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ESTIMATION OF p-ADIC SIZES OF PARTIAL DERIVATIVE FOR CERTAIN QUARTIC POLYNOMIAL

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ABSTRACT. The objective of this paper is to estimate the p-adic sizes of common zeros of partial derivative polynomials associated with a certain quartic polynomial at any point of neighbourhood restricted to some conditions by using Newton polyhedron technique. The p-adic sizes of such common zeros can be obtained from intersection points of indicator diagrams associated with the polynomials. Subsequently, p-adic sizes of common zeros will be determined explicitly in terms of coefficients of dominant terms of polynomials.

1. INTRODUCTION

In our discussion, we use notation of $\mathbb{Z}_p$ as the ring of p-adic integers , (Ω_p) is the completion of algebraic closure of $\mathbb{Q}_p$ the field of rational p-adic numbers and $(ord_p x)$ as the highest power of p which divides x. It follows that for any rational number x and y, $ord_p x = \infty$ if and only if $ord_p x = 0$; $ord_p(xy) = ord_p x + ord_p y$ and $ord_p(x + y) \geq \min\{ord_p x, ord_p y\}$, with equality if $ord_p x \neq ord_p y$. Let $\underline{x} = (x_1, x_2, x_3, \ldots, x_n)$ denote a vector in the space $\mathbb{Z}^n$ where $\mathbb{Z}$ denotes the ring of integers. Let q be a positive integer and f a polynomial in $\mathbb{Z}[x]$. The multiple exponential sums associated with f is defined as

$$S(f;q) = \Sigma_{x mod q} e^{((2pif(x))/q)},$$

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where the sum is taken over a complete set of $\underline{x}$ mod q. Obtaining the precise upper bound of multiple exponential sums have been the interest of many number theorist. Earlier works of some researchers related to a polynomial $f(x,y)$ over $\mathbb{Z}_p$, showed that such estimations can be obtained from the number and p-adic sizes of the common zeros of partial derivative polynomials associated with the $f(x,y)$ considered. Mohd Atan [4] first showed that the p-adic sizes of the zeros of a polynomial can be obtained by using Newton polyhedral method. Subsequently, Mohd Atan [5] determined the p-adic sizes of the common zeros to two polynomials by studying the intersection points of the indicator diagrams associated with the polynomials considered.

Newton polyhedron technique a tool developed by Mohd Atan and Loxton [6] is an analogue of Newton polygon as defined by Koblitz [3]. In order to ovecome the problem of construction of such a Newton polyhedron, Mohd Atan and Loxton [6] introduced the indicator diagram as a tool that captures the essentials of Newton polyhedron and represents it in a simpler form. Researchers such as Mohd Atan [1], Chan [2], Sapar and Mohd Atan [7,8] have employed the Newton polyhedron method to obtain estimations of p-adic sizes of the common zeros of partial derivative polynomials associated with two variable polynomials. Aminuddin [9] concentrating of finding the cardinality of the set of solution associated to a polynomial of cubic form. Lasaraiya [10] give an estimation the p-adic sizes of common zeros of partial derivative polynomials associated with certain class of polynomial of degree eleven.

2. p-ADIC ORDERS OF ZEROS OF A POLYNOMIALS

In this section, we focus on finding the p-adic sizes of common zeros of polynomials associated with quartic polynomial restricted with conditions of $ord_p ac^2 > ord_p b^3$. We need the following definitions and theorem developed by [4].

Definition 2.1. *Let $f(x,y) = \sum a_{ij}x^iy^j$ be a polynomial of degree n in $\Omega_p[x,y]$. By mapping the terms $T_ij = a_ijx^iy^j$ of $f(x,y)$ to the points $P_{ij} = a_{ij}x^iy^j$ in the three-dimensional Euclidean space R^3. The set of points P_{ij} is called as the Newton diagram of $f(x,y)$.*

Definition 2.2. *Let $f(x,y) = \sum a_{ij}x^iy^j$ be a polynomial of degree n in $\Omega_p[x,y]$. By mapping the terms $T_{ij} = a_{ij}x^iy^j$ of $f(x,y)$ to the points $P_{ij} = a_{ij}x^iy^j$ in the Euclidean* *space, the Newton polyhedron of $f(x,y)$ is defined to be the lower convex hull of the set S of points P_{ij}, $0 < i,j < n$. It is the highest convex connected surface which passes through or below the points in S. If $a_{ij} = 0$ for some (i,j) then $ord_p a_{ij} = \infty$.*

Definition 2.3. *The set of lines associated with the Newton polyhedron is denoted by N_f. Let $(\mu_i, \lambda_i, 1)$ be the normalized upward-pointing normals to the faces $F(i)$ of N_f for a polynomial $f(x,y)$ in $\Omega_p[x,y]$. The point $(\mu_i, \lambda_i, 1)$ is mapping to the point (μ_i, λ_i) in the $x-y$ plane. If F_r and F_s are adjacent faces in N_f, sharing a common edge, we construct the straight line joining (μ_r, λ_r) and (μ_s, λ_s). If F_r shares a common edges with a vertical face F say $\alpha x + \beta y = \gamma$ in N_f, we construct the straight line segment joining (μ_r, λ_r) and the appropriate point at infinity that corresponds to the normal F, that is the segment along a line with a slope $-\alpha/\beta$.*

Theorem 2.1. *Let p be a prime. Suppose f and g are polynomials in $\mathbb{Z}_p[x,y]$. Let (μ, λ) be a point of intersection of the indicator diagrams associated with f and g at the vertices or simple points of intersections. Then, there are ξ and η in Ω_p^2 satisfying $f(\xi,\eta) = g(\xi,\eta) = 0$ and $ord_p\xi = \mu_1, ord_p\eta = \mu_2$.*

3. MAIN RESULT

In this section, we find the p-adic sizes of common zeros for the certain quartic polynomial of the form $f(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + rx + sy + t$ in the neighbourhood of (x_0, y_0) subject to the condition $ord_p\frac{b}{c} > ord_p\lambda > ord_p\frac{a}{b}$. Two cases will be shown in this section, that is $ord_p\lambda = \frac{1}{2}ord_p\frac{a}{b}$ and $ord_p\lambda = \frac{1}{2}ord_p\frac{c}{e}$. From this study, the result is in the following theorem:

Theorem 3.1. *Let $f(x,y) = ax^4+bx^3y+cx^2y^2+dxy^3+ey^4+rx+sy+t$ be a polynomial in $\mathbb{Z}_p[x,y]$ with $p > 3$. Let $\alpha > 0$, $\delta = max\{ord_pa, ord_pb, ord_pc, ord_pd, ord_pe\}$ and $ord_p(36ae - c^2)^2 > ord_p9(6be - cd)(6ad - bc)$. Suppose $(x_0, y_0) \in \omega_p^2$, $ord_pb^2 > ord_pac$, $CEI - DEH - AI^2 = 0$ and $DE^2 - BEI + AHI = 0$ where $A = 108b^2e - 8c^3$, $B = 864abe + 27b^2d - 36bc^2$, $C = 1728a^2e + 216abd - 54b^2c$, $D = 432a^2d - 27b^3$, $E = 9bd - 4c^2$, $H = 6(6ad - bc)$ and $I = 3(4ac - 3b^2)$. If $ord_pf_x(x_0,y_0), ord_pf_y(x_0,y_0) \geq \alpha > \delta$ and $ord_p\lambda = \frac{1}{2}ord_p\frac{a}{c}$, then there exists (ξ,η) such that $f_x(\xi,\eta) = 0, f_y(\xi,\eta) = 0$ and $ord_p(\xi - x_0) > \frac{1}{3}(\alpha - 2\delta)$, $ord_p(\eta - y_0) > \frac{1}{3}(\alpha - 2\delta)$.*

In order to prove this theorem, we begin with several lemmas and corollaries before arriving at the estimation of p-adic sizes of common zeros. It can be shown that all these lemmas and corollaries are true.

In the following lemma, we show that $ord_p\gamma_i = ord_p(3b + 2\lambda c) - ord(4a + \lambda b)$ and $ord_p(\gamma_1 - \gamma_2) = ord_p(8ac - 3b^2) - 2ord_p(4a + \lambda b)$ where $\gamma_i = \frac{3b+2\lambda_i c}{3(4a+\lambda_i b)}$ for $i = 1, 2$ and λ is either λ_1 or λ_2 the roots of $k(x)$. This lemma will then be applied in the proof of Lemma 3.3.

Lemma 3.1. *Let $p > 3$ be a prime and a, b, c, d and e in $\mathbb{Z}_p$. Let λ_1, λ_2 be the zeros of $k(x)$. Let $\gamma_i = \frac{(3b+2\lambda_i c)}{3(4a+\lambda_i b)}$ for $i = 1, 2$. If $ord_p(36ae - c^2)^2 > ord_p 9(6be - cd)(6ad - bc)$, then $ord_p\gamma_i = ord_p(3b + 2\lambda c) - ord_p(4a + \lambda b)$ and $ord_p(\gamma_1 - \gamma_2) = ord_p(8ac - 3b^2) - 2ord_p(4a + \lambda b)$ where λ is either λ_1 or λ_2.*

In Lemma 3.2, the sizes of ord_px and ord_py are given in terms of W,γ_1 and γ_2. This assertion will be applied in the proof of Lemma 3.3.

Lemma 3.2. *Suppose $p > 3$ be a prime. Let (x, y) be a point in Ω_p^2 and $U = x + \gamma_1 y$, $V = x + \gamma_2 y$ where γ_i are rational numbers for $i = 1, 2$. Then $ord_px \geq ord_pW - ord_p(\gamma_1 - \gamma_2)$ and $ord_py \geq ord_pW - ord_p(\gamma_1 - \gamma_2)$ where W is either U or V and γ is either γ_1 or γ_2.*

In the lemma below, we show that $ord_p\frac{b}{c} > \frac{1}{2}ord_p\frac{a}{c} > ord_p\frac{a}{b}$ can be obtained from the condition $ord_pb^2 > ord_pac$.

Lemma 3.3. *Let $p > 3$ be a prime and a, b and c in $\mathbb{Z}_p$. If $ord_pb^2 > ord_pac$, then $ord_p\frac{b}{c} > \frac{1}{2}ord_p\frac{a}{c} > ord_p\frac{a}{b}$.*

In the following lemma, we apply the condition $ord_p\frac{b}{c} > ord_p\lambda > ord_p\frac{a}{b}$ in the estimate of common zeros in terms of a, c, λ and H_0, where λ is the roots of $k(x)$ and $H_0 \in \Omega_p$.

Lemma 3.4. *Let $p > 3$ be an odd prime, λ_i be the roots of $k(x)$ for $i = 1, 2$, and a, b, c be integers. Let $\gamma_i = \frac{(3b+2\lambda_i c)}{3(4a+\lambda_i b)}$ for $i = 1, 2$ and λ be either λ_1 or λ_2. Suppose $ord_p\frac{b}{c} > ord_p\lambda > ord_p\frac{a}{b}$. Let (μ, η) be a common solution of $U = x + \gamma_1 y$ and $V = x + \gamma_2 y$. If $ord_p(\mu + \gamma_i\eta) = \frac{1}{3}ord_p\frac{H_0}{(4a+\lambda b)}$ for $i = 1, 2$ where $H_0 \in \Omega_p$, then $ord_p\mu \geq \frac{1}{3}(ord_pH_0 - ord_pa)$ and $ord_p\eta \geq \frac{1}{3}(ord_pH_0 + 2ord_pa - 3ord_pc - 3ord_p\lambda)$.*

Corollary below is a consequence of Lemma 3.4 where $ord_p\lambda = \frac{1}{2}ord_p\frac{a}{c}$.

Corollary 3.1. *Let p be an odd prime and λ_i be the roots of $k(x)$, $\gamma_i = \frac{(3b+2\lambda_i c)}{3(4a+\lambda_i b)}$ for $i = 1, 2$ and λ be either λ_1 or λ_2. (μ, η) be a common solution of $U = x + \gamma_1 y$ and $V = x + \gamma_2 y$. Suppose $ord_pb^2 > ord_pac$. If $ord_p = \frac{1}{2}ord_p\frac{a}{c}$ and $ord_p(\mu + \gamma_i\eta) = \frac{1}{3}ord_p\frac{H_0}{(4a+\lambda b)}$ for $i = 1, 2$ where $H_0 \in \Omega_p$, then $ord_p\mu \geq \frac{1}{3}(ord_pH_0 - ord_pa)$ and $ord_p\eta \geq \frac{1}{3}(ord_pH_0 + \frac{1}{2}ord_pa - \frac{3}{2}ord_pc)$.*

Corollary below is obtained from Lemma 3.4 by considering $ord_p\lambda = \frac{1}{2}ord_p\frac{c}{e}$. Its result will be applied in Lemma 3.6.

Corollary 3.2. *Let p be an odd prime and λ_i be the roots of $k(x)$ for $i = 1, 2$. Let $\gamma_i = \frac{(3b+2\lambda_i c)}{3(4a+\lambda_i b)}$ for $i = 1, 2$ and λ be either λ_1 or λ_2 where $ord_p\frac{b}{c} > ord_p\lambda > ord_p\frac{a}{b}$. Let (μ, η) be a common solution of $U = x + \gamma_1 y$ and $V = x + \gamma_2 y$. If $ord_p\lambda = \frac{1}{2}ord_p\frac{c}{e}$ and $ord_p(\mu + \gamma_i\eta) = \frac{1}{3}ord_p\frac{H_0}{(4a+\lambda b)}$ for $i = 1, 2$ where $H_0 \in \omega_p$, then $ord_p\mu \geq \frac{1}{3}(ord_pH_0 - ord_pa)$ and $ord_p \geq \frac{1}{3}(ord_pH_0 + 2ord_pa - \frac{9}{2}ord_pc + \frac{3}{2}ord_pe)$.*

Suppose $\alpha > 0$ and δ is maximum of the p-adic orders for the coefficients of the dominant terms of $f(x, y)$. Now, by using the condition of $ord_pb^2 > ord_pac$, $ord_p\lambda = \frac{1}{2}ord_p\frac{a}{c}$ and Corollary 3.1, we give the p-adic orders of common zeros in terms of α and δ in the following lemma.

Lemma 3.5. *Let p be an odd prime and $f(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + rx + sy + t$ in $\mathbb{Z}_p[x, y]$. Suppose λ_i for $i = 1, 2$ are the roots of $k(x)$. Let $\gamma_i = \frac{(3b+2\lambda_i c)}{3(4a+\lambda_i b)}$ for $i = 1, 2$ and (μ, η) be a common solution of $U = x + \gamma_1 y$ and $V = x + \gamma_2 y$. Suppose $\alpha > 0$, $\delta = max\{ord_pa, ord_pb, ord_pc, ord_pd, ord_pe\}$ and $ord_p(\mu + \gamma_i\eta) = \frac{1}{3}ord_pH_0(4a + \lambda b)$ for $i = 1, 2$ where $H_0 = f_x(x_0, y_0) + \lambda f_y(x_0, y_0)$and λ is either λ_1 or λ_2. Suppose $ord_pb^2 > ord_pac$. If $ord_pf_x(x_0, y_0), ord_pf_y(x_0, y_0) \geq \alpha > \delta$, $ord_p\lambda = \frac{1}{2}ord_p\frac{a}{c}$, then $ord_p\mu$, $ord_p\eta > \frac{1}{3}(\alpha - 2\delta)$.*

In the following assertion, we give the p-adic orders of common zeros in terms of α and δ under the condition $ord_p\frac{b}{c} > ord_p\lambda > ord_p\frac{a}{b}$, $ord_p\lambda = \frac{1}{2}ord_p\frac{c}{e}$.

Lemma 3.6. *Let p be an odd prime and $f(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + rx + sy + t$ in $\mathbb{Z}_p[x, y]$, λ_i be the roots of $k(x)$, $\gamma_i = \frac{(3b+2\lambda_i c)}{3(4a+\lambda_i b)}$ for $i = 1, 2$ and λ be either λ_1 or λ_2. Suppose $ord_p\frac{b}{c} > ord_p\lambda > ord_p\frac{a}{b}$. Let (μ, η) be a common solution of $U = x + \gamma_1 y$ and $V = x + \gamma_2 y$. Suppose $\alpha > 0$, $\delta = max\{ord_pa, ord_pb, ord_pc, ord_pd, ord_pe\}$ and $\alpha > \delta$. If $ord_pf_x(x_0, y_0), ord_pf_y(x_0, y_0) \geq \alpha$, $ord_p\lambda = \frac{1}{2}ord_p\frac{c}{e}$ and $ord_p(\mu + \gamma_i\eta) = \frac{1}{3}ord_p\frac{H_0}{(4a+?b)}$ for $i = 1, 2$ where $H_0 = f_x(x_0, y_0) + \lambda f_y(x_0, y_0)$, $H_0 \in \omega_p$, then $ord_p\mu \geq \frac{1}{3}(\alpha - 2\delta)$, $ord_p\eta > \frac{1}{3}(\alpha - 3\delta)$ or $ord_p\eta > \frac{1}{3}(\alpha - 4\delta)$.*

In Lemma 3.7, we show that the partial derivative polynomials associated with $f(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + rx + sy + t$ can be rewritten into a simpler form. Note that this lemma will be used repeatedly in the proof of theorem in this section.

Lemma 3.7. *Let $f(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + rx + sy + t$ be a polynomial in $\mathbb{Z}_p[x,y]$ with $p > 3$. Suppose λ is a constant such that $\frac{(2c+3\lambda d)}{(4a+\lambda b)} - 3\left[frac(3b+2\lambda c)3(4a+\lambda b)\right]^2 = 0$ and $\frac{(d+4\lambda e)}{(4a+\lambda b)} - \left[\frac{(3b+2\lambda c)}{3(4a+\lambda b)}\right]^3 = 0$. Then $(f_x+\lambda f_y)(x,y) = (4a+\lambda b)\left[x + \frac{(3b+2\lambda c)}{3(4a+\lambda b)}y\right]^3 + r + \lambda s$.*

Lemma below gives the condition that ensure the existence of common zeros for $m(x)$ and $n(x)$.

Lemma 3.8. *Let $m(x) = Ax^3+Bx^2+Cx+D$ and $n(x) = Ex^2+Hx+I$ be polynomials in $\mathbb{Z}_p[x,y]$ with $p > 3$. If $CEI - DEH - AI^2 = 0$ and $DE^2 - BEI + AHI = 0$, then $m(x)$ and $n(x)$ have two common roots.*

Lemma below shows the p-adic orders of common zeros of $f(U,V) = U^3+aU^2+bU+c$ and $g(U,V) = V^3+rV^2+sV+t$ can be obtained from the combination of indicator diagrams associated with the Newton polyhedra of $f(x,y)$ and $g(x,y)$. Note that both U and V are in terms of X and Y as stated in the proof of Theorem 3.1.

Lemma 3.9. *Suppose $f(U,V) = U^3+aU^2+bU+c$ and $g(U,V) = V^3+rV^2+sV+t$ are polynomials in $\mathbb{Z}_p[U,V]$. Let (μ,λ) be a point of intersection of the indicator diagrams associated with the Newton polyhedra of $f(U,V)$ and $g(U,V)$. Then there exists (α,β) in ω_p^2 such that $f(\alpha,\beta) = 0, g(\alpha,\beta) = 0$, $ord_p\alpha = \mu = \frac{1}{3}ord_p c$ and $ord_p\beta = \lambda = \frac{1}{3}ord_p t$.*

Proof of Theorem 3.1

Proof. Given $f(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + rx + sy + t$. By taking the partial derivatives of the polynomial $f(x,y)$ with respect to x and y, it can be shown that $(f_x + \lambda f_y)(x,y) = (4a+\lambda b)\left(x + \frac{(3b+2\lambda c)}{3(4a+\lambda b)}y\right)^3 + r + s$ where λ is a constant. Let $X = x - x_0$ and $Y = y - y_0$. Then

$$(h+\lambda g)(X+x_0, Y+y_0) = (4a+\lambda b)\left[X + x_0 + \frac{3b+2\lambda c}{3(4a+\lambda b)}(Y+y_0)\right]^3 + r + \lambda s. \tag{3.1}$$

If

$$\frac{2c+3\lambda d}{4a+\lambda b} - 3\left(\frac{3b+2\lambda c}{3(4a+\lambda b)}\right)^2 = 0 \tag{3.2}$$

and

$$\frac{d+4\lambda e}{4a+\lambda b}-\left(\frac{3b+2\lambda c}{3(4a+\lambda b)}\right)^3=0, \tag{3.3}$$

by expanding equations (3.2) and (3.3), we obtain $m(\lambda)$ and $n(\lambda)$ in the form $m(\lambda)=A\lambda^3+B\lambda^2+C\lambda+D$ and $n(\lambda)=E\lambda^2+H\lambda+I$ where $A=108b^2e-8c^3$, $B=(864abe+27b^2d-36bc^2$, $C=1728a^2e+216abd-54b^2c$, $D=432a^2d-27b^3$, $E=9bd-4c^2$, $H=6(6ad-bc)$ and $I=3(4ac-3b^2)$. By the conditions $CEI-DEH-AI^2=0$ and $DE^2-BEI+AHI=0$, there exists at most two common roots for $m(x)$ and $n(x)$ by Lemma 3.8. Now by substituting equation (3.2) into (3.3), we have

$$\frac{d+4\lambda e}{4a+\lambda b}=\left(\frac{2c+3\lambda d}{3(4a+\lambda b}\right)\left(\frac{3b+2\lambda c}{3(4a+\lambda b}\right).$$

By simplifying the equation above, we have $(36be-6cd)\lambda^2+(144ae-4c^2)\lambda+36ad-6bc=0$. Then, dividing the equation by 2, we obtain $k(\lambda)=3(6be-cd)\lambda^2+2(36ae-c^2)\lambda+3(6ad-bc)=0$. Since $ord_p(36ae-c^2)^2>ord_p9(6be-cd)(6ad-bc)$, then $m(\lambda)$ and $n(\lambda)$ have two distinct common roots, λ_1 and λ_2, we have

$$\lambda_1=\frac{-(36ae-c^2)+\sqrt{(36ae-c^2)^2-9(6be-cd)(6ad-bc)}}{3(6be-cd)}$$

and $\lambda_2=\overline{\lambda_1}$. Let

$$U=X+\frac{3b+2\lambda_1 c}{3(4a+\lambda_1 b)}Y\ ,\ u_0=x_0+\frac{(3b+2\lambda_1 c)}{3(4a+\lambda_1 b)}y_0, \tag{3.4}$$

$$V=X+\frac{3b+2\lambda_2 c}{3(4a+\lambda_2 b)}Y\ ,\ v_0=x_0+\frac{3b+2\lambda_2 c}{3(4a+\lambda_2 b)}y_0. \tag{3.5}$$

By substituting (3.4) and (3.5) into (3.1), we have polynomials in (U,V) as follows:

$$F(U,V)=(4a+\lambda_1 b)(U+u_0)^3+r+\lambda_1 s \ \text{ and} \tag{3.6}$$

$$G(U,V)=(4a+\lambda_2 b)(V+v_0)^3+r+\lambda_2 s. \tag{3.7}$$

From (3.6) and (3.7), we obtain $F(U,V)=(4a+\lambda_1 b)(U^3+3u_0U^2+3u_0^2U)+F_0$ and $G(U,V)=(4a+\lambda_2 b)(V^3+3v_0V^2+3v_0^2V)+G_0$ where $F_0=f_x(x_0,y_0)+\lambda_1 f_y(x_0,y_0)$ and $G_0=f_x(x_0,y_0)+\lambda_2 f_y(x_0,y_0)$. By Lemma 3.9, there exists $(\widehat{U},\widehat{V})$ in Ω_p^2 such that $F(\widehat{U},\widehat{V})=0$, $G(\widehat{U},\widehat{V})=0$ where $ord_p\widehat{U}=\mu'=\frac{1}{3}ord_p\frac{F_0}{4a+\lambda_1 b}$ and $ord_p\widehat{V}=\lambda'=\frac{1}{3}ord_p\frac{G_0}{4a+\lambda_2 b}$. By equations (3.4) and (3.5), there exists $(\widehat{X},\widehat{Y})$ such that

$\widehat{U} = \widehat{X} + \lambda_1\widehat{Y}$, $\widehat{V} = \widehat{X} + \lambda_2\widehat{Y}$ where $\gamma_i = \frac{3b+2\lambda_i c}{3(4a+\lambda_i b}$ for $i = 1, 2$. Since $\widehat{U} = \widehat{X} + \gamma_1\widehat{Y}$, $\widehat{V} = \widehat{X} + \gamma_2\widehat{Y}$, $ord_p b^2 > ord_p ac$ and $ord_p\lambda = \frac{1}{2}ord_p\frac{a}{c}$, we have from Lemma 3.5,

$$ord_p\widehat{X} > \frac{1}{3}(\alpha - 2\delta), \quad ord_p\widehat{Y} > \frac{1}{3}(\alpha - 2\delta).$$

Let $\xi = \widehat{X} + x_0$ and $\eta = \widehat{Y} + y_0$, then $\widehat{X} = \xi - x_0$ and $\widehat{Y} = \eta - y_0$. Thus, we have

$$ord_p(\xi - x_0) > \frac{1}{3}(\alpha - 2\delta), \quad ord_p(\eta - y_0) > \frac{1}{3}(\alpha - 2\delta).$$

By back substitution in (3.4), (3.5) and (3.1), we have $g(\xi, \eta) = f_x(\xi, \eta) = 0$ and $h(\xi, \eta) = f_y(\xi, \eta) = 0$. □

Let $f(x, y)$ be in $\mathbb{Z}_p[x, y]$ and λ be the roots of $k(\lambda)$ of f_x and f_y. In Theorem 3.2, we give the p-adic sizes of common zeros in the neighbourhood of (x_0, y_0) under the condition $ord_p\frac{b}{c} > ord_p\lambda > ord_p\frac{a}{b}$ with $ord_p\lambda = \frac{1}{2}ord_p\frac{c}{e}$.

Theorem 3.2. *Let*

$$f(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + rx + sy + t$$

be a polynomial in $\mathbb{Z}_p[x, y]$ *with* $p > 3$. *Let* $\alpha > 0$*,*

$$\delta = max\,\{ord_p a, ord_p b, ord_p c, ord_p d, ord_p e\}$$

and $ord_p(36ae - c^2)^2 > ord_p 9(6be - cd)(6ad - bc)$. *Suppose* $(x_0, y_0) \in \omega_p^2$*,* $ord_p\frac{b}{c} > ord_p\lambda > ord_p\frac{a}{b}$*,* $CEI - DEH - AI^2 = 0$ *and* $DE^2 - BEI + AHI = 0$ *where*

$A = 108b^2e - 8c^3$*,* $B = 864abe + 27b^2d - 36bc^2$*,*

$C = 1728a^2e + 216abd - 54b^2c$*,* $D = 432a^2d - 27b^3$*,*

$E = 9bd - 4c^2$*,* $H = 6(6ad - bc)$ *and* $I = 3(4ac - 3b^2)$.

If $ord_p f_x(x_0, y_0)$*,* $ord_p f_y(x_0, y_0) \geq \alpha > \delta$ *and* $ord_p\lambda = \frac{1}{2}ord_p\frac{c}{e}$*, then there exists* (ξ, η) *such that* $f_x(\xi, \eta) = 0$*,* $f_y(\xi, \eta) = 0$ *and* $ord_p(\xi - x_0) \geq \frac{1}{3}(\alpha - 2\delta)$*,* $ord_p(\eta - y_0) > \frac{1}{3}(\eta - 3\delta)$ *or* $ord_p(\eta - y_0) > \frac{1}{3}(\alpha - 4\delta)$.

Proof. The proof is similar to Theorem 3.1 by using Lemmas 3.6, 3.7, 3.8 and 3.9. □

4. Conclusion

In this paper, the p-adic sizes of partial derivative polynomials associated with quartic polynomial is considered. Then, by using these results, we find the estimation of cardinality of the set $(f_x, f_y; p^\alpha)$ and also exponential sums of the polynomial.

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