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A GEOMETRIC APPROACH TO GOLDBACH'S CONJECTURE AND POLIGNAC'S CONJECTURE

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ABSTRACT. In this paper, we show that Goldbach's conjecture and Polignac's conjecture are equivalent by using a geometric approach. Our method is different from that of Jian Ye and Chenglian Liu [9]. First, we generalize two conjectures. The Goldbach conjecture is replaced by the line $y + x = 2n$, and the Polignac conjecture is replaced by the line $y - x = 2m$. Then, we show that there are infinitely many even numbers which are the sum of two primes. It is then also proved that Goldbach's conjecture implies Polignac's conjecture and vice versa. Lastly, a unified conjecture is obtained that combines the two conjectures

1. INTRODUCTION

The Goldbach and Polignac conjectures have close similarities. The former states that every even number greater than or equal to 4 is the sum of two primes and the latter states for every even number $2m$, there are infinitely many consecutive primes such that their difference is $2m$. The Goldbach conjecture originated from the correspondence between Goldbach and Euler in 1742. They proposed the ternary and binary conjectures. The ternary or weak Goldbach conjecture states that every odd integer greater than 5 is the sum of three primes. The binary or strong Goldbach conjecture, on the other hand, states that every even integer greater than or equal to 4 is the sum of two primes. In 2013, Helfgott [2,3] proved

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the ternary Goldbach conjecture. While in 1849, Polignac [7] stated the Polignac conjecture.

The binary Goldbach conjecture is still an open problem while their is some progress in the cases of Polignac conjecture. Two conjectures have common point where they are connected the twin primes (see [1,5,6]). In 2013, Yitang Zhan [10] proved that $2m < 70,000,000$. This bond was further improved by Maynard [4] and reduced to less than 600. The bounded gap is further refined and reduced to less than or equal to 246 by Polymath [8] in 2014.

The goal of this paper is to generalize the two conjectures and make a connection between them. A closely related paper is [9]. However, our approach is entirely new and is based on geometric analysis.

2. A GEOMETRIC APPROACH

Here we replace the Goldbach and Polignac conjectures by their geometric counterparts.

Definition 2.1 (Generalized Goldbach's conjecture). *The Goldbach conjecture states that every even number greater than or equal to 4 is the sum of two primes. This statement is replaced by a linear equation of the form of*

$$y + x = 2n\,, \tag{2.1}$$

where n is a positive integer and $x, y \in \mathbb{R}$.

We note that the slope of eq. (2.1) is -1, its $x-$ and $y-$intercepts is $2n$. In the first quadrant, there are finitely many lattice points which are the solutions of eq. (2.1), where lattice point is a point in $xy-$plane with integral coordinates.

Theorem 2.1. *There are infinitely many even numbers which are the sum of two primes.*

Proof. We treat $y + x = 4$ separately, as $2 + 2 = 4$ is trivially true. Now draw a line at $x = 3$. This line is represented by a vertical dotted line in Figure 1. Next, draw all the lines at $y = 3, 5, 7, \ldots$. These lines are shown by the horizontal dashed lines. These lines intersect at $(3, 3), (3, 5), (3, 7), \ldots$. But $(3, 3)$ is also a point on the line $y + x = 2n$, for some n. Similarly $(3, 5)$ is a point on another line of the form of $y + x = 2n$, for some different n. The lines $y + x = 2n$ are shown in figure 1 by the solid lines, which are actually parallel lines. It proves the theorem as every point of the form of $(3, p)$, lies on some line of the form of $y + x = 2n$, where p is a prime greater than 2. $\square$

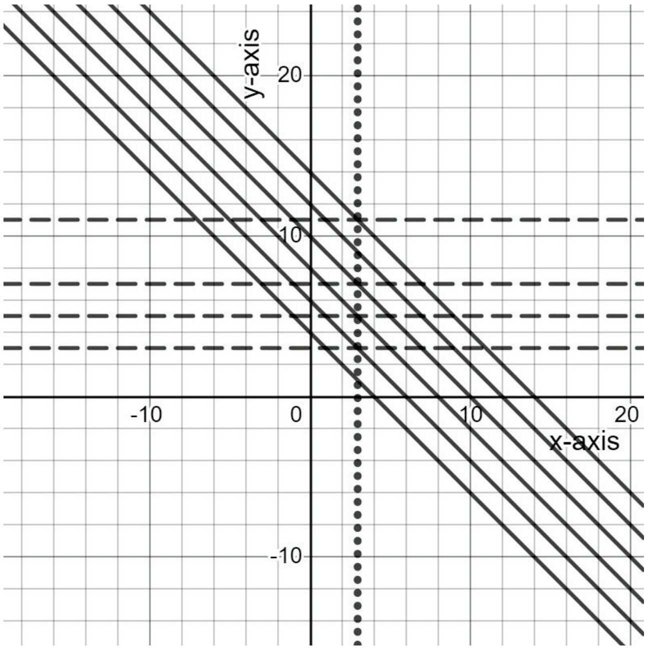

FIGURE 1. Here $x = 3$ is represented by the dotted vertical line, and $y = 3, 5, 7, \ldots$ are represented by the horizontal dashed lines, while the solid lines represent $y + x = 2n$. The image is drawn in online Desmos graphing calculator.

Definition 2.2 (Generalized Polignac's conjecture). *The Polignac conjecture states that for every even number $2m$, there are infinitely many consecutive primes whose difference is $2m$. This statement is replaced by the linear equation of the form of*

$$y - x = 2m\,, \tag{2.2}$$

where $m \in \mathbb{Z}$ and $x, y \in \mathbb{R}$.

Here we do not require that m is necessarily a positive integer. It can take any integral value. The slope of eq. (2.2) is 1. Hence eqs. (2.1) and (2.2) are perpendicular at each other.

Definition 2.3. *Let p_i, p_j be primes. Then $(p_i, p_j)_G$ is said to be Goldbach point if it is the solution of eq.* (2.1). *Similarly, $(p_i, p_j)_P$ is said to be Polignac point if it is the solution of eq.* (2.2).

Theorem 2.2. *The Goldbach pair is symmetric about the line $y = x$.*

Proof. The proof is obvious. If (p_i, p_j) is a solution of eq. (2.1), then (p_j, p_i) is also a solution of the same equation, because eq. (2.1) is symmetric about x and y. One notices that the two points lie symmetrically about $y = x-$line. $\square$

Theorem 2.3. *The Goldbach and Polignac equations intersect at integral coordinates.*

Proof. Solving eqs. (2.1) and (2.2), we get

$$y = n + m\,,$$

and

$$x = n - m\,.$$

Since the solution returns integral values, it proves the theorem. $\square$

Theorem 2.4. *The Goldbach line has finitely many lattice points, whereas the Polignac line has infinitely many lattice points.*

Proof. Since in the first quadrant ($x \geq 0$ and $y \geq 0$) the Goldbach line is bounded by the $x-$intercept $(2n, 0)$ and $y-$intercept $(0, 2n)$, therefore there are finitely many lattice points that lie on the Goldbach line. Whereas the Polignac line is unbounded in the first quadrant, therefore there are infinitely many lattice points that lie on this line. $\square$

In the following theorem, we prove that the Goldbach conjecture implies the de Polignac conjecture and vice versa.

Theorem 2.5. *Suppose Q is a point. Then Q is the Goldbach point if and only if Q is also the Polignac point.*

Proof. Draw the vertical lines $x = 3, 5, 7, \ldots$ and the horizontal lines $y = 3, 5, 7, \ldots$, as shown in Figure 2. These lines intersect at the points

$$\{(p_i, p_j)|\; p_i, p_j \text{ are primes greater than or equal to } 3\}$$

Then these points also lie on eqs. (2.1) and (2.2). This proves the theorem both directly and conversely. $\square$

It is possible to combine the Goldbach and Polignac conjectures into a single conjecture.

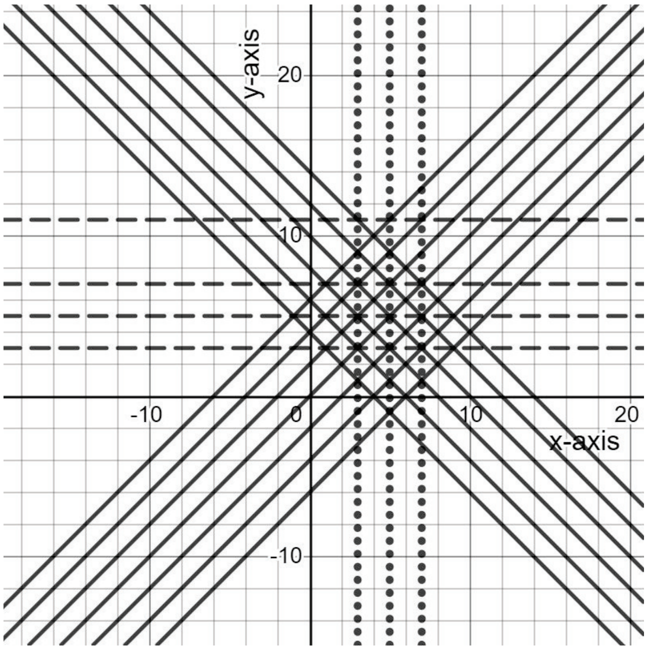

FIGURE 2. Here $x = 3, 5, 7, \ldots$ are represented by the dotted vertical lines, and $y = 3, 5, 7, \ldots$ are represented by the horizontal dashed lines, while the solid lines represent $y + x = 2n$ and $y - x = 2m$. The image is drawn in online Desmos graphing calculator.

Definition 2.4 (Unification of the Goldbach and Polignac conjectures). *The generalized Goldbach and Polignac conjectures are given by eq.* (2.1) *and eq.* (2.2)*, respectively. The simultaneous solution of the two equations is*

$$y = n + m\,,$$

and

$$x = n - m\,.$$

Then the unified conjecture states that there are infinitely many m and n such that $n + m$ and $n - m$ are primes.

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