha = \frac{-(\lambda + \mu) + \sqrt{(\lambda - \mu)^2 + 4\lambda\mu\theta^2}}{2}$ and $\beta = \frac{-(\lambda + \mu) - \sqrt{(\lambda - \mu)^2 + 4\lambda\mu\theta^2}}{2}$.

Now $\hat{G}(\theta, s) = \frac{s + \lambda\theta + \mu}{(s - \alpha)(s - \beta)} = \frac{\alpha + \lambda\theta + \mu}{\alpha - \beta} \frac{1}{s - \alpha} - \frac{\beta + \lambda\theta + \mu}{\alpha - \beta} \frac{1}{s - \beta}$. Inverting, we obtain

$$\begin{aligned} G(\theta, t) &= \frac{1}{\alpha - \beta} \{(\alpha + \lambda\theta + \mu) e^{\alpha t} - (\beta + \lambda\theta + \mu) e^{\beta t}\} \\ &= \frac{e^{-\frac{(\lambda + \mu)}{2}t}}{\sqrt{(\lambda - \mu)^2 + 4\lambda\mu\theta^2}} \left\{ \sqrt{(\lambda - \mu)^2 + 4\lambda\mu\theta^2} \cosh \left(\sqrt{\frac{(\lambda - \mu)^2 + 4\lambda\mu\theta^2}{2}} t \right) \right. \\ &\quad \left. + (\mu - \lambda + 2\lambda\theta) \sinh \left(\sqrt{\frac{(\lambda - \mu)^2 + 4\lambda\mu\theta^2}{2}} t \right) \right\}. \end{aligned}$$

In the particular case $f(t) = g(t) = \lambda e^{-\lambda t}$, we have $G(\theta, t) = e^{-\lambda t + \lambda\theta t}$ and hence we get the classical result $p_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$, $n = 0, 1, 2, \dots$

We observe that $E[X_t + Y_t] = \alpha t + \beta \int_0^t E[(-1)^{N_u}] du = \alpha t + \frac{\beta}{2\lambda}(1 - e^{-2\lambda t})$.

Setting $Z_t = X_t + Y_t - \alpha t - \frac{\beta}{2\lambda}(1 - e^{-2\lambda t})$, we have $E[Z_t/F_s] = E[X_t/F_s] + E[Y_t/F_s] - \alpha t - \frac{\beta}{2\lambda}(1 - e^{-2\lambda t})$.

But by direct computation,

$$\begin{aligned} E[X_t/F_s] &= E \left[\alpha t + \beta \int_0^t (-1)^{N_u} du / F_s \right] \\ &= \alpha t + \beta E \left[\int_0^s (-1)^{N_u} du + \int_s^t (-1)^{N_u} du / F_s \right] \\ &= \alpha t + \beta \int_0^s (-1)^{N_u} du + \beta \int_s^t E[(-1)^{N_u}] du \end{aligned}$$

$$= \alpha t + X_s - \alpha s + \beta \int_s^t e^{-2\lambda u} du = \alpha t + X_s - \alpha s + \beta \left\{ \frac{e^{-2\lambda s} - e^{-2\lambda t}}{2\lambda} \right\}.$$

Similarly,

$$\begin{aligned} E[Y_t/F_s] &= E \left[\gamma W_t + \delta \int_0^t (-1)^{N_u} dW_u / F_s \right] \\ &= \gamma W_s + \delta E \left[\int_0^s (-1)^{N_u} dW_u + \int_s^t (-1)^{N_u} dW_u / F_s \right] \\ &= \gamma W_s + \delta \int_0^s (-1)^{N_u} dW_u + \int_s^t (-1)^{N_u} E[dW_u] \\ &= \gamma W_s + \delta \int_0^s (-1)^{N_u} dW_u = Y_s. \end{aligned}$$

Hence,

$$\begin{aligned} E[Z_t/F_s] &= \alpha t + X_s - \alpha s + \beta \left\{ \frac{e^{-2\lambda s} - e^{-2\lambda t}}{2\lambda} \right\} + Y_s - \alpha t - \frac{\beta}{2\lambda} (1 - e^{-2\lambda t}) \\ &= X_s + Y_s - \alpha s - \frac{\beta}{2\lambda} (1 - e^{-2\lambda s}) = Z_s. \end{aligned}$$

This establishes that Z_t is a martingale.

3. CONCLUSION

Hence, stochastic models of financial markets that allow continuous trading have been proposed and analyzed. Continuous time stochastic finance models within the context of pricing of capital asset have led to an extensive switching type stochastic volatility. We have consistency with the data of financial institutions underlying asset returns as diffusions coupled with stochastic jumps. Hence it can be extended and modelled that the motion of asset returns as diffusions coupled with stochastic volatility and determine the arbitrage-free price of a call option by assuming the squared volatility as log-normal diffusion independent of the stock price.

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