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OSCILLATIONS IN SECOND-ORDER DAMPED DIFFERENCE EQUATIONS WITH A SUPERLINEAR NEUTRAL TERM

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ABSTRACT. This paper concerns the oscillatory behavior of the solutions to second-order damped nonlinear difference equations with a superlinear neutral term. We obtain oscillatory criteria by a Riccati type transformation as well as summation averaging conditions. We provide examples, illustrating the results and discuss extensions of this work, for future research.

1. INTRODUCTION

This paper deals with the oscillation of the solutions to second-order neutral difference equations with a superlinear neutral term and a damping term, having the general form

$$\Delta(b(n)\Delta u(n)) + d(n)\Delta u(n) + q(n)x^{\beta}(n-\sigma) = 0, \quad n \geq n_0, \tag{E}$$

where n_0 is a positive integer, and $u(n) = x(n) + p(n)x^{\alpha}(n-\tau)$.

Through the rest of the paper, we assume that the following conditions are satisfied:

- (H_1) α and β are ratios of odd positive integers with $\alpha \geq 1$;
- (H_2) $\{b(n)\}$, $\{p(n)\}$, $\{d(n)\}$ and $\{q(n)\}$ are positive real sequences with $p(n) \geq 1$, $p(n) \neq 1$ for large n;
- (H_3) τ and σ are positive integers with $\tau < \sigma$.

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Let $\theta = \max\{\tau, \sigma\}$. By a solution of (E), we mean a real sequence $\{x(n)\}$ defined for all $n \geq n_0 - \theta$, and satisfying (E) for all $n \geq n_0$. A nontrivial solution of (E) is called $oscillatory$ if it is neither eventually positive nor eventually negative, and $nonoscillatory$ otherwise. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

The study of the oscillatory behavior of the solutions of various classes of second-order neutral difference equations without damping terms has been a very active area of research over the years; for recent contributions see for example [1–5, 8–10, 12, 15–17, 19, 20] and the references cited therein. However, while reviewing the literature it becomes clear that results on the oscillation of the solutions of second-order neutral difference equations with damping terms are relatively scarce, see [11, 13, 18] for typical results in this area. Even though the work in [11, 13, 18] deals with second-order neutral difference equations with a damping term, the results obtained in these papers cannot be applied to the case where $p(n) > 1$ and/or $p(n) \to \infty$ as $n \to \infty$ and $\alpha > 1$.

To the best of our knowledge, there are no results for second-order difference equations with a superlinear neutral term and a damping term in the case where $p(n) \to \infty$ as $n \to \infty$. Thus the aim in the present paper is to investigate the oscillatory behavior of (E) and establish new results that extend and generalize the existing criteria. In this sense, this paper constitutes a valid contribution to the theory of the oscillatory behavior of the solutions of second-order damped difference equations with a superlinear neutral term.

2. Oscillation Results

In this section, we present sufficient conditions for the oscillation of all solutions of (E) when

$$\sum_{n=n_0}^{\infty} \frac{1}{b(n)E(n)} = \infty, \quad E(n) = \prod_{s=n_0}^{n-1} \left(\frac{b(s)}{b(s) - d(s)} \right) \tag{2.1}$$

and

$$b(n) - d(n) > 0, \tag{2.2}$$

for all $n \geq n_0$.

Define

$$B(n) = \sum_{s=n_1}^{n-1} \frac{1}{b(s)},$$

for all $n_1 \geq n_0$. We start with the following lemma.

Lemma 2.1. *Let* (2.1) *and* (2.2) *hold and* $\{x(n)\}$ *be an eventually positive solution of* (E). *Then, the following inequalities simultaneously hold, for all sufficiently large* n*:*

$$u(n) > 0,\ \Delta u(n) > 0, \quad \Delta(b(n)\Delta u(n)) < 0. \tag{2.3}$$

Proof. Assume that $x(n) > 0$, $x(n-\tau) > 0$ and $x(n-\sigma) > 0$ for all $n \geq n_1 \geq n_0$. Then $u(n) > 0$ and either $\{\Delta u(n)\}$ is oscillatory or $\{\Delta u(n)\}$ is nonoscillatory for all $n \geq n_1$. Let $\{\Delta u(n)\}$ be oscillatory. Then, there exists $n_1 > n_0$ such that either $\Delta u(n_1) < 0$ or $\Delta u(n_1) = 0$. First assume $\Delta u(n_1) < 0$. From equation (E), we have

$$\Delta u(n_1+1) - \frac{(b(n_1) - d(n_1))\Delta u(n_1)}{b(n_1+1)} = -\frac{q(n_1)}{b(n_1+1)} x_1^{\beta}(n_1 - \sigma) < 0,$$

or

$$\Delta u(n_1+1) < \frac{(b(n_1) - d(n_1))\Delta u(n_1)}{b(n_1+1)}, \tag{2.4}$$

which by (2.2) implies that $\Delta u(n_1+1) < 0$. By induction we have

$$\Delta u(n) < 0 \ \text{ for all } \ n \geq n_1.$$

If $\Delta u(n_1) = 0$, then from (2.4) we obtain $\Delta u(n_1+1) < 0$. Using a similar argument as above, we get $\Delta u(n) < 0$ for all $n \geq n_1$. Hence in both cases we obtain $\Delta u(n) < 0$ eventually which is a contradiction. Thus $\Delta u(n) > 0$ or $\Delta u(n) < 0$ eventually.

Assume $\Delta u(n) < 0$ for all $n \geq n_1 \geq n_0$. Letting $z(n) = -b(n)\Delta u(n) > 0$, we get from (E) that

$$\Delta z(n) + \frac{d(n)}{b(n)} z(n) \geq 0, \quad n \geq n_1,$$

or

$$z(n+1) - \left(1 - \frac{d(n)}{b(n)}\right) z(n) \geq 0,$$

which implies that

$$z(n) \geq \frac{z(n_1)}{E(n)}.$$

Then

$$\Delta u(n) \leq \frac{b(n_1)\Delta u(n_1)}{b(n)E(n)}, \quad n \geq n_1.$$

Summing up the last inequality from n_1 to n and taking (2.1) into account, we obtain

$$u(n+1) \leq u(n_1) + b(n_1)\Delta u(n_1) \sum_{s=n_1}^{n} \frac{1}{b(s)E(s)} \to -\infty \text{ as } n \to \infty$$

which contradicts that $u(n)$ is positive. Hence $\Delta u(n) > 0$ for $n \geq n_1$. From (E), we see that (2.3) holds. The proof of the lemma is complete. □

Lemma 2.2. *Let* (2.1) *and* (2.2) *hold and* $\{x(n)\}$ *be an eventually positive solution of* (E) *such that* (2.3) *holds. Then*

$$u(n) \geq B(n)b(n)\Delta u(n), \quad n \geq n_1 \geq n_0 \tag{2.5}$$

and $\{\frac{u(n)}{B(n)}\}$ *is eventually decreasing.*

Proof. The proof is contained in Lemma 2.2 in [10], and thus, the details are omitted.

For convenient, we use the following notation:

$$\Pi(n) = \begin{cases} 1 & \text{if } \beta = \alpha \\ a_1 & \text{if } \beta > \alpha \\ a_2 B^{\frac{\beta}{\alpha}-1}(n) & \text{if } \beta < \alpha, \end{cases}$$

where a_1, a_2 are positive real constants, and for any positive real sequence $\xi(n)$, we define

$$\eta(n) = \frac{b(n)\Delta\xi(n) - \xi(n)d(n)}{b(n)\xi(n+1)}.$$

To prove our main results, we use the additional condition:

(H_4) For every positive constant δ, we have

$$\phi(n) = \frac{1}{p(n+\tau)}\left[1 - \left(\frac{B(n+2\tau)}{B(n+\tau)}\right)^{\frac{1}{\alpha}} \frac{\delta^{\frac{1}{\alpha}-1}}{p^{\frac{1}{\alpha}}(n+2\tau)}\right] > 0$$

for all sufficiently large n.

Note that if $\alpha > 1$, the above condition requires $\lim_{n\to\infty} p(n) = \infty$. □

Theorem 2.1. *Let conditions* $(H_1)-(H_4)$ *and* (2.1), (2.2) *hold. If*

$$\sum_{n=n_1}^{\infty} q(n)\phi^{\frac{\beta}{\alpha}}(s-\sigma)=\infty \tag{2.6}$$

for all $n_1 \geq n_0$, *then* (E) *is oscillatory.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of (E). Without loss of generality, we may assume $x(n)>0$, $x(n-\tau)>0$ and $x(n-\sigma)>0$ for all $n \geq n_1 \geq n_0$, where n_1 is chosen so that (2.3) holds for all $n \geq n_1$.

From the definition of $u(n)$, we have

$$x^{\alpha}(n-\tau)=\frac{1}{p(n)}(u(n)-x(n)) \leq \frac{u(n)}{p(n)},$$

or

$$x(n) \leq \frac{u^{1/\alpha}(n+2\tau)}{p^{1/\alpha}(n+2\tau)}.$$

Using this in the definition of $u(n)$, we obtain

$$x^{\alpha}(n) \geq \frac{1}{p(n+\tau)}\left[u(n+\tau)-\frac{u^{\frac{1}{\alpha}}(n+2\tau)}{p^{\frac{1}{\alpha}}(n+2\tau)}\right]. \tag{2.7}$$

Since $\frac{u(n)}{B(n)}$ is decreasing, we get

$$\frac{B(n+2\tau)u(n+\tau)}{B(n+\tau)} \geq u(n+2\tau). \tag{2.8}$$

Using (2.8) in (2.7), we obtain

$$x^{\alpha}(n) \geq \frac{u(n+\tau)}{p(n+\tau)}\left[1-\left(\frac{B(n+2\tau)}{B(n+\tau)}\right)^{\frac{1}{\alpha}}\frac{u^{\frac{1}{\alpha}-1}(n+2\tau)}{p^{\frac{1}{\alpha}}(n+2\tau)}\right]. \tag{2.9}$$

Since $\{u(n)\}$ is positive and increasing for $n \geq n_1$, there exists an integer $n_2 \geq n_1$ and a constant $\delta>0$ such that

$$u(n) \geq \delta \text{ for } n \geq n_2. \tag{2.10}$$

Substituting (2.10) in (2.9), we get

$$x^{\alpha}(n) \geq \phi(n)u(n+\tau), \quad n \geq n_2,$$

or

$$x^{\alpha}(n-\sigma) \geq \phi(n-\sigma)u(n+\tau-\sigma), \quad n \geq n_3 \geq n_2. \tag{2.11}$$

Using (2.11) in (E), we obtain

$$\Delta(b(n)\Delta u(n)) + d(n)\Delta u(n) + q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)u^{\frac{\beta}{\alpha}}(n+\tau-\sigma) \le 0, \quad n \ge n_3. \tag{2.12}$$

Since $d(n) > 0$ and $\Delta u(n) > 0$, it follows from (2.12) that

$$\Delta(b(n)(\Delta u(n)) + q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)u^{\frac{\beta}{\alpha}}(n+\tau-\sigma) \le 0. \tag{2.13}$$

Summing up (2.13) from n_3 to $n-1$ and using (2.10), we obtain

$$b(n)\Delta u(n) \le b(n_3)\Delta u(n_3) - \delta^{\frac{\beta}{\alpha}} \sum_{s=n_0}^{n-1} q(s)\phi^{\frac{\beta}{\alpha}}(s-\sigma) \to -\infty \text{ as } n \to \infty$$

which contradicts the fact that $\Delta u(n)$ is positive. The proof of the theorem is complete. □

Theorem 2.2. *Let conditions* $(H_1) - (H_4)$ *and* (2.1)*,* (2.2) *hold. If there exists a positive real sequence* $\{\xi(n)\}$ *such that*

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n} \left[\xi(s)q(s)\phi^{\frac{\beta}{\alpha}}(s-\sigma)\Pi(s+\tau-\sigma)\frac{B(s+\tau-\sigma)}{B(s)} - \frac{b(s)\eta^2(s)\xi^2(s+1)}{4\xi(s)}\right] = \infty \tag{2.14}$$

for all $n_1 \ge n_0$*, then* (E) *is oscillatory.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of (E). Without loss of generality there is an integer $n_1 \ge n_0$ such that $x(n) > 0$, $x(n-\tau) > 0$ and $x(n-\sigma) > 0$ for all $n \ge n_1$, where n_1 is chosen so that $u(n)$ satisfies condition (2.3) for all $n \ge n_1$.

Proceeding as in the proof of Theorem 2.1, we arrive at (2.12) for $n \ge n_3$, which can be written as

$$\Delta(b(n)\Delta u(n))) + d(n)\Delta u(n) + q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)u^{\frac{\beta}{\alpha}-1}(n+\tau-\sigma)u(n+\tau-\sigma) \le 0, \tag{2.15}$$

for $n \ge n_3$. Since $\frac{u(n)}{B(n)}$ is decreasing, there exists a constant $c > 0$ such that

$$u(n) \le cB(n), \quad n \ge n_3. \tag{2.16}$$

In view of (2.10) and (2.16), inequality (2.15) can be written as

$$\Delta(b(n)\Delta u(n))) + d(n)\Delta u(n) + q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)\Pi(n+\tau-\sigma)u(n+\tau-\sigma) \le 0, \tag{2.17}$$

for $n \ge n_3$. Using the Riccati substitution, we define the sequence $\{w(n)\}$,

$$w(n) = \xi(n)\frac{b(n)\Delta u(n)}{u(n)}, \quad \text{for } n \ge n_3. \tag{2.18}$$

Clearly $w(n) > 0$, and from (2.17) and (2.18), we see that

$$\begin{aligned}
\Delta w(n) &= \frac{\Delta\xi(n)}{\xi(n)}w(n+1) + \frac{\xi(n)}{u(n)}\Delta(b(n)\Delta u(n)) \\
&\quad -\frac{\xi(n)}{u(n)u(n+1)}b(n+1)\Delta u(n+1)\Delta u(n) \\
&\leq \frac{\Delta\xi(n)}{\xi(n)}w(n+1) + \frac{\xi(n)}{u(n)}\left[-d(n)\Delta u(n)\right. \\
&\quad \left.-q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)\Pi(n+\tau-\sigma)u(n+\tau-\sigma)\right] - \frac{\xi(n)}{b(n)\xi^2(n+1)}w^2(n+1) \\
&\leq \eta(n)w(n+1) - \xi(n)q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)\Pi(n+\tau-\sigma)\frac{u(n+\tau-\sigma)}{u(n)} \\
&\quad -\frac{\xi(n)}{b(n)\xi^2(n+1)}w^2(n+1). \qquad (2.19)
\end{aligned}$$

Since $\frac{u(n)}{B(n)}$ is decreasing, we have

$$\frac{u(n+\tau-\sigma)}{u(n)} \geq \frac{B(n+\tau-\sigma)}{B(n)}.$$

Substituting this in (2.19), we obtain

$$\begin{aligned}
\Delta w(n) &\leq \eta(n)w(n+1) - \xi(n)q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)\Pi(n+\tau-\sigma)\frac{B(n+\tau-\sigma)}{B(n)} \\
&\quad -\frac{\xi(n)}{b(n)\xi^2(n+1)}w^2(n+1). \qquad (2.20)
\end{aligned}$$

Completing the square with respect to w, we get

$$\Delta w(n) \leq -\xi(n)q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)\Pi(n+\tau-\sigma)\frac{B(n+\tau-\sigma)}{B(n)} + \frac{b(n)\xi^2(n+1)\eta^2(n)}{4\xi(n)}$$

for $n \geq n_3$. Summing up the last inequality from n_3 to n yields,

$$\sum_{s=n_3}^{n}\left[\xi(s)q(s)\phi^{\frac{\beta}{\alpha}}(s-\sigma)\Pi(s+\tau-\sigma)\frac{B(s+\tau-\sigma)}{B(s)} - \frac{b(s)\xi^2(s+1)\eta^2(s)}{4\xi(s)}\right] < w(n_3),$$

which contradicts (2.14). The proof of the theorem is complete. □

Theorem 2.2 enables us to propose various conditions, for the oscillation of (E) by different choices of $\{\xi(n)\}$. For example, letting $\xi(n) = 1$, we obtain the following corollary.

Corollary 2.1. *Let conditions* $(H_1)-(H_4)$ *and* (2.1), (2.2) *hold. If*

$$\limsup_{n\to\infty}\sum_{s=n_1}^{n}\left[q(s)\phi^{\frac{\beta}{\alpha}}(s-\sigma)\Pi(s+\tau-\sigma)\frac{B(s+\tau-\sigma)}{B(s)}-\frac{d^2(s)}{4b(s)}\right]=\infty \tag{2.21}$$

for all $n_1\geq n_0$, *then* (E) *is oscillatory.*

Next, we present a new oscillation result for (E), assuming that $\eta(n)\leq 0$ for all $n\geq n_0$.

Theorem 2.3. *Let conditions* $(H_1)-(H_4)$ *and* (2.1), (2.2) *hold. If there exists a positive real sequence such that* $\eta(n)\leq 0$ *for all* $n\geq n_0$, *and*

$$\limsup_{n\to\infty}\sum_{s=n_1}^{n}\xi(s)q(s)\phi^{\frac{\beta}{\alpha}}(s-\sigma)\Pi(s+\tau-\sigma)\frac{B(s+\tau-\sigma)}{B(s)}=\infty \tag{2.22}$$

for all $n_1\geq n_0$, *then* (E) *is oscillatory.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of (E). Without loss of generality, we may assume $x(n)>0$, $x(n-\tau)>0$ and $x(n-\sigma)>0$ for all $n\geq n_1$ for some integer $n_1\geq n_0$. Then from Lemma 2.1, $\{u(n)\}$ satisfies condition (2.3) for all $n\geq n_2\geq n_1$. Proceeding as in the proof of Theorem 2.2, we arrive at (2.20) for $n\geq n_2$. Since $\eta(n)\leq 0$ and $w(n)>0$, the inequality (2.20) can be written as

$$\Delta w(n)\leq -\xi(n)q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)\Pi(n+\tau-\sigma)\frac{B(n+\tau-\sigma)}{B(n)},\quad n\geq n_2.$$

Summing up the last inequality from n_3 to n, we get

$$\sum_{s=n_3}^{n}\xi(s)q(s)\phi^{\frac{\beta}{\alpha}}(s-\sigma)\Pi(s+\tau-\sigma)\frac{B(s+\tau-\sigma)}{B(s)}<w(n_2),$$

which contradicts (2.22).The proof of the theorem is complete. □

Theorem 2.4. *Let conditions* $(H_1)-(H_4)$ *and* (2.1), (2.2) *hold. If*

$$\Delta V(n)+q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)B^{\frac{\beta}{\alpha}}(n+\tau-\sigma)V^{\frac{\beta}{\alpha}}(n+\tau-\sigma)=0 \tag{2.23}$$

is oscillatory, then (E) *is oscillatory.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of (E). Without loss of generality, we may assume $x(n)>0$, $x(n-\tau)>0$ and $x(n-\sigma)>0$ for all $n\geq n_1$ for some integer $n_1\geq n_0$. Then by Lemma 2.1, the sequence $\{u(n)\}$ satisfies (2.3) for all

$n \geq n_2 \geq n_1$. Proceeding as in the proof of Theorem 2.1, we arrive at (2.13), that is

$$\Delta(b(n)(\Delta u(n)) + q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)u^{\frac{\beta}{\alpha}}(n+\tau-\sigma) \leq 0. \tag{2.24}$$

From (2.5), we obtain

$$u(n+\tau-\sigma) \geq B(n+\tau-\sigma)b(n+\tau-\sigma)\Delta u(n+\tau-\sigma), \quad n \geq n_2. \tag{2.25}$$

Using (2.25) in (2.24) yields

$$\Delta(b(n)(\Delta u(n))) + q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)B^{\frac{\beta}{\alpha}}(n+\tau-\sigma)(b(n+\tau-\sigma)\Delta u(n+\tau-\sigma))^{\frac{\beta}{\alpha}} \leq 0.$$

Let $V(n) = b(n)\Delta u(n) > 0$ be a positive solution of the inequality

$$\Delta V(n) + q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)B^{\frac{\beta}{\alpha}}(n+\tau-\sigma)V^{\frac{\beta}{\alpha}}(n+\tau-\sigma) \leq 0.$$

Then by Lemma 1 of [6], the corresponding equation (2.23) also has a positive solution, which is a contradiction. Now, the proof is complete. □

Corollary 2.2. *Let conditions* $(H_1) - (H_4)$ *and* (2.1)*,* (2.2) *hold. If*

$$\liminf_{n\to\infty} \sum_{s=n+\tau-\sigma}^{n-1} q(s)\phi(s-\sigma)B(s+\tau-\sigma) > \left(\frac{\sigma-\tau}{\sigma-\tau+1}\right)^{\sigma-\tau+1} \quad \text{, when } \alpha = \beta \tag{2.26}$$

and

$$\sum_{n=n_1}^{\infty} q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)B^{\frac{\beta}{\alpha}}(n+\tau-\sigma) = \infty \quad \text{when } \alpha > \beta \tag{2.27}$$

for all $n_1 \geq n_0$*, respectively, then* (E) *is oscillatory.*

Proof. Assume (2.26) holds. Then by Theorem 7.6.1 in [7], equation (2.23) is oscillatory. So by Theorem 2.4, equation (E) is oscillatory. Now assume (2.27) holds. Then by Theorem 1 in [14], equation (2.23) is oscillatory and thus, by Theorem 2.4, we conclude that (E) is oscillatory. The proof of the corollary is complete. □

Corollary 2.3. *Let conditions* $(H_1) - (H_4)$ *and* (2.1)*,* (2.2) *hold. If* $\alpha < \beta$ *and there exists a constant* $\lambda > \frac{1}{\sigma-\tau}\ln\frac{\beta}{\alpha}$ *such that*

$$\liminf_{n\to\infty} \left[q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma)B^{\frac{\beta}{\alpha}}(n+\tau-\sigma)\exp(-e^{\lambda n})\right] > 0 \tag{2.28}$$

then (E) *is oscillatory.*

Proof. Assume (2.28) holds. Then by Theorem 2 in [14], equation (2.23) is oscillatory and thus, by Theorem 2.4, equation (E) is oscillatory. □

3. EXAMPLES

In this section, we illustrate our results with two examples.

Example 1. *Consider the difference equation with a superlinear neutral term and a damping term*

$$\Delta^2 u(n) + \frac{1}{n+1}\Delta u(n) + \frac{(n+2)}{2}x^5(n-2) = 0, \quad n \geq 1, \tag{3.1}$$

with

$$u(n) = x(n) + nx^5(n-1).$$

Here $b(n) = 1$, $p(n) = n$, $d(n) = \frac{1}{n+1}$, $q(n) = \frac{(n+2)}{2}$, $\alpha = 5$, $\beta = 5$, $\tau = 1$, and $\sigma = 2$. Then, it is easy to see that $E(n) = n$ and $B(n) = n-1$. Therefore conditions $(H_1) - (H_4)$ and (2.1), (2.2) hold. Furthermore

$$\phi(n) = \frac{1}{(n+2)}\left[1 - \frac{1}{\delta^{\frac{4}{5}}(n+1)^{\frac{1}{5}}}\right].$$

Thus, it follows from (2.6) that

$$\sum_{n=n_0}^{\infty} q(n)\phi^{\frac{\beta}{\alpha}}(n-\sigma) = \sum_{n=1}^{\infty}\frac{1}{2}\left[1 - \frac{1}{\delta^{\frac{4}{5}}(n+1)^{\frac{1}{5}}}\right] = \infty,$$

that is, condition (2.6) holds. Hence, all conditions of Theorem 2.1 hold, and consequently, equation (3.1) is oscillatory.

Example 2. *Consider the difference equation with a linear neutral term and a damping term*

$$\Delta^2 u(n) + \frac{1}{(n+1)}\Delta u(n) + n^2 x^3(n-2) = 0, \quad n \geq 1 \tag{3.2}$$

with

$$u(n) = x(n) + 2x(n-1).$$

Here $\alpha = 1$, $\beta = 3$, $p(n) = 2$, $d(n) = \frac{1}{n+1}$, $b(n) = 1$, $q(n) = n^2$, $\tau = 1$, and $\sigma = 2$. Then, it is easy to see that $E(n) = n$ and $B(n) = n - 1$. Therefore, conditions $(H_1) - (H_4)$ and (2.1), (2.2) hold. Furthermore $\phi(n) = \frac{n-1}{4n}$. Now, condition (2.21) becomes

$$\limsup_{n\to\infty} \sum_{s=4}^{n} \left(\frac{a_1 s^2}{4^3} \left(\frac{s-3}{s-2} \right)^3 \left(\frac{s-2}{s-1} \right) - \frac{1}{4(s+1)^2} \right)$$
$$\approx \limsup_{n\to\infty} \sum_{s=4}^{n} \left(\frac{a_1 s^2}{64} - \frac{1}{4(s+1)^2} \right) = \infty.$$

Thus, all conditions of Corollary 2.1 are satisfied and hence equation (3.2) is oscillatory.

4. Conclusion

The results obtained in this paper are original and extend the existing results, in the literature. Moreover, it is easy to see that these results also apply to second order difference equation with a superlinear neutral term and a damping term

$$\Delta(b(n)(\Delta u(n))^\gamma) + d(n)(\Delta u(n))^\gamma + q(n)x_{n-\sigma}^\beta = 0, \quad n \geq n_0 > 0,$$

under the condition

$$\sum_{n=n_0}^{\infty} \frac{1}{(b(n)E(n))^{\frac{1}{\alpha}}} = \infty.$$

The details are left to the reader. Of interest for future work, is to study equation (E) in the case where $p(n) \to -\infty$ as $n \to \infty$.

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