ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.12, 10989–11002
ISSN: 1857-8365 (printed): 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.12.82

APPLICATION OF FRACTIONAL CALCULUS ON A NEW DIFFERENTIAL PROBLEM OF DUFFING TYPE

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ABSTRACT. In this paper, we study a new nonlinear sequential fractional differential problem of Duffing oscillator type. The considered problem involves two fractional order operators: Riemann and Liouville integral, and the derivative of Caputo, it is also with new nonlocal conditions. We prove an existence and uniqueness result. Also, we prove a new existence result using Schaefer theorem. We end our paper by presenting an illustrative example.

1. INTRODUCTION

In recent years, the fractional differential equations have attracted great attention. These equations can be used for modeling phenomena in mechanics, chemistry, biology, etc. For more information, we cite the research papers [3–5, 7, 8, 10, 12, 15, 16]. Moreover, nonlinear fractional differential equations are one of the most important mathematical tools used to model real-world problems in many domains of science, the reader is invited to consult [2, 6, 13, 14, 17–19]. In particular, one of these nonlinear equations, called the Duffing equation which has become very important in engendering sciences [1, 23]. In this context, many authors have paid attention to the question of existence and uniqueness of solutions for certain types of such equation. For more details, we refer the interested reader to [9, 11, 21].

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2020 *Mathematics Subject Classification.* 30C45, 39B72, 39B82.

Key words and phrases. Duffing equation, Caputo derivative, fixed point, existence of solution.

We note that the standard Duffing oscillator model is:

$$y^{''}(t) + ay^{'}(t) + f(t, y(t)) = h(t), \qquad t \in [0,1], a > 0,$$

with $y(0) = a_1,\ y^{'}(0) = a_2$, and a_1 and a_2 are constants, f, g are continuous real functions.

Let us also give some research works that have motivated the present "Duffing" paper. We begin by [11], where the authors have discussed the application of numerical methods to a forced Duffing problem which is:

$$\begin{cases} D^{\beta}u(t) + \delta D^{\alpha}u(t) + \rho u(t) + \mu u^3(t) = \lambda \sin(\omega t), \\ u(0) = A^* \in \mathbb{R}, \quad D^{\alpha}u(0) = B^* \in \mathbb{R}, \\ 0 < \alpha < 1, \ \ 1 < \beta < 2, \ \ t \in [0,1], \end{cases}$$

where D^{α}, D^{β} are for Caputo and $\delta, \rho, \mu, \lambda > 0$.

In their recent work, P. Pirmohabbati with his co authors [22] have investigated the following initial value problem:

$$\begin{cases} D^{\beta}r(t) + aD^{\alpha}r(t) + f(t, r(t)) = h(t), \\ r(y_0) = x_0, \quad r^{'}(y_0) = x_1, \\ 0 < \alpha < 1, \ \ 1 < \beta < 2, \end{cases}$$

and also, in [9], the authors have been concerned with the following Duffing problem:

$$\begin{cases} D^{\beta}(D^{\alpha}x(t)) + kf(t, D^{\alpha}x(t)) + g(t, x(t), D^{p}x(t)) = h(t), \\ x(0) = A^* \in \mathbb{R}, \quad D^{\alpha}x(0) = B^* \in \mathbb{R}, \quad x(1) = C^* \in \mathbb{R}, \\ 0 < p < \alpha < 1, \ \ 1 < \beta < 2, \ \ t \in I, \end{cases}$$

where $D^{\alpha}, D^{\beta}, D^{p}$ are of Caputo, $I = [0,1]$, k is a real constant, also f, g and h are continuous. The existence of solutions and their stabilities of Ulam have been discussed by the authors.

Motivated by the papers [9, 22], we shall study the following three sequential fractional problem of Duffing type:

$$
(1.1)\qquad \begin{cases} D^{\alpha}(D^{\beta}(D^{\delta}y(t))) + f(t, y(t), D^{p}y(t)) + g(t, y(t), I^{q}y(t)) \\ \qquad +h(t, y(t)) = l(t), \\ y(0) = \xi \in \mathbb{R}, \\ y(1) = \int_0^{\eta} y(s)ds,\ 0 < \eta < 1, \\ I^{e}y(\theta) = D^{\delta}y(1),\ 0 < u < 1, \\ 0 < \alpha, \beta, \delta, p \leq 1, \quad q > 0, \quad t \in J, \end{cases}
$$

where $J := [0, 1]$, $D^{\alpha}, D^{\beta}, D^{\delta}, D^{p}$ are of Caputo, I^{q} denotes the Riemann-Liouville fractional integral of order q, and $f, g : J \times \mathbb{R}^2 \to \mathbb{R}$ are two given functions, also $h : J \times \mathbb{R} \to \mathbb{R}$ is a given function and l is a function which is defined on J.

The above problem is important since it includes several standard and fractional models of Duffing type; it easy for any reader to observe that the equation of Duffing can be derived from (1.1) but under some special data cases. Also we see that Equat.1 includes clearly the problem in [9] under some particular data cases of (1.1).

To the best of our knowledge, this is the first time in the literature where such problem will be considered.

2. Fractional Calculus

We recall some definitions and lemmas [20].

Definition 2.1. *Let $\alpha > 0$, and $f : [0, 1] \longmapsto \mathbb{R}$ be a continuous function. The Riemann-Liouville integral of order $\alpha > 0$ is defined by:*

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau)d\tau,$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u}u^{\alpha-1}du.$

Definition 2.2. *For a function $f \in C^{n}([0, 1], \mathbb{R})$ and $n - 1 < \alpha \leq n$, the Caputo fractional derivative is defined by:* $D^{\alpha}f(t) = I^{n-\alpha}\dfrac{d^n}{dt^n}(f(t))$.

To study (1.1) we need the following two lemmas [20]:

Lemma 2.1. *Let $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$. Then, the solutions of the equation $D^\alpha y(t) = 0; t \in [0,1]$ are: $y(t) = \sum_{i=0}^{n-1} c_i t^i$, where $c_i \in \mathbb{R}$, $i = 0,1,2,..,n-1$.*

Lemma 2.2. *If $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$, then, we have $I^\alpha D^\alpha y(t) = y(t) + \sum_{i=0}^{n-1} c_i t^i$, such that $c_i \in \mathbb{R}$, $i = 0,1,2,..,n-1$.*

Now, we prove what follows:

Lemma 2.3. *Let $G \in C([0,1])$. Then, the problem*

$$\begin{cases} D^\alpha(D^\beta(D^\delta y(t))) = G(t), \\ y(0) = \xi \in \mathbb{R}, \\ y(1) = \int_0^\eta y(s)ds, \ 0 < \eta < 1, \\ I^e y(\theta) = D^\delta y(1), \ 0 < u < 1, \\ 0 < \alpha, \beta, \delta, p \leq 1, \ , \ q > 0, \ , \ t \in [0,1], \end{cases}$$

has

$$\begin{aligned} y(t) &= I^{\delta+\beta+\alpha}G(t) - \frac{\phi_2\varphi_1 - \phi_1\varphi_2}{(\phi_1\varphi_3 - \phi_3\varphi_1)\Gamma(\beta+\delta+1)} t^{\beta+\delta} \\ &\quad - \Big[\frac{\varphi_2}{\varphi_1} + \frac{\varphi_3(\phi_2\varphi_1 - \phi_1\varphi_2)}{\varphi_1(\phi_1\varphi_3 - \phi_3\varphi_1)}\Big]\frac{1}{\Gamma(\delta+1)} t^\delta + \xi \\ &= I^{\delta+\beta+\alpha}G(t) + \Big[\frac{\phi_1 t^{\beta+\delta}}{(\phi_1\varphi_3 - \phi_3\varphi_1)\Gamma(\beta+\delta+1)} \\ &\quad + \frac{\varphi_3\phi_1 t^\delta}{\varphi_1(\phi_1\varphi_3 - \phi_3\varphi_1)\Gamma(\delta+1)} - \frac{t^\delta}{\varphi_1\Gamma(\delta+1)}\Big]\varphi_2 \\ &\quad - \Big[\frac{\varphi_1 t^{\beta+\delta}}{(\phi_1\varphi_3 - \phi_3\varphi_1)\Gamma(\beta+\delta+1)} + \frac{\varphi_3 t^\delta}{(\phi_1\varphi_3 - \phi_3\varphi_1)\Gamma(\delta+1)}\Big]\phi_2, \end{aligned} \tag{2.1}$$

as integral solution; where,

$$\varphi_1 = \frac{\theta^{\delta+e}}{\Gamma(\delta+e+1)} - 1, \quad \varphi_2 = I^{e+\delta+\beta+\alpha}G(\theta) - I^{\beta+\alpha}G(1) + \frac{\xi\theta^e}{\Gamma(e+1)},$$

$$\varphi_3 = \frac{1}{\Gamma(\beta+1)} - \frac{\theta^{\beta+\delta+e}}{\Gamma(\beta+\delta+e+1)}, \quad \phi_1 = \frac{\eta^{\delta+1}}{\Gamma(\delta+2)} - \frac{1}{\Gamma(\delta+1)},$$

$$\phi_2 = \int_0^\eta I^{\delta+\beta+\alpha}G(s)ds - I^{\delta+\beta+\alpha}G(1) + \xi(\eta-1),$$

$$\phi_3 = \frac{1}{\Gamma(\beta+\delta+1)} - \frac{\eta^{\beta+\delta+1}}{\Gamma(\beta+\delta+2)},$$

and

$$\theta^{\delta+1} \neq \Gamma(\delta+e+1), \quad \eta^{\delta+1} \neq \delta+1, \quad \phi_1\varphi_3 \neq \phi_3\varphi_1.$$

Proof. We shall use Lemma 2.2 to see that

$$D^\delta y(t) = I^\beta(I^\alpha G(t)) - \frac{c_0}{\Gamma(\beta+1)} t^\beta - c_1,$$

$$y(t) = I^\delta(I^\beta(I^\alpha G(t))) - \frac{c_0}{\Gamma(\beta+\delta+1)} t^{\beta+\delta} - \frac{c_1}{\Gamma(\delta+1)} t^\delta - c_2. \tag{2.2}$$

Therefore, we have

$$\begin{aligned}
y(0) = \xi \ &\Rightarrow \ -c_2 = \xi, \\
I^e y(\theta) = D^\delta y(1) \ &\Rightarrow \ \Big[\frac{\theta^{\delta+e}}{\Gamma(\delta+e+1)} - 1\Big] c_1 \\
&= \Big[I^e(I^\delta(I^\beta(I^\alpha G(\theta)))) - I^\beta(I^\alpha G(1)) + \frac{\xi\theta^e}{\Gamma(e+1)}\Big] \\
&\quad + \Big[\frac{1}{\Gamma(\beta+1)} - \frac{\theta^{\beta+\delta+e}}{\Gamma(\beta+\delta+e+1)}\Big] c_0 \\
y(1) = \int_0^\eta y(s)ds \ &\Rightarrow \ \Big[\frac{\eta^{\delta+1}}{\Gamma(\delta+2)} - \frac{1}{\Gamma(\delta+1)}\Big] c_1 \\
&= \Big[\int_0^\eta I^\delta(I^\beta(I^\alpha G(s))ds - I^\delta(I^\beta(I^\alpha G(1)) + \\
&\quad +\xi(\eta-1)\Big]\Big[\frac{1}{\Gamma(\beta+\delta+1)} - \frac{\eta^{\beta+\delta+1}}{\Gamma(\beta+\delta+2)}\Big] c_0
\end{aligned}$$

$$c_0 = \frac{\phi_2\varphi_1 - \phi_1\varphi_2}{(\phi_1\varphi_3 - \phi_3\varphi_1)\Gamma(\beta+\delta+1)}, \qquad c_1 = \frac{\varphi_2}{\varphi_1} + \frac{\varphi_3(\phi_2\varphi_1 - \phi_1\varphi_2)}{\varphi_1(\phi_1\varphi_3 - \phi_3\varphi_1)}.$$

By considering the values of c_0, c_1 and c_2 in (2.2), we get (2.1). □

In what follows, we use fixed point theory to study the problem

$$X := \{x \in C(J,\mathbb{R}), D^p x \in C(J,\mathbb{R})\},$$

and the norm: $\|x\|_X = Max\{\|x\|_\infty, \|D^p x\|_\infty\}$, where,

$$\|x\|_\infty = \sup_{t\in J} |x(t)|, \|D^p x\|_\infty = \sup_{t\in J} |D^p x(t)|.$$

Then, we take the nonlinear operator $H : X \to X$ that is defined by:

$$\begin{aligned}
Hy(t) =& \frac{1}{\Gamma(\delta+\beta+\alpha)} \int_0^t (t-s)^{\delta+\beta+\alpha-1} \big(l(s) - h(s,y(s)) - f(s, D^p y(s), I^q y(s)) \\
& - g(s, D^p y(s), I^q y(s))\big) ds + \Big[\frac{\phi_1 t^{\beta+\delta}}{(\phi_1\varphi_3 - \phi_3\varphi_1)\Gamma(\beta+\delta+1)} \\
& + \frac{\varphi_3 \phi_1 t^{\delta}}{\varphi_1(\phi_1\varphi_3 - \phi_3\varphi_1)\Gamma(\delta+1)} - \frac{t^\delta}{\varphi_1 \Gamma(\delta+1)}\Big] \\
& \cdot \Big[\frac{1}{\Gamma(e+\delta+\beta+\alpha)} \int_0^\theta (\theta - s)^{e+\delta+\beta+\alpha-1} \big(l(s) - h(s,y(s)) - f(s, D^p y(s), \\
& \quad I^q y(s)) - g(s, D^p y(s), I^q y(s))\big) ds + \frac{1}{\Gamma(\beta+\alpha)} \int_0^1 (1-s)^{\beta+\alpha-1} \big(l(s) \\
& - h(s,y(s)) - f(s, D^p y(s), I^q y(s)) - g(s, D^p y(s), I^q y(s))\big) ds + \frac{\xi \theta^e}{\Gamma(e+1)}\Big] \\
& - \Big[\frac{\varphi_1 t^{\beta+\delta}}{(\phi_1\varphi_3 - \phi_3\varphi_1)\Gamma(\beta+\delta+1)} + \frac{\varphi_3 t^\delta}{(\phi_1\varphi_3 - \phi_3\varphi_1)\Gamma(\delta+1)}\Big] \\
& \cdot \Big[\int_0^\eta \frac{1}{\Gamma(\delta+\beta+\alpha)} \times \int_0^s (s-\tau)^{\delta+\beta+\alpha-1} \big(l(\tau) - h(\tau, y(\tau)) \\
& - f(\tau, D^p y(\tau), I^q y(\tau)) - g(\tau, D^p y(\tau), I^q y(\tau))\big) d\tau ds \\
& - \frac{1}{\Gamma(\delta+\beta+\alpha)} \int_0^1 (1-s)^{\delta+\beta+\alpha-1} \big(l(s) - h(s,y(s)) - f(s, D^p y(s), I^q y(s)) \\
& - g(s, D^p y(s), I^q y(s))\big) ds + \xi(\eta - 1)\Big].
\end{aligned}$$

3. Main Results

We consider the following hypotheses:

$(Q1)$: The functions f and g defined on $[0,1] \times \mathbb{R}^2$ are continuous, and h defined on $[0,1] \times \mathbb{R}$ is also continuous, and l are continuous over J.

$(Q2)$: There exist nonnegative constants $\nu_{f1}, \nu_{f2}, \nu_{g1}, \nu_{g2}$, such that for any t $\in J$, $x_i, x_i^* \in \mathbb{R}$,

$$\begin{aligned}
|f(t, x_1, x_2) - f(t, x_1{}^*, x_2{}^*)| &\leq \sum_{i=1}^{2} \nu_{fi} |x_i - x_i{}^*|, \\
|g(t, x_1, x_2) - g(t, x_1{}^*, x_2{}^*)| &\leq \sum_{i=1}^{2} \nu_{gi} |x_i - x_i{}^*|,
\end{aligned}$$

and for any t $\in J$, $u, v \in \mathbb{R}$, $|h(t,u) - h(t,v)| \leq r|u - v|$. It is to note that we take: $\Lambda_f := Max(\nu_{f1}, \nu_{f2}), \Lambda_g := Max(\nu_{g1}, \nu_{g2})$.

$(Q3)$: There exist non negative constants M_f, M_g, M_h, such that, for any $t \in J$, $x, y \in \mathbb{R}$, we have $|f(t,x)| \leq \Delta_f, |g(t,x)| \leq \Delta_g, |h(t,y)| \leq \Delta_h$.

$(Q4)$: The function l satisfies: $\|l\|_\infty = \Delta_l$.

Also, we consider the quantities:

$$\begin{aligned}
\Upsilon_1 \;=\; & \left[r + 2\Lambda_f + \Lambda_g + \frac{\Lambda_g}{\Gamma(q+1)}\right]\left[\frac{1}{\Gamma(\delta+\beta+\alpha+1)}\right. \\
& + \left(\frac{|\phi_1|}{|\phi_1\varphi_3 - \phi_3\varphi_1|\Gamma(\beta+\delta+1)} + \frac{|\varphi_3\phi_1|}{|\varphi_1(\phi_1\varphi_3 - \phi_3\varphi_1)|\Gamma(\delta+1)}\right. \\
& \left. + \frac{1}{|\varphi_1|\Gamma(\delta+1)}\right) \cdot \left(\frac{\theta^{e+\delta+\beta+\alpha}}{\Gamma(e+\delta+\beta+\alpha+1)} + \frac{1}{\Gamma(\beta+\alpha+1)}\right) \\
& + \left(\frac{|\varphi_1|}{|\phi_1\varphi_3 - \phi_3\varphi_1|\Gamma(\beta+\delta+1)} + \frac{|\varphi_3|}{|\phi_1\varphi_3 - \phi_3\varphi_1|\Gamma(\delta+1)}\right) \\
& \left. \cdot \left(\frac{\eta^{\delta+\beta+\alpha+1}}{\Gamma(\delta+\beta+\alpha+2)} + \frac{1}{\Gamma(\delta+\beta+\alpha+1)}\right)\right],
\end{aligned}$$

$$\begin{aligned}
\Upsilon_2 \;=\; & \left[r + 2\Lambda_f + \Lambda_g + \frac{\Lambda_g}{\Gamma(q+1)}\right]\left[\frac{1}{\Gamma(\delta+\beta+\alpha-p+1)}\right. \\
& + \left(\frac{|\phi_1|}{|\phi_1\varphi_3 - \phi_3\varphi_1|\Gamma(\beta+\delta-p+1)}\right. \\
& \left. + \frac{|\varphi_3\phi_1|}{|\varphi_1(\phi_1\varphi_3 - \phi_3\varphi_1)|\Gamma(\delta-p+1)} + \frac{1}{|\varphi_1|\Gamma(\delta-p+1)}\right) \\
& \cdot \left(\frac{\theta^{e+\delta+\beta+\alpha}}{\Gamma(e+\delta+\beta+\alpha+1)} + \frac{1}{\Gamma(\beta+\alpha+1)}\right) \\
& + \left(\frac{|\varphi_1|}{|\phi_1\varphi_3 - \phi_3\varphi_1|\Gamma(\beta+\delta-p+1)} + \frac{|\varphi_3|}{|\phi_1\varphi_3 - \phi_3\varphi_1|\Gamma(\delta-p+1)}\right) \\
& \left. \times \left(\frac{\eta^{\delta+\beta+\alpha+1}}{\Gamma(\delta+\beta+\alpha+2)} + \frac{1}{\Gamma(\delta+\beta+\alpha+1)}\right)\right].
\end{aligned}$$

We pass to establish the following result:

Theorem 3.1. *Assume that* $(Q_2), (Q_3), (Q_4)$ *are satisfied. Then, the problem* (1.1) *has a unique solution, provided that* $\Upsilon < 1$, *where* $\Upsilon := \max\{\Upsilon_1, \Upsilon_2\}$.

Proof. We proceed to prove that H is a contraction mapping. For $(x,y)\in X^2$, we can write

$$\begin{aligned}
&\|Hy-Hx\|_\infty\\
\leq\ & \left[r+2\Lambda_f+\Lambda_g+\frac{\Lambda_g}{\Gamma(q+1)}\right]\Bigg[\frac{1}{\Gamma(\delta+\beta+\alpha+1)}+\Bigg(\frac{|\phi_1|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta+1)}\\
&+\frac{|\varphi_3\phi_1|}{|\varphi_1(\phi_1\varphi_3-\phi_3\varphi_1)|\Gamma(\delta+1)}+\frac{1}{|\varphi_1|\Gamma(\delta+1)}\Bigg)\Bigg(\frac{\theta^{e+\delta+\beta+\alpha}}{\Gamma(e+\delta+\beta+\alpha+1)}\\
&+\frac{1}{\Gamma(\beta+\alpha+1)}\Bigg)+\Bigg(\frac{|\varphi_1|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta+1)}+\frac{|\varphi_3|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\delta+1)}\Bigg)\\
&\times\Bigg(\frac{\eta^{\delta+\beta+\alpha+1}}{\Gamma(\delta+\beta+\alpha+2)}+\frac{1}{\Gamma(\delta+\beta+\alpha+1)}\Bigg)\Bigg]\|y-x\|_X.
\end{aligned}$$

On the other hand, we can write

$$\begin{aligned}
&\|D^pHy-D^pHx\|_\infty\\
\leq\ & \left[r+2\Lambda_f+\Lambda_g+\frac{\Lambda_g}{\Gamma(q+1)}\right]\Bigg[\frac{1}{\Gamma(\delta+\beta+\alpha-p+1)}\\
&+\Bigg(\frac{|\phi_1|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta-p+1)}+\frac{|\varphi_3\phi_1|}{|\varphi_1(\phi_1\varphi_3-\phi_3\varphi_1)|\Gamma(\delta-p+1)}\\
&+\frac{1}{|\varphi_1|\Gamma(\delta-p+1)}\Bigg)\Bigg(\frac{\theta^{e+\delta+\beta+\alpha}}{\Gamma(e+\delta+\beta+\alpha+1)}+\frac{1}{\Gamma(\beta+\alpha+1)}\Bigg)\\
&+\Bigg(\frac{|\varphi_1|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta-p+1)}+\frac{|\varphi_3|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\delta-p+1)}\Bigg)\\
&\times\Bigg(\frac{\eta^{\delta+\beta+\alpha+1}}{\Gamma(\delta+\beta+\alpha+2)}+\frac{1}{\Gamma(\delta+\beta+\alpha+1)}\Bigg)\Bigg]\|y-x\|_X.
\end{aligned}$$

Consequently, we observe that $\|Hy-Hx\|_X\leq\Upsilon\,\|x-y\|_X.$ □

Now, we pass to prove the following theorem:

Theorem 3.2. *Assume that hypotheses* $(Q1)$, $(Q3)$ *and* $(Q4)$ *are satisfied. Then,* (1.1) *admits at least one solution.*

Proof. Let us prove the result by proceeding into the steps:
Step 1: It is clear that H is continuous on X.
Step 2: Can we say that H maps bounded sets into bounded sets in X? Let us take $r>0$ and $B_r:=\{x\in X;\|x\|_X\leq r\}$. For y $\in B_r$, thanks to the hypotheses

$(Q3)$ and $(Q4)$, we can write

$$\begin{aligned}
&\|Hy\|_\infty \leq \\
&\left[\Delta_l+\Delta_h+\Delta_f+\Delta_g\right]\Bigg[\frac{1}{\Gamma(\delta+\beta+\alpha+1)} \\
&+\bigg(\frac{|\phi_1|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta+1)} \\
&+\frac{|\varphi_3\phi_1|}{|\varphi_1(\phi_1\varphi_3-\phi_3\varphi_1)|\Gamma(\delta+1)}+\frac{1}{|\varphi_1|\Gamma(\delta+1)}\bigg)\bigg(\frac{\theta^{e+\delta+\beta+\alpha}}{\Gamma(e+\delta+\beta+\alpha+1)} \\
&+\frac{1}{\Gamma(\beta+\alpha+1)}+\frac{|\xi|\theta^e}{\Gamma(e+1)}\bigg)+\bigg(\frac{|\varphi_1|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta+1)} \\
&+\frac{|\varphi_3|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\delta+1)}\bigg) \\
&\times\bigg(\frac{\eta^{\delta+\beta+\alpha+1}}{\Gamma(\delta+\beta+\alpha+2)}+\frac{1}{\Gamma(\delta+\beta+\alpha+1)}+|\xi(\eta-1)|\bigg)\Bigg] < +\infty,
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&\|D^\delta Hy\|_\infty \leq \\
&\left[\Delta_l+\Delta_h+\Delta_f+\Delta_g\right]\Bigg[\frac{1}{\Gamma(\delta+\beta+\alpha-p+1)} \\
&+\bigg(\frac{|\phi_1|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta-p+1)} \\
&+\frac{|\varphi_3\phi_1|}{|\varphi_1(\phi_1\varphi_3-\phi_3\varphi_1)|\Gamma(\delta-p+1)}+\frac{1}{|\varphi_1|\Gamma(\delta-p+1)}\bigg) \\
&\bigg(\frac{\theta^{e+\delta+\beta+\alpha}}{\Gamma(e+\delta+\beta+\alpha+1)}+\frac{1}{\Gamma(\beta+\alpha+1)}+\frac{|\xi|\theta^e}{\Gamma(e+1)}\bigg) \\
&+\bigg(\frac{|\varphi_1|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta-p+1)} \\
&+\frac{|\varphi_3|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\delta-p+1)}\bigg)\times\bigg(\frac{\eta^{\delta+\beta+\alpha+1}}{\Gamma(\delta+\beta+\alpha+2)} \\
&+\frac{1}{\Gamma(\delta+\beta+\alpha+1)}+|\xi(\eta-1)\bigg)\Bigg] < +\infty.
\end{aligned} \tag{3.2}$$

So, we have $\|Hy\|_X < +\infty$, Consequently, H is uniformly bounded on B_r.

Step 3: Can one confirm that H maps bounded sets into equicontinuous sets of X? Let $t_1, t_2 \in [0,1], t_1 < t_2$ and let B_r be any bounded set of X. So by considering $x \in B_r$, we can state that for each $t \in [0,1]$, we have

$$
\begin{aligned}
&|Hx(t_1)-Hx(t_2)|\leq\\
&\Big[\Delta_l+\Delta_h+\Delta_f+\Delta_g\Big]\Big[\frac{|{t_1}^{\delta+\beta+\alpha}-{t_2}^{\delta+\beta+\alpha}|+2|t_1-t_2|^{\delta+\beta+\alpha}}{\Gamma(\delta+\beta+\alpha+1)}\\
&+\Big(\frac{|\phi_1||{t_1}^{\delta+\beta}-{t_2}^{\delta+\beta}|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta+1)}+\frac{|\varphi_3\phi_1||{t_1}^{\delta}-{t_2}^{\delta}|}{|\varphi_1(\phi_1\varphi_3-\phi_3\varphi_1)|\Gamma(\delta+1)}+\frac{|{t_1}^{\delta}-{t_2}^{\delta}|}{|\varphi_1|\Gamma(\delta+1)}\Big)\\
&\times\Big(\frac{\theta^{e+\delta+\beta+\alpha}}{\Gamma(e+\delta+\beta+\alpha+1)}+\frac{1}{\Gamma(\beta+\alpha+1)}+\frac{|\xi|\theta^e}{\Gamma(e+1)}\Big)\\
&+\Big(\frac{|\varphi_1||{t_1}^{\delta+\beta}-{t_2}^{\delta+\beta}|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta+1)}+\frac{|\varphi_3||{t_1}^{\delta}-{t_2}^{\delta}|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\delta+1)}\Big)\\
&\times\Big(\frac{\eta^{\delta+\beta+\alpha+1}}{\Gamma(\delta+\beta+\alpha+2)}+\frac{1}{\Gamma(\delta+\beta+\alpha+1)}+|\xi(\eta-1)|\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&|D^pHx(t_1)-D^pHx(t_2)|\leq\\
&\Big[\Delta_l+\Delta_h+\Delta_f+\Delta_g\Big]\Big[\frac{|{t_1}^{\delta+\beta+\alpha-p}-{t_2}^{\delta+\beta+\alpha-p}|+2|t_1-t_2|^{\delta+\beta+\alpha-p}}{\Gamma(\delta+\beta+\alpha-p+1)}\\
&\quad+\Big(\frac{|\phi_1||{t_1}^{\delta+\beta-p}-{t_2}^{\delta+\beta-p}|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta-p+1)}+\frac{|\varphi_3\phi_1||{t_1}^{\delta-p}-{t_2}^{\delta-p}|}{|\varphi_1(\phi_1\varphi_3-\phi_3\varphi_1)|\Gamma(\delta-p+1)}\\
&\quad+\frac{|{t_1}^{\delta-p}-{t_2}^{\delta-p}|}{|\varphi_1|\Gamma(\delta-p+1)}\Big)\Big(\frac{\theta^{e+\delta+\beta+\alpha}}{\Gamma(e+\delta+\beta+\alpha+1)}+\frac{1}{\Gamma(\beta+\alpha+1)}+\frac{|\xi|\theta^e}{\Gamma(e+1)}\Big)\\
&\quad+\Big(\frac{|\varphi_1||{t_1}^{\delta+\beta-p}-{t_2}^{\delta+\beta-p}|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta-p+1)}+\frac{|\varphi_3||{t_1}^{\delta-p}-{t_2}^{\delta-p}|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\delta-p+1)}\Big)\\
&\quad\times\Big(\frac{\eta^{\delta+\beta+\alpha+1}}{\Gamma(\delta+\beta+\alpha+2)}+\frac{1}{\Gamma(\delta+\beta+\alpha+1)}+|\xi(\eta-1)\Big)\Big].
\end{aligned}
$$

By Steps 1,2,3 and also with the Arzela-Ascoli theorem, we conclude that H is completely continuous.

Step 4: Is $A:=\{x\in X: x=\varsigma\ Hx,\varsigma\in]0,1[\}$ bounded? Let $y\in A,$ Then, we have $y=\varsigma Hy$ for some $0<\varsigma<1$. Hence, we can write

$$\begin{aligned}
\|y\|_\infty \;&\le \varsigma\Bigg(\Big[\Delta_l+\Delta_h+\Delta_f+\Delta_g\Big]\Big[\frac{1}{\Gamma(\delta+\beta+\alpha+1)}\\
&+\Bigg(\frac{|\phi_1|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta+1)}\\
&+\frac{|\varphi_3\phi_1|}{|\varphi_1(\phi_1\varphi_3-\phi_3\varphi_1)|\Gamma(\delta+1)}+\frac{1}{|\varphi_1|\Gamma(\delta+1)}\Bigg)\\
&\cdots\Bigg(\frac{\theta^{e+\delta+\beta+\alpha}}{\Gamma(e+\delta+\beta+\alpha+1)}+\frac{1}{\Gamma(\beta+\alpha+1)}\\
&+\frac{|\xi|\theta^e}{\Gamma(e+1)}\Bigg)+\Bigg(\frac{|\varphi_1|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta+1)}+\frac{|\varphi_3|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\delta+1)}\Bigg)\\
&\times\Bigg(\frac{\eta^{\delta+\beta+\alpha+1}}{\Gamma(\delta+\beta+\alpha+2)}+\frac{1}{\Gamma(\delta+\beta+\alpha+1)}+|\xi(\eta-1)|\Bigg)\Big]\Bigg).
\end{aligned}$$

We have also

$$\begin{aligned}
\|D^p y\|_\infty \le\;& \varsigma\Bigg(\Big[\Delta_l+\Delta_h+\Delta_f+\Delta_g\Big]\Big[\frac{1}{\Gamma(\delta+\beta+\alpha-p+1)}\\
&+\Bigg(\frac{|\phi_1|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta-p+1)}+\frac{|\varphi_3\phi_1|}{|\varphi_1(\phi_1\varphi_3-\phi_3\varphi_1)|\Gamma(\delta-p+1)}\\
&+\frac{1}{|\varphi_1|\Gamma(\delta-p+1)}\Bigg)\Bigg(\frac{\theta^{e+\delta+\beta+\alpha}}{\Gamma(e+\delta+\beta+\alpha+1)}\\
&+\frac{1}{\Gamma(\beta+\alpha+1)}+\frac{|\xi|\theta^e}{\Gamma(e+1)}\Bigg)+\Bigg(\frac{|\varphi_1|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\beta+\delta-p+1)}\\
&+\frac{|\varphi_3|}{|\phi_1\varphi_3-\phi_3\varphi_1|\Gamma(\delta-p+1)}\Bigg)\\
&\times\Bigg(\frac{\eta^{\delta+\beta+\alpha+1}}{\Gamma(\delta+\beta+\alpha+2)}+\frac{1}{\Gamma(\delta+\beta+\alpha+1)}\\
&+|\xi(\eta-1)\Bigg)\Big]\Bigg).
\end{aligned}$$

Using (3.1) and (3.2), we state that $\|y\|_X < \infty$. The set is thus bounded.

We deduce by Schaeffer theorem that H has a fixed point which is a solution of the problem (1.1). □

Example 1. *We consider the following problem:*

$$\begin{cases} D^{\frac{1}{2}}(D^{\frac{2}{5}}(D^{\frac{4}{5}}y(t))) + \dfrac{1}{20e^{t^2+4}}\left(\dfrac{|y(t)|}{\pi(1+|y(t)|)} + \cos D^{\frac{1}{10}}y(t)\right) \\ + \dfrac{|y(t)| + |I^{\frac{1}{2}}y(t)|}{(t+300)(e^t + |y(t)| + |I^{\frac{1}{2}}y(t)|)} \\ + \dfrac{1}{120\pi e^{t+2}}\left(\sin y(t) + \ln(t+2)\right) = \dfrac{t}{4}, \\ y(0) = 3,\ y(1) = \displaystyle\int_0^{\frac{1}{5}} y(s)ds,\ I^{\frac{1}{5}}y(\frac{1}{2}) = D^{\frac{4}{5}}y(1), \quad t \in [0,1], \end{cases}$$

where, we take:

$$f(t,u,v) = \frac{1}{20e^{t^2+4}}\left(\frac{|u|}{\pi(1+|u|)} + \cos v\right),\ g(t,u,v) = \frac{|u| + |v|}{(t+300)(e^t + |u| + |v|)},$$
$$h(t,u) = \frac{1}{120\pi e^{t+2}}\left(\sin u + \ln(t+2)\right),\ l(t) = \frac{t}{4},$$
$$\Upsilon_1 = 0.1178, \quad \Upsilon_2 = 0.1219,\ \Upsilon = max\left\{\Upsilon_1, \Upsilon_2\right\} = 0.1219.$$

The conditions of Theorem 3.1 *hold. Therefore, our example has a unique solution on* $[0,1]$.

ACKNOWLEDGMENT

This paper is supported by DGRSDT Algerie: www. dgrsdt.dz.

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