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THE GENERALIZED WAVELET TRANSFORM ON SOBOLEV TYPE SPACES

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ABSTRACT. The Generalized Wavelet transform is studied on the Sobolev type space$B_k^{\omega}\left(R^n\right)$ Boundedness results in this Sobolev space is obtained. Compactly supported wavelets on distribution space are also studied. Approximation properties of the generalized wavelet transform will also be discussed.

1. INTRODUCTION

Let $\psi \in L^2\left(\mathbf{R}^n\right)$ be the analyzing wavelet and $f \in L^2\left(\mathbf{R}^n\right)$ be any function. We define the translation operator τ_b by

$$\tau_b\psi\left(x\right) = \psi\left(x-b\right),\ b \in \mathbf{R}^n,$$

and the dilation operator D_a by

$$D_a\psi\left(x\right) = \left|a\right|^{-1/2}\psi\left(\frac{x}{a}\right),\ a \in \mathbf{R}^n.$$

A unitary transformation $W\left(b,a\right) : L^2\left(\mathbf{R}^n, dt\right) \to L^2\left(\mathbf{R}^n, dt\right)$ is defined by

$$W\left(b,a\right)\psi\left(x\right) = \left(\tau_b D_a\psi\right)\left(x\right) = \left|a\right|^{-1/2}\psi\left(\frac{x-b}{a}\right);\ \left(b,a\right) \in \mathbf{R}^n \times \mathbf{R}^n,$$

and the Fourier transform of $f \in L^1\left(\mathbf{R}^n\right)$ by

$$\hat{f}\left(\xi\right) = \int_{\mathbf{R}^n} f\left(x\right)e^{-i\langle\xi,x\rangle}dx;\ \xi, x \in \mathbf{R}^n.$$

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Definition 1.1. *A function $\psi \in L^2(\mathbf{R}^n, dt)$ is admissible only if ψ is not identical to zero and*

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \left|\langle W(b,a)\psi, \psi\rangle_0\right|^2 \frac{dadb}{a^2} < \infty.$$

Lemma 1.1. *$\psi \in L^2(\mathbf{R}^n, dt)\backslash\{0\}$ is admissible if and only if the integral $\int_{\mathbf{R}^n} \frac{\left|\hat{\psi}(\xi)\right|^2}{|\xi|} d\xi$ exists.*

Proof. See [5, p.877]. □

Lemma 1.2. *Let ψ be admissible and $f \in L^2(\mathbf{R}^n, dt)$. Let*

$$C_\psi = \int_{\mathbf{R}^n} \frac{\left|\hat{\psi}(\xi)\right|^2}{|\xi|} d\xi.$$

The integral

$$(1.1)\qquad \begin{aligned}(W_\psi f)(b,a) =& \tilde{f}(b,a) = \frac{1}{\sqrt{C_\psi}} \langle f, W(b,a)\psi\rangle_0 \\ =& \frac{1}{\sqrt{C_\psi}} \frac{1}{\sqrt{|a|}} \int_{\mathbf{R}^n} \bar{\psi}\left(\frac{t-b}{a}\right) f(t)\, dt\end{aligned}$$

defines an element of $L^2\left(R^n \times R^n, \frac{dbda}{a^2}\right)$. Moreover,

$$W_\psi : L^2(\mathbf{R}^n, dt) \to L^2\left(\mathbf{R}^n \times \mathbf{R}^n, \frac{dbda}{a^2}\right)$$

is an isometry.

In this paper, we extend the continuous wavelet transform, which we defined on $L^2(\mathbf{R}^n, dt)$, to Sobolev type space $\mathbf{B}_k^\omega(\mathbf{R}^n)$ and boundedness properties will be investigated. Approximation properties for small dilation parameter will also be studied.

2. The Sobolev type Space $\mathbf{B}_k^\omega(\mathbf{R}^n)$

In this section, we recall definitions and properties of certain function and distribution spaces introduced by Björck [1]. Let $\mathbf{M}$ be the set of continuous and real valued functions ω on $\mathbf{R}^n$, satisfying the following conditions:

(i) $0 = \omega(0) \le \omega(\xi + n) \le \omega(\xi) + \omega(\eta)\,;\ \xi, \eta \in \mathbf{R}^n$,

(ii) $\int_{\mathbf{R}^n} \frac{\omega(\xi)}{(1+|\xi|)^{n+1}} < \infty$,

(iii) $\omega(\xi) \geq a + b\log(1+|\xi|)\,;\ \xi \in \mathbf{R}^n,$

for some real number a and position real number b. We denote by $\mathbf{M}_c$ the set of all $\omega \in \mathbf{M}$ satisfying condition $\omega(\xi) = \omega(|\xi|)$ with a concave function ω on $[0,\infty)$. We suppose $\omega \in \mathbf{M}_c$ from now on.

Let $\omega \in \mathbf{M}_c$. We denote by $\mathbf{S}_\omega$ the set of all functions $\phi \in L^1(\mathbf{R}^n)$ with the property that ϕ and $\hat{\phi} \in C^\infty$ and

$$p_{\alpha,\lambda}(\phi) = \sup_{x\in\mathbf{R}^n} e^{\lambda\omega(x)} |D^\alpha \phi(x)| < \infty,$$

$$\pi_{\alpha,\lambda}(\phi) = \sup_{\xi\in\mathbf{R}^n} e^{\lambda\omega(\xi)} \left|D^\alpha \hat{\phi}(\xi)\right| < \infty.$$

The topology of $\mathbf{S}_\omega$ is defined by the semi-norms $p_{\alpha,\lambda}$ and $\pi_{\alpha,\lambda}$. The dual of $\mathbf{S}_\omega$ is denoted by $\mathbf{S}'_\omega$ the elements of which are called ultra-distributions. We may refer to [1] for its various properties. We note that for $\omega(\xi) = \log(1+|\xi|)\,,\ \mathbf{S}_\omega$is reduces to $\mathbf{S}$, the Schwartz space.

We also recall the definition of test function space $\mathbf{D}\phi$. The space $\mathbf{D}\phi$ is the set of all ϕ in $L^1(\mathbf{R}^n)$ such that ϕ has compact support and $\|\phi\|_\lambda < \infty$ for all $\lambda > 0$ and

$$\|\phi\|_\lambda = \int_{\mathbf{R}^n} \left|\hat{\phi}(\xi)\right| e^{\lambda\omega(\xi)} d\xi.$$

Now, let $\omega \in \mathbf{M}_c$. Then $\mathbf{K}_\omega$ is defined to be the set of positive function k in $\mathbf{R}^n$ with the following property. There exists $k > 0$ such that

$$k(\xi+\eta) \leq e^{\lambda\omega(-\xi)} k(\eta)\,\forall\, \xi,\eta \in \mathbf{R}^n.$$

Let $\omega \in \mathbf{M}_c$, $k \in \mathbf{K}_\omega$ and $1 \leq p < \infty$. Then Sobolev type space $\mathbf{B}^\omega_k(\mathbf{R}^n)$ is defined to be the space of all ultra-distributions $f \in \mathbf{S}_\omega$ such that

$$\|f\|_k = \left(\int_{\mathbf{R}^n} \left|k(\xi)\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} < \infty$$

and

$$\|f\|_{\infty,k} = ess\sup k(\xi)\left|\hat{f}(\xi)\right|.$$

An inner product in $\mathbf{B}^\omega_k(\mathbf{R}^n)$ is given by

$$\langle f,g\rangle_k = \int_{\mathbf{R}^n} \hat{f}(\xi)\,\hat{g}(\xi)\,(k(\xi))^2\,d\xi.$$

3. The Continuous Wavelet Transform on Sobolev type Space $\mathbf{B}_k^{\omega}(\mathbf{R}^n)$

In this section, we define the space ζ_k of all measurable functions f on $\mathbf{R}^n \times \mathbf{R}^n$ such that

$$\|f(b,a)\|_{\zeta_k} = \left(\int_{\mathbf{R}^n} \|f(b,a)\|_k^2 \frac{da}{a}\right)^{1/2} < \infty. \tag{3.1}$$

Theorem 3.1. *Assume that analyzing wavelet $\psi \in L^2$ satisfies the following admissibility condition:*

$$C_\psi = \int_{\mathbf{R}^n} \frac{\left|\hat{\psi}(\xi)\right|^2}{|\xi|} d\xi < \infty.$$

Let $(W_\psi f)(b,a)$ be the wavelet transform of the function $f \in \mathbf{B}_k^{\omega}$ with respect to the analyzing wavelet $\psi \in L^2$. Then

$$\|(W_\psi f)(b,a)\|_{\zeta_k}^2 = \|f\|_k .$$

Proof. From (3.1), we have

$$\begin{aligned}
\|(W_\psi f)(b,a)\|_{\zeta_k}^2 &= \int_{\mathbf{R}^n} \|(W_\psi f)(b,a)\|_k^2 \frac{da}{a} \\
&= \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} |k(\xi)|^2 \left|(L_\psi f)(b,a)^\wedge(\xi)\right|^2 d\xi\right) \frac{da}{a} \\
&= \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} |k(\xi)|^2 \left(\frac{1}{C_\psi}\right) a \left|\hat{\psi}(-a\xi)\right|^2 \left|\hat{f}(\xi)\right|^2 d\xi\right) \frac{da}{a} \\
&= \left(\frac{1}{C_\psi}\right) \int_{\mathbf{R}^n} \frac{\left|\hat{\psi}(-u)\right|^2}{|u|} du \int_{\mathbf{R}^n} \left|k(\xi)\hat{f}(\xi)\right|^2 d\xi \\
&= \left(\frac{1}{C_\psi}\right) \int_{\mathbf{R}^n} \frac{\left|\hat{\psi}(u)\right|^2}{|u|} du \,\|f\|_k \\
&= \left(\frac{1}{C_\psi}\right) C_\phi \|f\|_k \\
&= \|f\|_k .
\end{aligned}$$

□

Theorem 3.2. *For admissible and integrable ψ_1, ψ_2 and $f, g \in \mathbf{B}_k^{\omega}(\mathbf{R}^n)$*

$$\|W_{\psi_1} f(b,a) - W_{\psi_2} g(b,a)\|_k \le \left(\left\|\frac{\psi_1}{\sqrt{C_{\psi_1}}} - \frac{\psi_2}{\sqrt{C_{\psi_2}}}\right\|_{L_1} \|f\|_k + \left\|\frac{\psi_2}{\sqrt{C_{\psi_2}}}\right\|_{L_2} \|f-g\|_k\right).$$

Proof. We have

$$\begin{aligned}\|W_{\psi_1} f(b,a) - W_{\psi_2} g(b,a)\|_k \le \|W_{\psi_1} f(b,a) - W_{\psi_2} f(b,a)\|_k + \\ \|W_{\psi_2} f(b,a) - W_{\psi_2} g(b,a)\|_k .\end{aligned} \tag{3.2}$$

Now,

$$\begin{aligned}&\|W_{\psi_1} f(b,a) - W_{\psi_2} g(b,a)\|_k \\ &= \left(\int_{\mathbf{R}^n} \left|(W_{\psi_1} f(b,a) - W_{\psi_2} f(b,a))^\wedge(\xi)\right|^2 |k(\xi)|^2 d\xi\right)^{1/2} \\ &= \left(\int_{\mathbf{R}^n} \left|\sqrt{\frac{1}{C_{\psi_1}}}\, |a|^{1/2}\, \hat{\psi}_1(-a\xi)\, \hat{f}(\xi)\right.\right. \\ &\quad \left.\left. - \sqrt{\frac{1}{C_{\psi_2}}}\, |a|^{1/2}\, \hat{\psi}_2(-a\xi)\, f(\xi)\right|^2 |k(\xi)|^2 d\xi\right)^{1/2} \\ &= \left(\int_{\mathbf{R}^n} |a|\, |k(\xi)|^2 \left|\hat{f}(\xi)\right|^2 d\xi \left|\left(\frac{\hat{\psi}_1}{\sqrt{C_{\psi_1}}} - \frac{\hat{\psi}_2}{\sqrt{C_{\psi_2}}}\right)(-a\xi)\right|^2\right)^{1/2} .\end{aligned} \tag{3.3}$$

Now, using the inequality

$$\left|\hat{\psi}(\xi)\right| \le \|\psi\|_{L^1} ,$$

we have

$$\left|\frac{\hat{\psi}_1}{\sqrt{C_{\psi_1}}} - \frac{\hat{\psi}_2}{\sqrt{C_{\psi_2}}}\right| \le \left\|\frac{\psi_1}{\sqrt{C_{\psi_1}}} - \frac{\psi_2}{\sqrt{C_{\psi_2}}}\right\|_{L^1} , \tag{3.4}$$

so that

$$\left|\frac{\hat{\psi}_1}{\sqrt{C_{\psi_1}}} - \frac{\hat{\psi}_2}{\sqrt{C_{\psi_2}}}\right|^2 \le \left\|\frac{\psi_1}{\sqrt{C_{\psi_1}}} - \frac{\psi_2}{\sqrt{C_{\psi_2}}}\right\|_{L^1}^2 .$$

Using (3.4) in (3.3), we have

$$\|W_{\psi_1} f(b,a) - W_{\psi_2} f(b,a)\|_k \le |a|^{1/2} \left\|\frac{\psi_1}{\sqrt{C_{\psi_1}}} - \frac{\psi_2}{\sqrt{C_{\psi_2}}}\right\|_{L^1} \|f\|_k . \tag{3.5}$$

Similarly, we can write

$$\|W_{\psi_2} f(b,a) - W_{\psi_2} g(b,a)\|_k \le |a|^{1/2} \left\|\frac{\psi_2}{\sqrt{C_{\psi_2}}}\right\|_{L^1} \|f - g\|_k .$$

Invoking (3.5), (3.2), we have

$$\|W_{\psi_1}f(b,a) - W_{\psi_2}g(b,a)\|_k \leq |a|^{1/2} \left\| \frac{\psi_1}{\sqrt{C_{\psi_1}}} - \frac{\psi_2}{\sqrt{C_{\psi_2}}} \right\|_{L^1} \|f\|_k + \left\| \frac{\psi_2}{\sqrt{C_{\psi_2}}} \right\|_{L^1} \|f-g\|_k .$$

This completes the proof of Theorem 3.2. □

4. Approximation Properties of Continuous Wavelet Transform for small dilation

Let us recall the equation (1.1):

$$W_\psi f(b,a) = \frac{1}{\sqrt{C_\psi}} \frac{1}{\sqrt{|a|}} \int_{\mathbf{R}^n} \psi\left(\frac{t-b}{a}\right) f(t)\, dt. \tag{4.1}$$

In what follows we assume that ψ is real valued and $a > 0$.

Let us define

$$\psi_a(x) = \frac{1}{a}\psi\left(\frac{x}{a}\right).$$

Let us use the notation

$$\theta_\psi f(b,a) = (\psi_a * f)(b) = \frac{1}{a}\int_{\mathbf{R}^n} \psi\left(\frac{b-t}{a}\right) f(t)\, dt. \tag{4.2}$$

From (4.1) and (4.2), we have

$$(\psi_a * f)(b) = (\theta_\psi f)(b,a) = \sqrt{\frac{C_\psi}{a}} W_\psi f(b,-a).$$

Let $P(D) = \sum_{\alpha \geq 0} C_\alpha D^\alpha$ be a differential operator, where P is any polynomial.

$$\tilde{P}(\xi) = \sum_{\alpha \geq 0} |D^\alpha P(\xi)|^2 .$$

Then, by Hörmander [2, p.10], we have

(i) $f \in \mathbf{B}_k^\omega(\mathbf{R}^n) \Rightarrow P(D) f \in \mathbf{B}_{\frac{k}{\tilde{P}}}^\omega(\mathbf{R}^n)$.

(ii) $\|P(D) f\|_{\frac{k}{\tilde{P}}} \leq \|f\|_k$.

Theorem 4.1. *Let* $f \in \mathbf{B}_k^{\omega}(\mathbf{R}^n)$ *and* $\psi \in L^1(\mathbf{R}^n)$ *with* $\int_{\mathbf{R}^n} \psi(t)\,dt = 1$. *Then*

(i) $\theta_\psi f(.,a) \to f(.)$ *in* $\mathbf{B}_k^{\omega}(\mathbf{R}^n)$ *as* $a \to 0$.

(ii) $P(D)(\theta_\psi f)(.,a) = \wedge_{P(a^{-1}D)\psi} f = \theta_\psi P(D) f$.

Proof. (i) In view of (4.2), we have

$$\|\psi_a * f - f\|_k^2 = \int_{\mathbf{R}^n} \left|(\psi_a * f - f)^\wedge(\xi)\right|^2 |k(\xi)|^2 d\xi$$

$$= \int_{\mathbf{R}^n} \left|(\psi_a * f)^\wedge(\xi) - \hat{f}(\xi)\right|^2 |k(\xi)|^2 d\xi$$

$$= \int_{\mathbf{R}^n} \left|\left(\frac{C_\psi}{a}\right)^{1/2} ((W_\psi f)(b,a))^\wedge(\xi) - \hat{f}(\xi)\right|^2 |k(\xi)|^2 d\xi$$

$$= \int_{\mathbf{R}^n} \left|\left(\frac{C_\psi}{a}\right)^{1/2} \left(\frac{1}{C_\psi}\right)^{1/2} |a|^{1/2} \hat{\psi}(a\xi) \hat{f}(\xi) - \hat{f}(\xi)\right|^2 |k(\xi)|^2 d\xi$$

$$= \int_{\mathbf{R}^n} \left|\hat{\psi}(a\xi) \hat{f}(\xi) - \hat{f}(\xi)\right|^2 |k(\xi)|^2 d\xi$$

$$= \int_{\mathbf{R}^n} \left|\hat{f}(\xi)\right|^2 |k(\xi)|^2 \left|1 - \hat{\psi}(a\xi)\right|^2 d\xi$$

$$= \int_{\mathbf{R}^n} \hat{f}(\xi) |k(\xi)|^2 \left|1 - \hat{\psi}(a\xi)\right|^2 d\xi$$

$$= \int_{\mathbf{R}^n} |I(a,\xi)|^2 d\xi,$$

where

$$|I(a,\xi)| = \left|k(\xi) \hat{f}(\xi) \left(1 - \hat{\psi}(a\xi)\right)\right|.$$

Then we have

$$\lim_{a\to 0} |I(a,\xi)| = 0 \text{ a.e..}$$

Let us now set $M = \sup_{\xi\in\mathbf{R}} \left|1 - \hat{\psi}(a\xi)\right|$, which is independent of a. Then

$$|I(a,\xi)| \le M \left|k(\xi) \hat{f}(\xi)\right|.$$

Now, applying the dominated convergence theorem, we have

$$(\psi_a * f) = \theta_\psi f(.,a) \to f(.) \ \text{in } \mathbf{B}_k^{\omega}(\mathbf{R}^n) \ \text{as } a \to 0.$$

(ii) Let$\{f_n\}_{n\in\mathbf{N}} \in \mathbf{S}_\omega(\mathbf{R}^n)$ converge to f in $\mathbf{B}_k^{\omega}(\mathbf{R}^n)$. Now, operating the differential operator $P(D) = \sum_{\alpha\ge 0} C_\alpha D^\alpha$ on both sides of the equation (4.2), we have following equality

$$P(D)\,\theta_\psi f_n(b,a) = \theta_{P(a^{-1}D)\psi} f_n = \theta_\psi P(D)\, f_n;\; f_n \in S_\omega(\mathbf{R}^n)\,.$$

Since $S_\omega(\mathbf{R}^n)$ in dense in $\mathbf{B}_k^\omega(\mathbf{R}^n)$ and the operators θ_ψ and $P(D)$ are continuous, hence limits of the above three are equal in $\mathbf{B}_k^\omega(\mathbf{R}^n)$. □

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