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EXTRAPOLATION OF TYPE-1 DIAGONALLY IMPLICIT MULTISTAGE INTEGRATION METHODS FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS

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This article is dedicated to the late Robert Chan for his support and interest in the study of extrapolation to numerical ordinary differential equations

ABSTRACT. Runge-Kutta (RK) methods and Linear Multistep Methods (LLM) are two methods that can be used to solve the ODEs. However, due the restriction of the stability and computational cost in solving challenging problem, Diagonally Implicit Multistage Integration Methods (DIMSIMs) was introduced that takes the combinations of RK and LLM methods. The research studies the extrapolation of DIMSIMs for solving the ordinary differential equations (ODEs) particularly non-stiff problems. Main idea of the research is to implement extrapolation technique for special case of DIMSIMs with the conditions of stage order equal to the order. Extrapolation is applied by considering the stability of the base method with step-size h and $h/2$. The numerical results for Van der Pol and Euler equations showed the efficiencies of extrapolation to these methods which therefore opens the door to extend the research for higher order DIMSIMs.

1. INTRODUCTION

Consider the given ordinary differential equations given by

$$y' = f(y(t)), \quad t \in [t_0, t_n], y(t_0) = y_0. \tag{1.1}$$

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The solution of initial value problem can be determined using any numerical methods particularly the traditional methods such as Runge-Kutta (RK) and Linear Multistep (LLM). There is also General Linear Methods(GLMs) that was given as a framework that unifies these traditional methods and not many methods have been developed that differ significantly from conventional methods [1]. Runge-Kutta and linear multistep methods are essentially the backbone for constructing general linear methods. GLMs are constructed upon taking into consideration all the drawback of RK and LLM methods. In recent years, general linear methods have been analyzed and developed using numerous theoretical and practical approaches [2].

Researchers have been trying different ways to construct an efficient method for solving the ordinary differential equations (ODE). General linear methods (GLMs) are preferred by researches due to the efficiency and accuracy in solving complicated problems or problems when RK methods and LLM methods fail to achieve greater accuracy [3].

In addition, there is also the generalization of general linear methods, known as implicit-explicit GLMs. These methods were introduced by Zhang in his PhD thesis [4] and the research has also been extended by other group of researchers [5]. The subclass of general linear methods is known as Diagonally Implicit Multistage Integration Methods, abbreviated as (DIMSIMs) which was first introduced by Butcher [6]. These methods are considered to have the advantages of RK methods and linear multistep methods. They overcome the disadvantages such as the high implementation of implicit RK methods and lack of A-stable by the linear multistep methods.

DIMSIMs is defined as follows:

$$\begin{array}{l} Y^{([n])} = h\left(A \otimes I_m\right) F\left(Y^{[n]}\right) + \left(U \otimes I_m\right) y^{[n-1]} \\ y^{([n])} = h\left(B \otimes I_m\right) F\left(Y^{[n]}\right) + \left(V \otimes I_m\right) y^{[n-1]} \end{array},$$

for $n = 1, 2, 3, \ldots N$ where $Nh = x_n$. The internal approximations $Y^{([n])}$ depends on the stage derivatives $f\left(Y^{[n]}\right)$.

To create a robust and efficient implementation in the variable stepsize and order environment, then equation (1.1) assumed in the following Nordsieck form

$$\begin{aligned} Y^{[n]} &= h\left(A \otimes I_m\right) F\left(Y^{[n]}\right) + \left(P \otimes I_m\right) z^{[n-1]}, \\ z^{[n]} &= h\left(G \otimes I_m\right) F\left(Y^{[n]}\right) + \left(Q \otimes I_m\right) z^{[n-1]}. \end{aligned}$$

In this article, the authors focused on DIMSIMs of Type-1. This method is chosen since the method give less computational cost as they are just of order-1. The new idea in this article is the implementation of extrapolation to these methods. Although several researchers have investigated the extrapolation of General Linear Methods (refer to Section 3), the extrapolation idea to the family of DIMSIMs is still new. This article main contribution is the implementation of extrapolations to Type-1 DIMSIMs.

In the next section, the extrapolation of DIMSIMs is discussed. Although the construction of DIMSIMs is not new, the idea of extrapolation to DIMSIMs is new and the authors wish to address it in the next section.

2. Extrapolation of DIMSIMs

Recent studies of extrapolation is given by Cardone et.al [7] investigated a new class of implicit-explicit RK methods for solving the ordinary differential equations with both stiff and nonstiff components. In their research, they used approach for constructing implicit-explicit methods that are based on a new type technique of extrapolation of the stage values at the current and previous steps. The structure of these methods starts with the formulation of singly diagonally-implicit RK (SDIRK) methods.

Although different approaches of extrapolation has been investigated, the idea of Richardson extrapolation remains the same where extrapolation is applied to improve the accuracy by using two different step-length such as h and $h/2$. In addition, the Richardson extrapolation can be applied in two ways such as active and passive extrapolation. The active case occurs if the extrapolated value is used to propagate the next iterations whereas it is known as passive if extrapolation is applied without propagating the extrapolated value.

This research investigates several efficient implementations of the extrapolation technique applied to order-1 DIMSIMs methods, that have become popular in recent years rather than RK methods among engineers and scientists. The idea of applying extrapolation to DIMSIMs is based on the research by Zlatev et. al., [8].

3. Implementation of Extrapolation

The extrapolation technique can be performed by using the following algorithm.

- Step I.
  Implement one large step with the stepsize h by applying y_{n-1} as a starting value to compute:
$$\mathbf{z_n} = \mathbf{R}(\mathbf{z})\mathbf{y_{n-1}}.$$
- Step II.
  Implement two small steps with the stepsize $0.5h$ by applying y_{n-1} as a starting value in the first of the two steps.
$$\bar{w}_n = R\left(\frac{z}{2}\right) y_{n-1}, \quad w_n = R\left(\frac{z}{2}\right) \bar{w}_n = \left[R\left(\frac{z}{2}\right)\right]^2 y_{n-1}.$$
- Step III.
  Compute the extrapolation technique, as follows,
$$\mathrm{y_n} = \frac{2^{\mathrm{p}}\mathrm{w_n} - \mathrm{z_n}}{2^p - 1} = \frac{2^{\mathrm{p}}\left[\mathrm{R}\left(\frac{\mathrm{z}}{2}\right)\right]^2 - \mathrm{R}(\mathrm{z})}{2^p - 1}\mathrm{y_{n-1}}.$$

To show the implementation of extrapolation to DIMSIMs consider the type-1 DIMSIMs of order-1 as follows:
$$\left[\begin{array}{c|c} A & P \\ \hline G & Q \end{array}\right] = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 1 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right].$$
The method has the stability matrix given by $M(z) = V + zB(I - zA)^{-1}U$ as
$$M = \begin{bmatrix} z+1 & 0 \\ z & 0 \end{bmatrix}.$$
The stability function is given by
$$\mathbf{R}(\mathbf{z}) = \mathbf{1} + \mathbf{z},$$
which is similar as to the Euler method.

Next by using the extrapolation of the stability equation, $\overline{\mathbf{R}}(\mathbf{z})$ with $p = 1$ is given by
$$\overline{\mathbf{R}}(\mathbf{z}) = \frac{2^p\mathbf{R}\left[\frac{\mathbf{z}}{2}\right]^2 - \mathbf{R}(\mathbf{z})}{2^p - 1} = 2\left(1 + \frac{z}{2}\right)^2 - (1+z).$$
The following Figure 1 the efficiency of stability of order one for explicit DIMSIMs with and without the extrapolation technique.

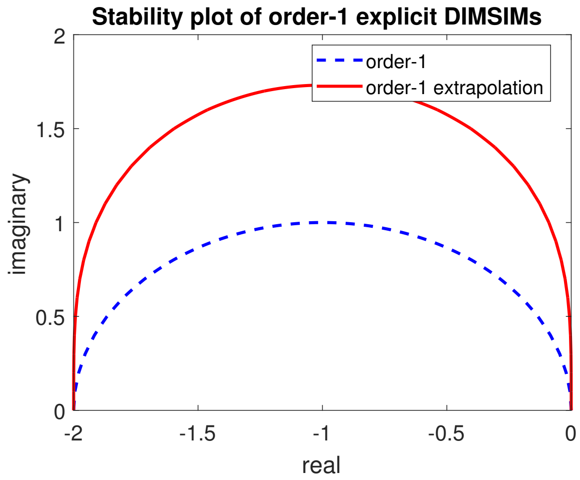

FIGURE 1. Stability plot of order-1 explicit DIMSIMs with and without the extrapolation

4. NUMERICAL EXPERIMENTS

The numerical experiments are based on some efficient test problems such as Van der Pol and the Euler equations for a rigid body problems [9]. Both problems are described in detailed on page 144 and page 463 in [9]. The numerical results are based on modified codes as given in [10] which are known as dim18 for explicit DIMSIMs and dim13s for implicit DIMSIMs. The numerical results were compared with ODE15s and ODE23s solvers. All numerical computations are done on a Lenovo ThinkPad with 1.6GHz Intel(R) Core i5-4200U CPU, using Matlab 2018. The global errors on all the graphs are computed by taking the maximum norm such that for different tolerances on which the tolerance his reduced by $1/10$ at every iterations. The starting tolerance used is $10^{(-5)}$. The legend used in all the figures is summarized in Table 1.

TABLE 1. Summary of the legend used in all the figures

Abbreviation	**Full name of the method**
ODE15s	Multistep solver of order 1-5
ODE23s	Modified Rosenbrock formula of order 2
dim1x	Order-1 explicit DIMSIM method
dim1passive	Order-1 explicit DIMSIM method with passive extrapolation
dim1active	Order-1 explicit DIMSIM method with active extrapolation

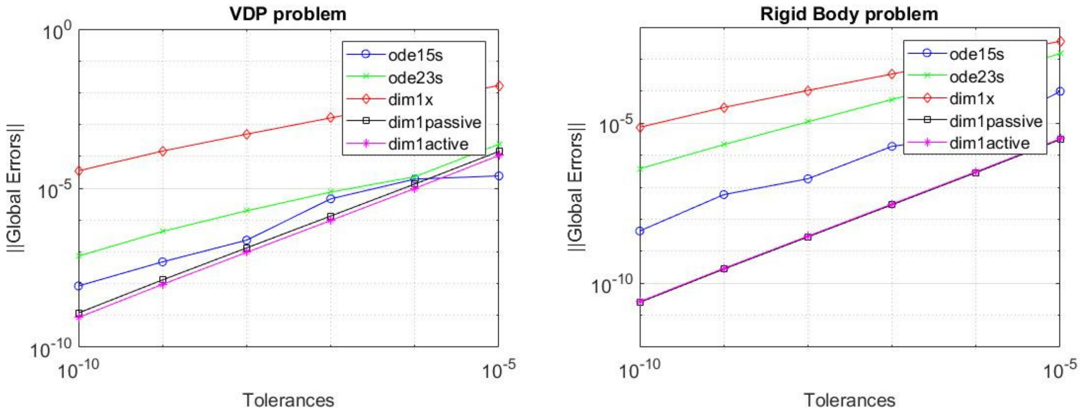

FIGURE 2. Numerical results for Van der Pol and Euler equations using order-1 DIMSIMs with passive and active extrapolations

In the given numerical results, we can see that order-1 DIMSIMs with extrapolation is efficient than the method itself without extrapolation. Both passive and active extrapolations gave greater accuracy than the ODE15 and ODE23s which are of higher order.

5. CONCLUSIONS

From the numerical results, we can conclude the extrapolation of DIMSIMs is very efficient than the method itself without extrapolation. Active extrapolation is shown to have slightly greater accuracy than passive extrapolation. The results were compared with ODE15s and ODE23s. Although we know that implicit DIMSIMs suffers from order reduction, the numerical results are still giving promising results with extrapolation. This opens the door to study the theoretical analysis of DIMSIMs with extrapolation in more detailed for the families of symmetric Runge-Kutta methods and investigate the stability aspects of the method when applied with extrapolation.

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