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# AN ALMOST-PRODUCT STRUCTURE ON THE FOLIATED MANIFOLD

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ABSTRACT. Let $M$ be a differentiable, connected, paracompact (resp. Compact) manifold of dimension $n+m$, of class $C^\infty$; $L$ a vector valued $1-$form with rank $m$ and of zero Nijenhuis torsion. Such a manifold $M$ endowed with this vector valued $1-$form defines a foliated manifold. In this paper, we propose to define an almost-product structure denoted $\Gamma$ and to give some properties of this almost-product structure by studying the Lie algebras which attach to it on the foliated manifold. We study in particular the ideals, the centralizers and normalizers associated with the almost-product structure $\Gamma$ such that we can adopt some results found in [9].

## 1. INTRODUCTION

The theory of connections to Finslerian geometry is not satisfactorily established as in Riemannian geometry. Several attempts were made to build an adequate theory. The only most important in this direction is Grifone's theory (in [4] and [5]). This theory is essentially based on the almost-tangent structure on the tangent fiber of a differentiable manifold. M. Anona in [1] generalized the almost-tangent structure by considering a vector valued $1-$form $L$ on a manifold (without being fiber) satisfying certain conditions. He investigated the $d_L-$cohomology induced by $L$ on $M$ and generalized some Grifone's results. N. L. Youssef adopted from M. Anona's point of view in [1] a generalization of Grifone's approach on nonlinear

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connections by considering a vector valued $1-$form $L$ on the manifold $M$ of constant rank such that $[L, L] = 0$ and $Im\left(L_z\right) = ker\left(L_z\right); z \in M$. He found that $L$ has properties similar to $J$ which allows him to systematically generalize important results of Grifone's theory. Grifone's theory is a very special case in these results on the tangent bundle of a differentiable manifold, and $L$ is the almost-tangent structure $J$ ($J^2 = 0$). In [9], M. Anona, P. Randriambololondrantomalala and H. S. G. Ravelonirina studied some properties for a vector valued $1-$form $\Gamma$ having an almost-product structure ($\Gamma^2 = I$), in the sense of Grifone on a differentiable manifold $N$ of dimension $n$ by studying certain Lie algebras attached to it. In this paper, we propose to establish a similar property for an almost-product structure on a foliated manifold which is not necessarily a fiber. We propose to study the theory of this almost-product structure on the foliated manifold by a vector valued $1-$form by considering the Lie algebras associated with it.

Let $M$ be a differentiable, connected, paracompact (resp. compact) manifold of dimension $n+m$ and of class $C^\infty$. All objects are assumed to be of class $C^\infty$ on $M$. The Frolicher-Nijenhuis formalism is a fundamental tool in this work. We endow the manifold $M$ by a vector valued $1-$form $L$ of rank $m$ and of zero Nijenuis torsion ($[L, L] = 0$). The nullity of the Nijenhuis torsion of $L$ defines a foliation on $M$ such that the image space of $L$ corresponds to the space tangent to the leaves. First, M. Anona, P. Randriambololondrantomalala and H.S.G. Ravelonirina are interested in the Lie algebra $\mathfrak{A}_\Gamma$ of vector fields on the tangent fiber $TN - \{0\}$ of a differentiable manifold $N$ of dimension $n$ whose corresponding Lie derivative with the almost-product structure $\Gamma$ (within the meaning of Grifone) is zero (see [9]). M. Anona in [1] found that if $L$ is a connection in the sense of Grifone $L^2 = I$ where $I$ is an identity matrix of order $n$, the Lie algebra $\mathfrak{A}_L$ is isomorphic to $\chi(N) \times \mathbb{R}^n$. In our study, we propose to define an almost-product structure $\Gamma$ ($\Gamma^2 = I$) on a manifold $M$ foliated by $L$, by studying the Lie algebra $\mathfrak{A}_\Gamma$ of vector fields of $\chi\left(L, \mathcal{F}\left(M\right)\right)$ whose Lie derivative corresponding to $\Gamma$ with respect to a vector field of $\chi\left(L, \mathcal{F}\left(M\right)\right)$ is zero (with $\chi\left(L, \mathcal{F}\left(M\right)\right) = \chi\left(M\right)$). We have found a system of partial differential equations similar to that in [9] in order to find all the vector fields of Lie algebra $\mathfrak{A}_\Gamma$ on the considered manifold. Then, F. Taken in [13] proved that any derivation of the Lie algebra $\chi(M)$ on the manifold $M$ is inner. A. Lichnerowicz [7] considered the Lie $L_{\mathfrak{F}}$ algebra of infinitesimal automorphisms of a foliation $\mathfrak{F}$ on a manifold $M$ and proved that, whatever the considered foliation is, any derivation of $L_{\mathfrak{F}}$ is inner. If $L$ is a vector valued $1-$form of the transversal fiber to a foliation, Lehmann-Lejeune in [8] proved that the derivations of $\mathfrak{A}_L$ are adjunct linear applications of the normalizer of $\mathfrak{A}_L$. In our work, we can ask ourselves the behaviors of the derivations of the Lie algebra $\mathfrak{A}_\Gamma$ ($\Gamma^2 = I$) on the considered foliated manifold. In the following section, M. Anona, P. Randriambololondrantomalala and H.S.G. Ravelonirina have studied the Lie algebra $\mathfrak{A}_\Gamma^h$ of the fields of $\mathfrak{A}_\Gamma$ in horizontal space, the Lie algebra $\mathfrak{A}_\Gamma^v$ of the fields of $\mathfrak{A}_\Gamma$ in vertical space and the Lie algebra $\mathfrak{N}_R^h$ of the horizontal nullity space of the curvature $R$. In this part, our work consists of studying the structures of Lie subalgebras, ideals of Lie algebra $\mathfrak{A}_\Gamma$, its associated normalizers and the parts of the nullity space of the curvature $R$ associated with $\Gamma$. In each part, we give some examples to illustrate the results found.

## 2. Preliminary

In the following section, we assume that all the objects are of class $C^\infty$. Let $M$ be a differentiable, connected, paracompact (resp. compact) manifold of dimension $n+m$; $L$ a vector valued 1-form of rank $m \geq 1$ on $M$ whose Nijenhuis torsion is zero ($[L; L] = 0$). The nullity of Nijenhuis torsion of $L$ defines on $M$ a distribution $\mathfrak{D} : z \in M \to L_z(T_z M)$ which is completely integrable. The manifold $M$ endowed with this vector valued $1-$form of constant rank $p \geq 1$ is called a foliated manifold by $L$ such that the image space of $L$ corresponds to the space tangent to the leaves (we will suppose that the leafs are regular or fibers on $M$) . The manifold $M$ is then decomposed into connected sub-manifolds of $m$ dimension such that each of them is called "leaf". According to Fröbenuis theorem, the manifold $M$ can thus be defined by an open covering $\mathcal{U}$ of $M$ and by the data of each $U \in \mathcal{U}$, of a coordinate system $(x^1, \ldots, x^n, y^1, \ldots, y^m)$ such that in $U$, $y^\beta = C^{te}, 1 \leq \beta \leq m$; along of leafs and, the $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^m}$ form a local basis tangent to the leaves.

**Definition 2.1.** *A foliation $\mathfrak{F}$ of $n-$codimensional is the data an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ and for all $i$ of a diffeomorphism $\phi_i : \mathbb{R}^{n+m} \to U_i$ such that for all non-empty intersection $U_i \cap U_j$ the diffeomorphism of coordinate change*

$$\phi_j^{-1} \otimes \phi_i : (z,t) \in \phi_i^{-1}(U_i \cap U_j) \to \left(z^{'}, t^{'}\right) \in \phi_j^{-1}(U_i \cap U_j)$$

*is the form $z^{'} = \phi_{ij}(z,t)$ and $t^{'} = \gamma_{ij}(t)$.*

Let us denote by $\mathfrak{F}$ the previous obtained foliation of $n-$codimensional defined by the atlas

$\mathcal{A} = \{U, (x^a, y^i)\}_{1\leq a\leq n, 1\leq i\leq m}$ whose transition functions verify $\frac{\partial x^a}{\partial y^i} = 0$. We always use the local coordinates in the adapted charts to the foliation when to the local expression of an element. For $U, U' \in \mathcal{U}$ of the respective coordinates systems $(x^1, \ldots, x^n, y^1, \ldots, y^m)$ and $\left(x'^1, \ldots, x'^n, y'^1, \ldots, y'^m\right)$ with $U \cap U' \neq \emptyset$ the Jacobi matrix of the coordinate change on $U \cap U'$ is defined by

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

where

(i) $A = (\frac{\partial x^{b'}}{\partial x^a})$ with $1 \leq a, b \leq n$ is a square matrix of order $n$,
(ii) $B = (\frac{\partial y^{i'}}{\partial x^a})_{1\leq i\leq m, 1\leq a\leq n}$ is a matrix of type $m \times n$,
(iii) $C = (\frac{\partial x^{b'}}{\partial y^j})_{1\leq b\leq n, 1\leq j\leq m} = 0$ a null matrix of type $n \times m$,
(iv) $D = (\frac{\partial y^{i'}}{\partial y^j})_{1\leq i,j\leq m}$ is a square matrix of order $m$.

So the distribution $\mathfrak{D}$ is defined by the equations $dx^\alpha = 0$ for $1 \leq \alpha \leq n$. By introducing a riemann metrix on $M$ we can define a supplementary distribution $\mathfrak{D}^s$ orthogonal to $\mathfrak{D}$ by the equations

$$\theta^\beta = dy^\beta + \Gamma^\beta_\alpha dx^\alpha = 0, 1 \leq \alpha \leq n, 1 \leq \beta \leq m, \tag{2.1}$$

where the $\Gamma^\beta_\alpha$ are functions of class $C^\infty$. Consequently, we obtain the decomposition of $TM$ by $T_zM = \mathfrak{D}_z \oplus \mathfrak{D}^s_z$ for all $z \in M$. It's abvious that the $(dx^\alpha, \theta^\beta), 1 \leq \alpha \leq n, 1 \leq \beta \leq m$ define a basis of $\mathcal{F}(M)-$module of scalar valued $p-$forms on $M$. The dual basis is

$$X^\alpha = \frac{\partial}{\partial x^\alpha} - \Gamma^\beta_\alpha \frac{\partial}{\partial y^\beta}, Y^\beta = \frac{\partial}{\partial y^\beta} \tag{2.2}$$

In $U \cap U'$ the system $dx^a$ must be equivalent to $dx^{a'}$; and the systems $\theta^i$ and $\theta^{i'}$ must be equivalent. We get

$$\Gamma^{i'}_{c'} = \frac{\partial y^{i'}}{\partial y^j}\frac{\partial x^b}{\partial x^{c'}}\Gamma^j_b - \frac{\partial y^{i'}}{\partial x^b}\frac{\partial x^b}{\partial x^{c'}}. \tag{2.3}$$

By othogonality we can then provide the foliated manifold $M$ an almost-product structure $\Gamma = (h, v)$ such that $h = \frac{1}{2}(I + \Gamma)$ and $v = \frac{1}{2}(I - \Gamma)$ where $h : \mathfrak{D}^s_z \to T_zM$ and $v : \mathfrak{D}_z \to T_zM$ are horizontal and vertical projectors of respective ranks $n$ and $m$ associated to $\Gamma$. So we have

(i) for $z \in M$, $h(T_zM) \subset T_zM$ and $v(T_zM) \subset T_zM$;
(ii) $T_zM = h(T_zM) \oplus v(T_zM)$;

(iii) $h$ and $v$ belong to the class $C^{\infty}$.

**Remark 2.1.** *The almost-product structure thus defined has a property of foliated manifold but non specially of foliated manifold by a vector valued* $1-$*form.*

**Definition 2.2.** *For an almost-product structure* $\Gamma$, *we define the Lie algebra* $\mathfrak{A}_{\Gamma}$ *of the vector fields of* $\chi(M)$ *whose Lie derivative corresponding to* $Gamma$ *is null. A vector field* $X$ *is therefore an element of* $\mathfrak{A}_{\Gamma}$ *if only if* $[X, \Gamma Y] = \Gamma [X, Y]$, *for* $Y \in \chi(M)$.

**Definition 2.3.** *[3] We define the curvature of the almost-product structure* $\Gamma$ *the vector valued* $2-$*form noticed* $R$ *by* $R = \frac{1}{2}[\Gamma, \Gamma]$ *such that for all* $X, Y \in \chi(M)$ *we have* $R(X,Y) = [\Gamma X, \Gamma Y] + [X,Y] - \Gamma[\Gamma X, Y] - \Gamma[X, \Gamma Y]$.

**Theorem 2.1.** *The curvature* $R$ *of the almost-product structure* $\Gamma$ *is null if and only if the distribution* $\mathfrak{D}^s$ *is completely integrable.*

*Proof.* Using the Frobenius's theorem let us prove that the bracket of $hX$ and $hY$ belongs to the distribution $\mathfrak{D}^s$. By definition $R = -\frac{1}{2}[h,h]$. Since $R$ is semi-basic we have $R(X,Y) = -\frac{1}{2}[h,h](X,Y) = -\frac{1}{2}[h,h](hX,hY)$ for all $X, Y \in \chi(M)$, that is, $R(X,Y) = -[hX,hY] + h[hX,hY]$. If $R=0$ we have $[hX,hY] = h[hX,hY]$. So the bracket of $hX$ and $hY$ belongs to $\mathfrak{D}^s$.

The converse implication is obvious. Indeed, if $\mathfrak{D}^s$ is completely integrable; so by definition the bracket of $hX$ and $hY$ belongs to $\mathfrak{D}^s$ for $X, Y \in \chi(M)$. Therefore we have $[hX,hY] = h[hX,hY]$ and $-[hX,hY] + h[hX,hY] = 0$. Thus we have $[h,h](hX,hY) = [h,h](X,Y) = 0$. Hence the result. □

The nullity space of the curvature $R$ associated to $\Gamma$ is a set $\mathfrak{N}_R = \{X \in \chi(M)$ such that $i_X R = 0\}$ where $i_X$ denotes the interior product in respect to a vector field $X$

**Proposition 2.1.** *[9] The Lie algebra* $\mathfrak{A}_{\Gamma}$ *leaves the nullity space* $\mathfrak{N}_R$ *of curvature* $R$ *stable.*

Let $\mathfrak{A}$ be a Lie algebra.

**Definition 2.4.** *A* $p-$*cochain* $C$ *of* $\mathfrak{A}$ *is an alternated application of* $\mathfrak{A}^p$ *in* $\mathfrak{A}$

$$\begin{aligned} C : \mathfrak{A} \times \mathfrak{A} \times \ldots \times \mathfrak{A} &\to \mathfrak{A} \\ X^1, \ldots, X^p &\to C\left(X^1, \ldots, X^p\right). \end{aligned}$$

*The* $0-$*cochains are elements of* $\mathfrak{A}$.

**Definition 2.5.** *The coboard operator $\partial$ makes corresponding to $p-$cochain $C$ the $(p+1)-$cochain $\partial C$ defined by*

$$\partial C\left(X^0,\ldots,X^p\right)=\delta_{0\ldots p}^{\lambda_0,\ldots,\lambda_p}\frac{1}{p!}\left[X^{\lambda_0},C\left(X^{\lambda_1},\ldots,X^{\lambda_p}\right)\right]-\delta_{0\ldots p}^{\lambda_0,\ldots,\lambda_p}\frac{1}{2\left(p-1\right)!}$$
$$C\left(\left[X^{\lambda_0},X^{\lambda_1}\right],X^{\lambda_2}\ldots,X^{\lambda_p}\right),$$

*where $\delta$ is antisymetric indicator of Kronecker and the $X^{\lambda_i}$ are elements of $\mathfrak{A}$.*
*For $p=0,\partial C=-ad_X$ where $ad_X:Y\to[X,Y]$ is an adjunct application.*
*For $p=1$, we have $\partial C\left(X,Y\right)=[X,C\left(Y\right)]+[C\left(X\right),Y]-C\left([X,Y]\right)$ for all $X,Y\in\mathfrak{A}$.*

We denote by $\mathfrak{C}^p\left(\mathfrak{A}^p,\mathfrak{A}\right)$ the set of $p-$cochains of $\mathfrak{A}^p$ onto $\mathfrak{A}$. Let $\Delta:\mathfrak{C}^p\to\mathfrak{C}^p$ be a linear transformation such that $\Delta=0$. We consider $Ker\Delta$ and $Im\Delta$ the respectives kernel and image space of differential operator $\Delta$. The element vectors of $Ker\Delta$ are $p-$coboards. The vectors which are elements of $Ker\Delta$ are called $p-$cocycles and the elements of $Im\Delta$ are $p-$cobords. Hence, the $1-$cocycles are only the derivation of $\mathfrak{A}$ and the exact $1-$cocycles are the inner derivations. Since $\Delta^2=0$ and $Im\Delta\subset Ker\Delta$ we can define the cohomology

$$\mathfrak{H}\left(\mathfrak{C}^p,\Delta\right):=\frac{Ker\Delta}{Im\Delta}.$$

Now, let $\mathfrak{M}$ be a $\mathfrak{A}-$module having an application $\vartheta:\mathfrak{A}\to End\left(\mathfrak{M}\right)$ such that $\vartheta\left([X,Y]\right)=\vartheta\left(X\right).\vartheta\left(Y\right)-\vartheta\left(Y\right).\vartheta\left(X\right)$ for all $X,Y\in\mathfrak{A}$. We define the linear applications space
$\mathcal{L}^p\left(\mathfrak{A},\mathfrak{M}\right):=Hom\left(\Lambda^p\mathfrak{A},\mathfrak{M}\right)$ which is isomorphic to $\Lambda^p\mathfrak{A}^*\otimes\mathfrak{M}$ called the $p-$forms space of $\mathfrak{A}$ in $\mathfrak{M}$. We define a differential application $d:\mathcal{L}^p\left(\mathfrak{A},\mathfrak{M}\right)\to\mathcal{L}^{p+1}\left(\mathfrak{A},\mathfrak{M}\right)$ verifying

(i) for $m\in\mathfrak{M},dm\left(X\right)=\vartheta\left(X\right)m$ for all $X\in\mathfrak{M}$
(ii) for $\alpha\in\mathfrak{A}^*,d\alpha\left(X,Y\right)=-\alpha\left([X,Y]\right)$ for all $X,Y\in\mathfrak{A}$
(iii) in $\Lambda\mathfrak{A}^*,d\left(\alpha\wedge\beta\right)=d\alpha\wedge\beta+\left(-1\right)^{|\alpha|}\alpha\wedge d\beta.$
(iv) $\Lambda\mathfrak{A}^*\otimes\mathfrak{M},d\left(\alpha\otimes m\right)=d\alpha\otimes m+\left(-1\right)^{|\alpha|}\alpha\wedge dm.$

We verify that $d^2m=0$ for $m\in\mathfrak{M}$ and $d^2\alpha=0$ for $\alpha\in\mathfrak{A}^*$. So we define the following differential sequence:

$$\ldots\to\mathcal{L}^{p-1}\left(\mathfrak{A},\mathfrak{M}\right)\xrightarrow{d}\mathcal{L}^{p}\left(\mathfrak{A},\mathfrak{M}\right)\xrightarrow{d}\mathcal{L}^{p+1}\left(\mathfrak{A},\mathfrak{M}\right)\to\ldots$$

called a Chevalley-Eilenberg's cohomology complex of $\mathfrak{A}$ with value in $\mathfrak{M}$. And the space

$$\mathfrak{H}^p(\mathfrak{A},\mathfrak{M}) = \frac{Kerd : \mathcal{L}^p(\mathfrak{A},\mathfrak{M}) \to \mathfrak{H}^{p+1}(\mathfrak{A},\mathfrak{M})}{Imd : \mathcal{L}^{p-1}(\mathfrak{A},\mathfrak{M}) \to \mathcal{L}^p(\mathfrak{M},\mathfrak{A})}$$

is called the Lie aLgebra of cohomology of $\mathfrak{A}$ with value in $\mathfrak{M}$. This Lie algebra is called Chevalley-Eilenberg's cohomology space of $\mathfrak{A}$ in $\mathfrak{M}$.

In the next section, we will suppose that $M$ is a foliated manifold by $L$.

## 3. Some properties of Lie algebras attached to an almost-product structure $\Gamma$ on the foliated manifold $M$

**Definition and Proposition 3.1.** *Taking into account an adapted chart of local coordinates* $(x^1,\dots,x^n,x^{n+1},\dots,x^{n+m})$ *of the domain* $U$*, we define an almost-product structure* $\Gamma$ *adapted to this chart by* $\Gamma = \Gamma_\alpha^\beta dx^\alpha \otimes \frac{\partial}{\partial x^\beta}, 1 \le \alpha,\beta \le n+m$ *where the* $\Gamma_\alpha^\beta$ *are functions of class* $C^\infty$ *in the equation (2.1), verifying:*

(i) $\Gamma_{\cdot}^\beta = 0$ *for* $1 \le \beta \le n$,
(ii) $\Gamma_\alpha^{\cdot} = 0$ *for* $n+1 \le \alpha \le n+m$,
(iii) *and* $\Gamma_\alpha^\beta \ne 0$ *for any* $1 \le \alpha,\beta \le n+m$.

*Proof.* The proof is immediate. We just need to adapt the almost-product structure $\Gamma$ by the change of the adapted charts on $M$ from the relation (2.3). □

**Proposition 3.1.** *Let* $U$ *be a domain of adapted chart to a local coordinates system* $(x^1,\dots,x^n,y^1,\dots,y^m)$*. All element* $X = X^\alpha\frac{\partial}{\partial x^\alpha} + Y^\beta\frac{\partial}{\partial y^\beta}, 1 \le \alpha \le n, 1 \le \beta \le m,$ *of* $\mathfrak{A}_\Gamma$ *where* $\Gamma = \Gamma_\alpha^\beta dx^\alpha \otimes \frac{\partial}{\partial x^\beta}, 1 \le \alpha,\beta \le n+m$ *such that* $\Gamma_\alpha^i = 0,$ *for* $1 \le i \le n$ *and* $\Gamma_j^\beta = 0$ *for* $n+1 \le j \le n+m$*, verifying:*

$$\frac{\partial X^\alpha}{\partial y^\beta} = 0, \tag{3.1}$$

$$\underset{1\le i\le n}{X^i\frac{\partial \Gamma_\alpha^\beta}{\partial x^i}} + \underset{1\le i\le n}{\Gamma_i^\beta\frac{\partial X^i}{\partial x^\alpha}} + \underset{1\le i\le m}{Y^i\frac{\partial \Gamma_\alpha^\beta}{\partial y^i}} - \underset{1\le i\le m}{\Gamma_\alpha^i\frac{\partial Y^\beta}{\partial y^i}} = 0, \tag{3.2}$$

*for* $1 \le \alpha \le n$ *and* $1 \le \beta \le m$.

*Proof.* We suppose that the adapted chart of domain $U$ has a local coordinates system $(x^i,x^j)_{1\le i\le n, n+1\le j\le n+m}$ with $x^{n+j} = y^j$ for $1 \le j \le m$. Now, let $X$ be a vector field of $U$ such that $X = X^i \frac{\partial}{\partial x^i}, 1 \leq i \leq n+m$. By definition, $X \in \mathfrak{A}_\Gamma$ if and only if the components of $X$ verify

$$X^i \frac{\partial \Gamma^j_k}{\partial x^i} + \Gamma^j_i \frac{\partial X^i}{\partial x^k} - \Gamma^i_k \frac{\partial X^j}{\partial x^i} = 0 \text{ for } 1 \leq j,k \leq n+m. \tag{3.3}$$

Case $1$, if $1 \leq j \leq n$ :

(1) if $1 \leq k \leq n$ we obtain in (3.3) and according to (3.1) we have $-\Gamma^i_k \frac{\partial X^j}{\partial x^i} = 0$ for $n+1 \leq i \leq n+m$. So $\frac{\partial X^j}{\partial x^i} = 0$ for $n+1 \leq i \leq n+m$,

(2) if $n+1 \leq k \leq n+m$ the equation (3.3) is immediately equal to zero.

Case $2$, if $n+1 \leq j \leq n+m$ :

(1) if $1 \leq k \leq n$ we have $\underset{1\leq i\leq n+m}{X^i \frac{\partial \Gamma^j_k}{\partial x^i}} + \underset{1\leq i\leq n}{\Gamma^j_i \frac{\partial X^i}{\partial x^k}} - \underset{n+1\leq i\leq n+m}{\Gamma^i_k \frac{\partial X^j}{\partial x^i}} = 0$

(2) and if $n+1 \leq k \leq n+m$ we have $\underset{1\leq i\leq n}{\Gamma^j_i \frac{\partial X^i}{\partial x^k}} = 0$. Thus $\underset{1\leq i\leq n}{\frac{\partial X^i}{\partial x^k}} = 0$.

□

**Corollary 3.1.** *All vector field $X$ of $\mathfrak{A}_\Gamma$ is projectable. That is, $\frac{\partial X^i}{\partial y^j} = 0$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.*

*Proof.* Indeed, according to the proposition 3.1 we get the corollary. □

We will denote by $\mathfrak{A}_\mathfrak{F}$ the set of infinitesimal automorphisms vector fields which leave the leaves invariant. According to Bruce L. Reinhart in [11] the projections $(h, v)$ which admit the following properties:

(i) $Lh = L;$ $\quad hL = 0$

(ii) $Lv = 0;$ $\quad vL = L$

(iii) $\Gamma h = h\Gamma = h;$ $\quad \Gamma = v\Gamma = -v.$

We consider by $\mathfrak{A}^h_\Gamma$ the module of all vector fields $X \in \chi(L, \mathcal{F}(M))$ such that $h(X) = X$ and $\mathfrak{A}^v_\Gamma$ that of all vector fields $X \in \chi(L, \mathcal{F}(M))$ such that $v(X) = X$. In dual basis $(X^\alpha, Y^\beta)$ to the equations of (2.2), locally we obtain $X \in \mathfrak{A}^h_\Gamma$ and it is equal to $X = X^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\beta_\alpha X^\alpha \frac{\partial}{\partial y^\beta}$ and for $Y \in \mathfrak{A}^v_\Gamma$ we have $Y = Y^\beta \frac{\partial}{\partial y^\beta}$. We immediately obtain the following corollary from the proposition (3.1)

**Corollary 3.2.** *We taking into account an adapted chart of local coordinates system $(x^1, \ldots, x^n, y^1, \ldots, y^m)$ of the domain $U$. Every element $X = Y^\beta \frac{\partial}{\partial y^\beta}, 1 \leq \beta \leq m$ of $\mathfrak{A}^v_\Gamma$ verifies*

$$\underset{1\leq i\leq m}{Y^i \frac{\partial \Gamma^\beta_\alpha}{\partial y^i}} - \underset{1\leq i\leq m}{\Gamma^i_\alpha \frac{\partial Y^\beta}{\partial y^i}} = 0, \textit{ for } 1 \leq \alpha \leq n \textit{ and } 1 \leq \beta \leq m.$$

*And all element* $X = \underset{1\leq\alpha\leq n}{X^{\alpha}\frac{\partial}{\partial x^{\alpha}}} - \underset{1\leq\beta\leq m}{\Gamma_{\alpha}^{\beta}X^{\alpha}\frac{\partial}{\partial y^{\beta}}} \in \mathfrak{A}_{\Gamma}^{h}$ *satisfies*

$$\underset{1\leq i\leq n}{X^{i}\frac{\partial\Gamma_{\alpha}^{\beta}}{\partial x^{i}}} - \underset{1\leq l\leq m}{\Gamma_{t}^{l}X^{t}\frac{\partial\Gamma_{\alpha}^{\beta}}{\partial y^{l}}} + \underset{1\leq i\leq n}{\Gamma_{i}^{\beta}\frac{\partial X^{i}}{\partial x^{\alpha}}} + \underset{1\leq i\leq m}{\Gamma_{\alpha}^{i}X^{t}\frac{\partial\Gamma_{t}^{\beta}}{\partial y^{i}}} = 0,$$

*for* $1 \leq \alpha, t \leq n$ *and* $1 \leq \beta \leq m$.

*Proof.* It's an immediate consequence of proposition 3.1. In fact, we just need to replace the elements of $\mathfrak{A}_{\Gamma}$ by those of $\mathfrak{A}_{\Gamma}^{h}$ and that of $\mathfrak{A}_{\Gamma}^{v}$. □

It's immediate to note that

**Proposition 3.2.** $\mathfrak{A}_{\Gamma}^{h}$ *and* $\mathfrak{A}_{\Gamma}^{v}$ *are ideals of Lie algebra* $\mathfrak{A}_{\Gamma}$ *whose it is a direct product of these ideals.*

*Proof.* We have a similary result of [9] where the almost-product structure $\Gamma$ is a Grifone's connection. □

**Proposition 3.3.** *[9] If the almost-product structure* $\Gamma$ *is flat then the Lie algebra* $\mathfrak{A}_{\Gamma}$ *is a direct product of* $\mathfrak{A}_{\Gamma}^{h}$ *with* $\mathfrak{A}_{\Gamma}^{v}$.

**Proposition 3.4.** *A vector field* $X$ *of* $M$ *is an element of* $\mathfrak{A}_{\Gamma}$ *if and only if* $X$ *leaves invariant the generalized distributions defined by Lie subalgebras* $\mathfrak{A}_{\Gamma}^{h}$ *and* $\mathfrak{A}_{\Gamma}^{v}$.

*Proof.* It's an immediate consequence of the proposition $3.1$ of [6] in the case where the vector valued $1-$form $L$ is an almost-product structure of the eigenvalues $1$ and $-1$. □

**Lemma 3.1.** *If* $M$ *is a compact manifold without board such that*

(i) $dimM = 2k+1, k \in \mathbb{N}$ *we obtain* $\mathfrak{A}_{\Gamma} = \{0\}$
(ii) $dimM = 2k$ *we have* $\mathfrak{A}_{\Gamma} \neq \{0\}$.

**Theorem 3.2.** *If* $M$ *is a compact manifold without board dimension* $2k$ *and for all adapted chart of local coordinates system* $(x^{1},\ldots,x^{n},y^{1},\ldots,y^{m})$ *of domain* $U$ *we have* $\mathfrak{A}_{\Gamma}^{v} = \{f\left(x^{1},\ldots,x^{n}\right)y^{i}\frac{\partial}{\partial y^{i}}, 1 \leq i \leq m\}$ *where* $f$ *is a function of class* $C^{\infty}$.

*Proof.* We suppose that $M$ is compact without board manifold of dimension $2k$ with $k \in \mathbb{N}$. According to the lemma 3.1 we have $\mathfrak{A}_{\Gamma} \neq \{0\}$. Since the Lie algebra $\mathfrak{A}_{\Gamma}$ is a direct product of $\mathfrak{A}_{\Gamma}^{h}$ and $\mathfrak{A}_{\Gamma}^{v}$ then those Lie subalgebras are supplementary as $\mathcal{F}\left(M\right)-$modules. So $\mathfrak{A}_{\Gamma}^{h}$ and $\mathfrak{A}_{\Gamma}^{v}$ are the basis which do not vanish naturally. Thus the elements of $\mathfrak{A}_\Gamma^h$ and $\mathfrak{A}_\Gamma^v$ are the generator systems. Naturally all element of $\mathfrak{A}_\Gamma^v$ can also be written $Y^i\frac{\partial}{\partial y^i}, 1 \leq i \leq m$ where the $Y^i$ are functions in $x^\alpha$ and $y^i$ for $1 \leq \alpha \leq n$ and $1 \leq i \leq m$. Let $X$ be $X \in \mathfrak{A}_\Gamma^v$. If $X \in \mathfrak{A}_\Gamma^v$ then $X$ is not generated by $\frac{\partial}{\partial x^\alpha}, 1 \leq \alpha \leq n$. We prove that $X \in \langle Y^i\frac{\partial}{\partial y^i}\rangle$ where $Y^i = f(x^1, \ldots, x^n)\, y^i$ for $1 \leq i \leq m$. According to the corollary 3.2, $X \in \mathfrak{A}_\Gamma^v$ verifies the equation

$$Y^i \underset{1\leq i\leq m}{\frac{\partial \Gamma_\alpha^\beta}{\partial y^i}} = \Gamma_\alpha^i \underset{1\leq i\leq m}{\frac{\partial Y^\beta}{\partial y^i}} \text{ for } 1 \leq \alpha \leq n \text{ and } 1 \leq \beta \leq m. \tag{3.4}$$

If $Y^i = f(x^1, \ldots, x^n)$ where $Y^i = f(y^1, \ldots, y^n)$ then it's impossible to obtain the equality (3.4), because the $Y^i$ are functions of $x^\alpha$ and $y^i$ with $1 \leq \alpha \leq n$ and $1 \leq i \leq m$. Let $Y^i = f(x^1, \ldots, x^n, y^1, \ldots, y^m)$. If $M$ is of dimension $2k$, passing by local coordinates system $(x^\alpha, y^i), 1 \leq \alpha \leq n, 1 \leq i \leq m$ of the domain $U$ of adapted chart to the foliation, the $Y^i$ is equal to the functions $f(x^1, \ldots, x^n)\, y^i, 1 \leq i \leq m$ according to the definition of almost-product structure $\Gamma$ in 3.1. Thus $X \in \langle f(x^1, \ldots, x^n)\, y^i\frac{\partial}{\partial y^i}\rangle$. Hence the result. □

**Proposition 3.5.** *If the curvature $R$ admits a nullity space then we have $\mathfrak{A}_\Gamma^h \neq \{0\}$.*

*Proof.* In fact, we suppose that the nullity space $\mathfrak{N}_R$ of $R$ isn't null. Let $X$ be $X \in \mathfrak{A}_\Gamma^h$ such that on an adapted local coordinates system $(x^\alpha, y^i)_{1\leq\alpha\leq n, 1\leq i\leq m}$ we have $X = X^\alpha\frac{\partial}{\partial x^\alpha} - \Gamma_\alpha^i X^\alpha\frac{\partial}{\partial y^i}$. If $X = 0$ then for all $\alpha \in \{1, \ldots, n\}$, we have $X^\alpha = 0$ with $\alpha \in \{1, \ldots, n\}$. But the space $\mathfrak{N}_R$ is generated by projectable fields $X^\alpha\frac{\partial}{\partial x^\alpha}$. Then $\mathfrak{N}_R$ is reduced to zero. This contradicts to the hypothesis. So necessarily, we have $X \neq 0$. Thus $\mathfrak{A}_\Gamma^h \neq \{0\}$. □

Consequently we have

**Corollary 3.3.** *If the almost-product structure $\Gamma$ is flat then immediately we have $\mathfrak{A}_\Gamma^h \neq \{0\}$.*

*Proof.* If the product strucutre $\Gamma$ is flat we have $\mathfrak{N}_R = \chi(L, \mathcal{F}(M))$. Hence The result. □

**Remark 3.1.** *If the associated curvature to $\Gamma$ isn't null the Lie algebra $\mathfrak{A}_\Gamma^h$ isn't necessarily reduced to zero.*

**Definition 3.1.** *[2] A derivation $D$ of Lie $\mathbb{R}-$algebra $\mathfrak{A}$ is a $\mathbb{R}-$linear application in $\mathfrak{A}$ to $\mathfrak{A}$ such that for alls $X, Y \in \mathfrak{A}, D[X,Y] = [DX,Y] + [X,DY]$. A derivation $D$*

*of $\mathfrak{A}$ is inner if for all $X \in \mathfrak{A}, D(X) = [X, .]$. In particular in [7], a derivation of Lie algebra is given by a first degree differential operator.*

**Proposition 3.6.** *[10] Let $D$ be a derivation of $\mathfrak{A}_\Gamma$ and $U$ a domain of the adapted chart of the considered manifold. If there is a field $X \in \mathfrak{A}_\Gamma$ such that $X|_U \equiv 0$ then we have $D(X)|_U \equiv 0$. That is, $D$ is local.*

*Proof.* Let $U$ be a domain of the adapted chart and suppose such that there is $X \in \mathfrak{A}_\Gamma$ such that $X|_U \equiv 0$ and $D(X)(x) \neq 0$ for $x \in U$. We consider a vector field $Y$ of $M$ such that $Supp(Y) \subset U$ and $[DX, Y](x) \neq 0$. By definition, $D[X, Y] = [DX, Y] + [X, DY]$. For $x \in U$ we have $D[X, Y](x) = 0 = [DX, Y](x) + [X, DY](x)$ and $[X, DY] = 0$. Then $0 = [DX, Y]$ and this contradicts the hypothesis.So necessarily $DX(x) = 0, x \in U$ and thus $DX|_U = 0$. □

**Definition 3.2.** *[10] We call Chevalley-Eilenberg Cohomology's first space of $\mathfrak{A}$ the quotient vectorial space:*

$$H^1(\mathfrak{A}) = Der(\mathfrak{A})/ad\mathfrak{A}$$

*where $Der(\mathfrak{A})$ (resp. $ad\mathfrak{A}$) is the Lie algebra of derivations (resp. of inner derivations) of $\mathfrak{A}$.*

**Definition 3.3.** *[2] We call derivative ideal of Lie algebra $\mathfrak{A}$ the submodule of $\mathfrak{A}$, noted $[\mathfrak{A}, \mathfrak{A}]$ generated by $[X, Y]$ such that for all $X, Y \in \mathfrak{A}$. We define by recurrence:*

$$D^1(\mathfrak{A}) = [\mathfrak{A}, \mathfrak{A}],$$
$$D^k(\mathfrak{A}) = D^1(D^{k-1}(\mathfrak{A})) \text{ for } k \geq 2.$$

It's obvious to obtain the following theorem

**Theorem 3.3.** *The derived ideal of Lie algebra $\mathfrak{A}^v_\Gamma$ is contained in $\mathfrak{A}^v_\Gamma$. That is, $[\mathfrak{A}^v_\Gamma, \mathfrak{A}^v_\Gamma] \subset \mathfrak{A}^v_\Gamma$.*

*Proof.* It's obvious. Indeed, the derived ideal of $\mathfrak{A}^v_\Gamma$ is a Lie subalgebra of $\mathfrak{A}^v_\Gamma$ which is stable of the Lie bracket. Hence the result. □

**Remark 3.2.** *On a foliated manifold, the derived ideal of Lie algebra $\mathfrak{A}^v_\Gamma$ is not necessarily coincided with $\mathfrak{A}_v$.*

**Theorem 3.4.** *All derivation $D$ of Lie algebra $\mathfrak{A}^v_\Gamma$ is equal to $D(.) = D'(.) + [Z, .], Z \in \mathfrak{A}^v_\Gamma$ where $D'$ is a derivation of $\mathfrak{A}_\Gamma$ which is not necessarily inner.*

*Proof.* Let $(x^1, x^n, x^{n+1}, \dots, x^{n+m})$ with $x^{n+\beta} = y^\beta, 1 \leq \beta \leq m$ be an adapted chart of the domain $U$, $D$ a derivation of $\mathfrak{A}_\Gamma^v$ and $X$ a vector field of $\chi(M)$. By definition

$$\begin{aligned} DX(z) &= z^\alpha \frac{\partial X^\beta}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} + z^\alpha \Gamma_\alpha^\beta(x, X)\frac{\partial}{\partial x^\beta} \text{ for } z \in U, \\ D\left[X, Y\right] &= \left[DX, Y\right] + \left[X, DY\right], \text{ for all } Y \in \chi(M). \end{aligned}$$

If $X \in \mathfrak{A}_\Gamma^v, Y \in \mathfrak{A}_\Gamma$ and for $z \in U$, by using thoses definitions we get

$$\begin{aligned} & z^i \left[\frac{\partial}{\partial x^i}\left(X^a \frac{\partial Y^b}{\partial x^a} - Y^b \frac{\partial X^a}{\partial x^b}\right)\right]\frac{\partial}{\partial x^a} + z^i \Gamma_i^a \frac{\partial}{\partial x^a} \\ & - \left[z^i\left(\frac{\partial X^a}{\partial x^i} + \Gamma_i^a\right)\frac{\partial}{\partial x^a} Y^b \frac{\partial}{\partial x^b} - Y^b \frac{\partial}{\partial x^b}\left(z^i\left(\frac{\partial X^a}{\partial x^i} + \Gamma_i^a\right)\frac{\partial}{\partial x^a}\right)\right] \\ & - \left[X^a \frac{\partial}{\partial x^a}\left(z^i\left(\frac{\partial Y^b}{\partial x^i} + \Gamma_i^b\right)\frac{\partial}{\partial x^b}\right) - z^i\left(\frac{\partial Y^b}{\partial x^i} + \Gamma_i^b\right)\frac{\partial}{\partial x^b} X^a \frac{\partial}{\partial x^a}\right] = 0, \end{aligned}$$

then we have

$$\begin{aligned} & z^i \frac{\partial}{\partial x^i}\left(X^b \frac{\partial Y^b}{\partial x^b}\right) - z^i \frac{\partial}{\partial x^i}\left(Y^b \frac{\partial X^b}{\partial x^b}\right) + z^i \Gamma_i^b - z^i \frac{\partial X^a}{\partial x^i}\frac{\partial Y^b}{\partial x^a} - z^i \Gamma_i^a \frac{\partial Y^b}{\partial x^a} \\ & + Y^c \frac{\partial}{\partial x^c}\left(z^i \frac{X^b}{\partial x^i}\right) + Y^c \frac{\partial}{\partial x^c}\left(z^i \Gamma_i^b\right) - X^a \frac{\partial}{\partial x^a}\left(z^i \frac{\partial Y^b}{\partial x^i}\right) - X^a \frac{\partial}{\partial x^a}\left(z^i \Gamma_i^b\right) \\ & + z^i \frac{\partial Y^b}{\partial x^i}\frac{\partial X^b}{\partial x^b} + z^i \Gamma_i^b \frac{\partial X^b}{\partial x^b} = 0, \end{aligned}$$

and

$$\begin{aligned} & z^i \frac{\partial X^a}{\partial x^i}\frac{\partial Y^b}{\partial x^a}\frac{\partial}{\partial x^a} + z^i X^a \frac{\partial^2 Y^b}{\partial x^i \partial x^a}\frac{\partial}{\partial x^a} - z^i \frac{\partial Y^b}{\partial x^i}\frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial x^a} - z^i Y^b \frac{\partial^2 X^a}{\partial x^i \partial x^b}\frac{\partial}{\partial x^a} \\ & + z^i \Gamma_i^a \frac{\partial}{\partial x^a} - z^i \frac{\partial X^a}{\partial x^i}\frac{\partial Y^b}{\partial x^a}\frac{\partial}{\partial x^b} - z^i \Gamma_i^a \frac{\partial Y^b}{\partial x^a}\frac{\partial}{\partial x^b} + z^i Y^b \frac{\partial^2 X^a}{\partial x^b \partial x^i}\frac{\partial}{\partial x^a} \\ & + z^i Y^b \frac{\partial \Gamma_i^a}{\partial x^b}\frac{\partial}{\partial x^a} - z^i X^a \frac{\partial^2 Y^b}{\partial x^a \partial x^i}\frac{\partial}{\partial x^b} - z^i X^a \frac{\partial \Gamma_i^b}{\partial x^a}\frac{\partial}{\partial x^b} + z^i \frac{\partial Y^b}{\partial x^i}\frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial x^a} \\ & + z^i \Gamma_i^b \frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial x^a} = 0. \end{aligned}$$

By eliminating the terms of second derivatives, we get

$$z^i \Gamma_i^a \frac{\partial}{\partial x^a} - z^i \Gamma_i^a \frac{\partial Y^b}{\partial x^a}\frac{\partial}{\partial x^b} + z^i Y^b \frac{\partial \Gamma_i^a}{\partial x^b}\frac{\partial}{\partial x^a} - z^i X^a \frac{\partial \Gamma_i^b}{\partial x^a}\frac{\partial}{\partial x^b} + z^i \Gamma_i^a \frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial x^a} = 0.$$

$$
\begin{aligned}
z^i\Gamma_i^a\frac{\partial}{\partial x^a} &= z^i\Gamma_i^a\frac{\partial Y^b}{\partial x^a}\frac{\partial}{\partial x^b} - z^iY^b\frac{\partial \Gamma_i^a}{\partial x^b}\frac{\partial}{\partial x^a} + z^iX^a\frac{\partial \Gamma_i^b}{\partial x^a}\frac{\partial}{\partial x^b} - z^i\Gamma_i^a\frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial x^a},\\
&= [Y,\Gamma]\frac{\partial}{\partial x^a}(z) + z^iX^a\frac{\partial}{\partial x^a}\Gamma_i^b\frac{\partial}{\partial x^b} - z^i\Gamma_i^b\frac{\partial X^a}{\partial x^b},\\
&= z^iX^a\frac{\partial}{\partial x^a}\Gamma_i^b\frac{\partial}{\partial x^b} - z^i\Gamma_i^b\frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial x^a},
\end{aligned}
$$

taking into account $Y \in \mathfrak{A}_\Gamma$.

$$
\begin{aligned}
DX(z) &= z^i\left(\frac{\partial X^k}{\partial x^i} - \Gamma_i^b\frac{\partial X^k}{\partial x^b} + X^a\frac{\partial \Gamma_i^k}{\partial x^a}\right)\frac{\partial}{\partial x^k}\\
&= z^i\left[\frac{\partial X^k}{\partial x^i} - \left(\Gamma_i^b\frac{\partial X^k}{\partial x^b} - X^a\frac{\partial \Gamma_i^k}{\partial x^a}\right)\right]\frac{\partial}{\partial x^k}\\
&= z^i\left[-X^a\frac{\partial}{\partial x^a}\left(1-\Gamma_i^k\right)\frac{\partial}{\partial x^k} + \left(1-\Gamma_i^b\right)\frac{\partial X^k}{\partial x^b}\frac{\partial}{\partial x^k}\right]\\
&= \left[\left(1-\Gamma_i^b\right)\frac{\partial}{\partial x^b}, X^a\frac{\partial}{\partial x^a}\right](z)\\
&= \left[\left(1-\Gamma_i^l\right)\frac{\partial}{\partial x^l} + \left(1-\Gamma_i^k\right)\frac{\partial}{\partial x^k}, X^a\frac{\partial}{\partial x^a}\right](z)\\
&= \left[\frac{\partial}{\partial x^l}, X^a\frac{\partial}{\partial x^a}\right](z) + \left[\left(1-\Gamma_i^k\right)\frac{\partial}{\partial x^k}, X^a\frac{\partial}{\partial x^a}\right](z),
\end{aligned}
$$

for $1 \leq l \leq n$, $n+1 \leq a \leq n+m$ and $1 \leq i, b \leq n+m$. Considering $Z = \left(1-\Gamma_i^k\right)\frac{\partial}{\partial x^k} \in \mathfrak{A}_\Gamma^v$, so we obtain

$$
DX(z) = \left[\frac{\partial}{\partial x^l}, X^a\frac{\partial}{\partial x^a}\right](z) + \left[Z, X^a\frac{\partial}{\partial x^a}\right](z). \tag{3.5}
$$

The relation (3.5) results that $DX(z) = \left[\frac{\partial}{\partial x^l}, X\right] + [Z,X](z), z \in M$. Hence the theorem. $\square$

**Lemma 3.2.** *Let $U$ a domain the adapted chart of the local coordinates system $(x^1,\ldots,x^n,x^{n+1},\ldots,x^{n+m})$ and $X \in \chi(U)$ a vector field. If $[X,Y]=0$ for $Y \in \mathfrak{A}_\Gamma^v$, necessarily we have $X \equiv 0$.*

*Proof.* Let $U$ be a domain of an adapted chart and $X \in \chi(U)$ a vector field. For $Y \in \mathfrak{A}_\Gamma^v$ we suppose that $[X,Y]=0$. Locally on the adapted chart of the domain $U$

we have $Y = Y^{\beta}\frac{\partial}{\partial x^{\beta}}, n+1 \leq \beta \leq n+m$. If $z \in U$ we have

$$z^{i}\left(X^{\alpha}\frac{\partial Y^{\beta}}{\partial x^{\alpha}} - Y^{\beta}\frac{\partial X^{\alpha}}{\partial x^{\beta}}\right) = 0,$$

$$X^{\alpha}\frac{\partial Y^{\beta}}{\partial x^{\alpha}} - Y^{\beta}\frac{\partial X^{\alpha}}{\partial x^{\beta}} = 0,$$

$$X^{\alpha}\frac{\partial Y^{\beta}}{\partial x^{\alpha}} = Y^{\beta}\frac{\partial X^{\alpha}}{\partial x^{\beta}}.$$

For $z \in U$ we can choose $Y \in \mathfrak{A}_{\Gamma}^{v}$ such that $Y(z) = 0$ but $\frac{\partial Y^{\beta}}{\partial x^{\alpha}}$ is arbitrary. So, it follows that the $X^{\alpha}$ is null for $1 \leq \alpha \leq n+m$. □

**Theorem 3.5.** *All derivation $D$ of Lie algebra $\mathfrak{A}_{\Gamma}$ with value in $\chi(M)$ is of the form $D(.) = [Z, .]$ where $Z \in \mathfrak{A}_{\Gamma}$. That is, all derivation of$\mathfrak{A}_{\Gamma}$ is inner. Thus its Chevalley-Eilenberg's cohomology first space is null.*

*Proof.* Let $D$ be a derivation of $\mathfrak{A}_{\Gamma}$ in value to $\chi(M)$. Let $D'$ be the restruction of $D$ to $\mathfrak{A}_{\Gamma}^{v}$. For $Y \in \mathfrak{A}_{\Gamma}^{v}$ we can find $Z \in \mathfrak{A}_{\Gamma}$ such that $D'Y = [Z, Y]$. Let $X \in \mathfrak{A}_{\Gamma}$, we have

$$[DX, Y] + [X, DY] - D[X, Y] = 0,$$

$$[DX, Y] + [X, [Z, Y]] - [Z, [X, Y]] = 0 \text{ for } Y \in \mathfrak{A}_{\Gamma}^{v}$$

According to Jacobi's identity $[DX - [Z, X], Y] = 0$. According to the lemma (3.2), we obtain $DX - [Z, X] = 0$ and thus $DX = [Z, X]$. Hence the theorem.
This theorem 3.5 is other than the similary result to that of Kanie in [13]. □

**Definition 3.4.** *[2] Let $\mathfrak{A}$ be a Lie algebra on $M$. We define the center (resp. the centralizer) $\mathfrak{C}$ of $\mathfrak{A}$ the set all vector fields $X \in \mathfrak{A}$ such that $[X, Y] = 0$ for all $Y \in \mathfrak{A}$ (resp. $Y \in \chi(M)$).*

**Proposition 3.7.** *If the almost-product structure $\Gamma$ is flat on $M$ then the centralizer of $\mathfrak{A}_{\Gamma}^{h}$ (resp. of $\mathfrak{A}_{\Gamma}^{v}$) in $\mathfrak{A}_{\Gamma}$ is reduced to zero (resp is generated by the $\frac{\partial}{\partial x^{i}}, 1 \leq i \leq n$).*

*Proof.* We suppose that the curvature of $\Gamma$ is null on domain $U$ of the adapted chart of the local coordinates $(x^{1}, \ldots, x^{n}, x^{n+1}, \ldots, x^{n+m})$ with $x^{n+j} = y^{j}, 1 \leq j \leq m$. According to the proposition 3.3 the Lie algebra $\mathfrak{A}_{\Gamma}^{h} \neq \{0\}$. In $U$ let $X = \underset{1\leq i\leq n}{X^{i}\frac{\partial}{\partial x^{i}}} + \underset{1\leq j\leq m}{X^{j}\frac{\partial}{\partial y^{j}}}$ such that $\frac{\partial X^{i}}{\partial y^{j}} = 0$ and $[X, Y] = 0$ for $Y \in \mathfrak{A}_{\Gamma}^{h}$. We have $\underset{1\leq k\leq n+m}{X^{k}\frac{\partial Y^{j}}{\partial x^{k}} - Y^{k}\frac{\partial X^{j}}{\partial x^{k}}} = 0$

for $1 \leq j \leq n+m$ and

$$X^k \frac{\partial Y^j}{\partial x^k}_{1\leq k\leq n+m} = Y^h \frac{\partial X^j}{\partial x^h}_{1\leq h\leq n+m}. \tag{3.6}$$

Since $Y \in \mathfrak{A}_\Gamma^h$ one of the $Y^h, 1 \leq h \leq n+m$ is at least non zero. Two cases are possible: if $Y^{k_0} = c^{te}$ non zero for some $1 \leq k_0 \leq k$, the equality (3.6) gives $\frac{\partial X^{j_0}}{\partial x^h} = 0$, for $1 \leq j_0 \leq j$ so $X^{j_0} = c^{te}$ non zero. And if $Y^{k_0} = f\left(x^i, 1 \leq i \leq n+m\right)$, (3.6) implies that $X^k \frac{\partial Y^{j_0}}{\partial x^k} \neq 0$ for any $1 \leq j_0 \leq j$ and $X^{k_0} \neq 0$ for any $1 \leq k_0 \leq k$. Hence the proposition. □

**Proposition 3.8.** *Let $U$ be a domain of the adapted chart of local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^m)$ where $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^m}$ form a tangent local basis to the leaves. If the almost-product structure $\Gamma$ is flat on $M$, the centralizer of $\mathfrak{A}_\Gamma^v$ in the generated part by the basis $\frac{\partial}{\partial y^i}, 1 \leq i \leq m$ of $\chi\left(L, \mathcal{F}\left(M\right)\right)$ (resp. of $\chi\left(L, \mathcal{F}\left(M\right)\right)$) is reduced to zero (resp. isn't reduced ever to zero).*

This proposition 3.8 is similary of a result in [9] if the almost-product structure $\Gamma$ is a connection in the sense of Grifone.

**Proposition 3.9.** *If the manifold $M$ is compact without board of dimension $2k$ with $k \in \mathbb{N}^*$ then the centralizer of $\mathfrak{A}_\Gamma$ isn't null.*

*Proof.* We suppose that $M$ is a compact without board manifold of dimension $2k$. According to the lemma 3.1 we have $\mathfrak{A}_\Gamma \neq \{0\}$. Let $Y$ be an element of centralizer of $\mathfrak{A}_\Gamma$ such that $[X, Y] = 0$ for $X \in \mathfrak{A}_\Gamma$. Let $(x^a, y^i)_{1\leq a\leq k, 1\leq i\leq k}$ be a coordinates system of the adapted chart of domain $U$ of $M$. Locally let $X, Y$ be such that $X = X^a \frac{\partial}{\partial x^a} + Y^i \frac{\partial}{\partial y^i}$ and $Y = X'^b \frac{\partial}{\partial x^b} + Y'^j \frac{\partial}{\partial y^j}$. We have

$$\begin{aligned} X^a \frac{\partial X'^b}{\partial x^a} \frac{\partial}{\partial x^b} - X'^b \frac{\partial X^a}{\partial x^b} \frac{\partial}{\partial x^a} + X^a \frac{\partial Y'^j}{\partial x^a} \frac{\partial}{\partial y^j} \\ - X'^b \frac{\partial Y^i}{\partial x^b} \frac{\partial}{\partial y^i} + Y^i \frac{\partial Y'^j}{\partial y^i} \frac{\partial}{\partial y^j} - Y'^j \frac{\partial Y^i}{\partial y^j} \frac{\partial}{\partial y^i} = 0, \end{aligned}$$

for $1 \leq a, b \leq k$ and $1 \leq i, j \leq k$. Then we get

$$X^a \frac{\partial X'^{b_0}}{\partial x^a} - X'^b \frac{\partial X^{b_0}}{\partial x^b} = 0 \tag{3.7}$$

and

$$X^a \frac{\partial Y'^{j_0}}{\partial x^a} - X'^b \frac{\partial Y^{j_0}}{\partial x^b} + Y^i \frac{\partial Y'^{j_0}}{\partial y^i} - Y'^j \frac{\partial Y^{j_0}}{\partial y^j} = 0 \tag{3.8}$$

for $1 \leq a, b_0 \leq k$ and $1 \leq i, j_0 \leq k$. Since $X \in \mathfrak{A}_\Gamma$ we can find $1 \leq a_0 \leq k$ such that $X^{a_0} \neq 0$. In (3.7) and (3.8), we can have $X'^{b_0} \neq 0$. Else, $X'^{b} \frac{\partial Y^{j_0}}{\partial x^b} - Y^{i} \frac{\partial Y'^{j_0}}{\partial y^i} + Y'^{j} \frac{\partial Y^{j_0}}{\partial y^j} = 0$. For any $1 \leq b, j_0 \leq k$ we have either $X'^{b} = 0$ or $Y'^{j_0} = 0$. Hence The result. □

**Definition 3.5.** *[2] Let $\mathfrak{A}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{A}$. We define the normalizer $\mathcal{N}$ of $\mathfrak{h}$ in $\mathfrak{A}$ the set of all vector fields $X \in \mathfrak{A}$ such that $[X, \mathfrak{h}] \subset \mathfrak{h}$.*

**Theorem 3.6.** *The normalizer of $\mathfrak{A}_\Gamma^v$ of all vector fields of $\chi(L, \mathcal{F}(M))$ is generated by the elements of $\mathfrak{A}_{\mathfrak{F}}$ wich are infinitesimal automorphisms fields to the leaves.*

*Proof.* We can calculate the normalizer of $\mathfrak{A}_\Gamma^v$ on the local coordinates system of the adapted chart of the considered domain $U$ and we obtain the result. □

**Proposition 3.10.** *If the almost-product structure $\Gamma$ is of regular curvature then the Lie algebra $\mathfrak{A}_{\mathfrak{F}}$ is the only normalizer of $\mathfrak{A}_\Gamma$ of all vector fields of $\chi(L, \mathcal{F}(M))$.*

*Proof.* It is immediate according to the definition 3.5. □

**Example 1.** *We consider the tore $\mathbb{T}^2$ given as the square $0 \leq \theta \leq 1$ and $0 \leq \beta \leq 1$ in the plan of local coordinates system of the adapted chart by $(\theta, \beta, \theta^1, \beta^1)$ of the domain $U$ and the equations:*

$$\cos^2(2\pi\beta)\, d\theta - \sin(2\pi\beta)\, d\beta = 0$$
$$\sin(2\pi\beta)\, d\theta + \cos^2(2\pi\beta)\, d\beta = 0,$$

*which define an almost-product structure on $\mathbb{T}^2$. The associated Christoffel coefficients are: $\Gamma_{11}^1 = \cos^2(2\pi\beta)$; $\Gamma_{12}^1 = -\sin(2\pi\beta) = \Gamma_{21}^1$; $\Gamma_{21}^2 = \sin(2\pi\beta) = \Gamma_{12}^2$; $\Gamma_{22}^2 = \cos^2(2\pi\beta)$.*

*However, the components of $\Gamma$ are: $\Gamma_1^1 = \theta^1 \cos^2(2\pi\beta) - \beta^1 \sin(2\pi\beta)$; $\Gamma_2^1 = -\theta^1 \sin(2\pi\beta)$; $\Gamma_1^2 = \beta^1 \sin(2\pi\beta)$; $\Gamma_2^2 = \theta^1 \sin(2\pi\beta) + \beta^1 \cos^2(2\pi\beta)$. We know that the almost-product structure $\Gamma$ is of regular curvature on $M$. We only obtain the Lie algebra $\mathfrak{A}_\Gamma = \{c\frac{\partial}{\partial\theta}\}, c \in \mathbb{R}^*$ which is resolvable.*

**Example 2.** *We construct an almost-product structure in the sphere $\mathbb{S}^3$ of dimension 3. This structure doesn't have any invariant group, nor is it of null torsion. For the construction we suppose $\mathbb{S}^3$ given by the equation $x^2+y^2+z^2+t^2 = 2$ in $\mathbb{E}^4$. Let us pass by spheric coordinates of the adapted local coordinates system $(\alpha, \theta, \varphi, \alpha^1, \theta^1, \varphi^1)$, the associated metric is $ds^2 = 2\left[d\alpha^2 + \sin^2(\alpha)\left(d\theta^2 + \sin^2(\theta) d\varphi^2\right)\right]$. We get the associated Christoffel symbols which are: $\Gamma_{22}^1 = -\cos(\alpha)\sin(\alpha)$; $\Gamma_{33}^1 = -\frac{1}{2}\sin(2\alpha)\sin^2(\theta)$;*

$\Gamma_{12}^2 = cotan(\alpha) = \Gamma_{21}^2$; $\Gamma_{33}^2 = -\cos(\theta)\sin(\theta)$; $\Gamma_{13}^3 = cotan(\alpha) = \Gamma_{31}^3$; $\Gamma_{23}^3 = cotan(\theta) = \Gamma_{32}^3$. *And the components of the almost-product structure* $\Gamma$ *are the followings:* $\Gamma_1^1 = 0$; $\Gamma_2^1 = -\alpha^1\sin(\theta)\cos(\theta)$; $\Gamma_3^1 = -\varphi^1\sin(2\alpha)\sin^2(\theta)$; $\Gamma_1^2 = \theta^1 cotan(\alpha)$; $\Gamma_2^2 = -\alpha^1 cotan(\alpha)$; $\Gamma_1^3 = \varphi^1 cotan(\alpha)$; $\Gamma_2^3 = \varphi^1 cotan(\theta)$*;* $\Gamma_3^3 = \alpha^1 cotan(\alpha) + \theta^1 cotan(\theta)$. *We find that* $\Gamma$ *is of regular curvature but, its Lie algebra* $\mathfrak{A}_\Gamma$ *is null. This example also proves that if the dimension of* $M$ *is odd then the Lie algebra isn't semisimple.*

**Example 3.** *Let* $U$ *be a domain of the adapted chart of local coordinates* $(x^1, x^2, y^1, y^2, y^3, y^4)$ *and* $\Gamma$ *an almost-product structure defined by* $\Gamma_2^1 = y^3 e^{x^1}$*,* $\Gamma_2^2 = -2 = \Gamma_2^4$*. According to the calculation, we obtain that the associated curvature* $R$ *to* $\Gamma$ *is null. That is, the nullity space of* $R$ *is equal to* $\chi(L, \mathcal{F}(M))$*. And the Lie algebra* $\mathfrak{A}_\Gamma^h = \{ae^{-x^1}\frac{\partial}{\partial x^1} + b\frac{\partial}{\partial x^2} + (2b - ay^3)\frac{\partial}{\partial y^2} + 2b\frac{\partial}{\partial y4}\}$ *where* $a \in \mathbb{R}^*$ *and* $b \in \mathbb{R}$*.*

**Example 4.** *Let* $U$ *be a domain of the adapted chart of local coordinates system* $(x^1, x^2, y^1, y^2, y^3, y^4)$ *and* $\Gamma$ *is the almost- product structure such that* $\Gamma_1^1 = e^{x^1 y^1}$*,* $\Gamma_1^2 = y^4 = -\Gamma_1^2$*,* $\Gamma_2^2 = e^{x^2}$*. We obtain that* $\mathfrak{A}_\Gamma^h = \{0\}$ *and* $\mathfrak{A}_\Gamma^v = \{f(x^1, x^2, y^3, y^4)\frac{\partial}{\partial y^3} + g(x^1, x^2, y^3, y^4)\frac{\partial}{\partial y^4}\}$*.*

## 4. Complete lift on the foliation

Let us consider the differentiable manifold $M$ connected, paracompact of dimension $n+m$ and of class $C^\infty$ providing the foliation $\mathfrak{F}$ of $n-$codimensional defined by the atlas $\mathcal{A} = \{U, (x^a, y^i)\}_{1\le a\le n, 1\le i\le m}$. Let $\mathfrak{Q}$ be the transversal fiber to the foliation $\mathfrak{F}$ and $T_{\mathfrak{F}}$ the Lie algebra of vector fields tangent to leaves of $\mathfrak{F}$ (see [12]). A form $w$ on $M$ is saw transversal if $i_X w = 0$ for all $X \in T_{\mathfrak{F}}; w$ is basic if $i_X w = i_X dw = 0$ for all $X \in T_{\mathfrak{F}}$. A vector field $X$ on $M$ is saw foliated if $Xf$ is basic for $f \in \mathcal{F}(M)$. Locally a foliated field $X$ has a local expression: $X = Y^i(y^i, x^a)\frac{\partial}{\partial y^i} + X^a(x^a)\frac{\partial}{\partial x^a}$ that of associated transversal field noted by $\tilde{X}$, is $\tilde{X} = X^a\frac{\tilde{\partial}}{\partial x^a}$ where the $\frac{\tilde{\partial}}{\partial x^a}$ are the local basis of $\mathfrak{Q}$. If $X$ and $Y$ are two foliated vector fields, their Lie bracket $[X, Y]$ is also a foliated vector field. Let us denote by $L_{\mathfrak{F}}$ the Lie algebra of the foliated vector fields of $\mathfrak{F}$ and $l_{\mathfrak{F}} = L_{\mathfrak{F}}/_{T_{\mathfrak{F}}}$. We know in [7] that $T_{\mathfrak{F}}$ is an ideal of $L_{\mathfrak{F}}$ and $l_{\mathfrak{F}}$ is a Lie algebra whose elements are called the transversal fields of the leaves of $\mathfrak{F}$.

Now, let $(\mathfrak{Q}^*, \pi, M)$ be the dual of the transversal fiber. The module of the sections of $\mathfrak{Q}^*$ can be identified to those of the transversal $1-$forms. In an adapted chart $\{U, (y^i, x^a)\}$, an element $w$ of $\mathfrak{Q}^*$ has a local expression $w = z^a dx^a$. We'll assume $(y^i, x^a, z^a)_{1\leq i\leq m, 1\leq a\leq n}$ as local coordonates in $\pi^{-1}(U)$ and thus, we obtain on $\mathfrak{Q}^*$, an atlas whose transition functions are of the form

$$y^{i'} = y^{i'}\left(y^i, x^a\right); \quad x^{a'} = x^{a'}\left(x^a\right); \quad z^{a'} = \frac{\partial x^a}{\partial x^{a'}} z^a.$$

**Definition 4.1.** *Let $X$ and $Y$ two be differentiable manifolds of class $C^\infty$. An application $p : X \to Y$ is saw a coating if the following conditions are verified:*

(i) *$p$ is surjective and a morphism of class $C^s, s \in \mathbb{N}^*$ of $X$ to $Y$,*

(ii) *for all $y \in Y$ there is an open $V$ of $Y$ containing $y$ such that $p^{-1}(V)$ admits a repartition of the form $p^{-1}(V) = \cup_{i\in I} U_i$ where the $U_i$ are the opens of $X$ such that for all $i \in I$, the restriction of $p$ to $U_i$ is a diffeomorphism of $U_i$ on $V$.*

**Example 5.** *The application $p : t \longmapsto (\cos(2\pi t), \sin(2\pi t))$ of $\mathbb{R}$ to $\mathbb{S}^1 \subset \mathbb{R}^2$ is a coating. Indeed, we know that $\mathbb{S}^1$ and $\mathbb{R}$ are manifolds of dimension $1$ and of class $C^\infty$. But the application :$t \longmapsto (\cos(2\pi t), \sin(2\pi t))$ is a $C^\infty-$morphism in $\mathbb{R}^2$, so into $\mathbb{S}^1$. We suppose $(x_0, y_0)$ a point of $\mathbb{S}^1, t_0$ such that $x_0 = \cos(2\pi t_0), y_0 = \sin(2\pi t_0)$ and $\alpha \in ]0, \frac{1}{2}[$. If $V = p\left(]t_0 - \alpha, t_0 + \alpha[\right)$, this is an open containing $(x_0, y_0)$, and $p^{-1}(V) = \cup_{k\in\mathbb{Z}} ]t_0 - \alpha + k, t_0 + \alpha + k[$. In addition, the restriction of $p$ in $]t_0 - \alpha + k, t_0 + \alpha + k[$ establishes a diffeomorphism of this interval on $V$.*

**Proposition 4.1.** *[12] $\mathfrak{Q}^*$ is provided with a foliation $\mathfrak{F}^*$ whose leafs are the coating of the leaves of $\mathfrak{F}$.*

*Proof.* The leafs of $\mathfrak{F}^*$ are locally defined by $x^a = C^t$ and $z^a = C^t$. Let $U$ be the domain of the local chart which is simple for the foliation $\mathfrak{F}$. Then $p^{-1}(U)$ is simple for $\mathfrak{F}^*$. Let $F^*$ be a leaf of $\mathfrak{F}^*$ and $F = p(F^*)$ its projection on $M$. Since $F^* \cap p^{-1}(U)$ is a plaque reunion then $F \cap U$ is a plaque reunion. So $F$ is a leaf of $\mathfrak{F}$. If $F_U$ is a plaque of $F \cap U, p^{-1}(F_U)$ is a plaque reunion whose each one is diffeomorphic to $F_U$. So $(F^*, p, F)$ is a coating. □

Let $\phi_t$ be a local group for one parameter of a foliated vector field $X$ on $M$. Since each $\phi_t$ is an automorphism of foliation then $(\phi_t)_*$ which induces an automorphism $\left(\overline{\phi_t}\right)_*^{-1}$ is an automorphism of $\mathfrak{Q}^*$. We denote $\overline{X}$ the vector field associated to the local group for a parameter $Transp\left(\left(\overline{\phi_t}\right)_*^{-1}\right)$. Locally $\overline{X}$ admits a local expression (see [12])

$$\overline{X} = Y^i\left(y^i, x^a\right)\frac{\partial}{\partial y^i} + X^a\left(x^a\right)\frac{\partial}{\partial x^a} - z^b\frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial z^a}.$$

Thus, $\overline{X}$ is a foliated vector field for the foliation $\mathfrak{F}^*$. Therefore, we have

**Definition 4.2.** *A complete lift of a vector field $X$ where locally, $X = Y^i\left(y^i, x^a\right)\frac{\partial}{\partial y^i} + X^a\left(x^a\right)\frac{\partial}{\partial x^a}$ on $M$ to $\mathfrak{Q}^*$, noted by $\overline{X}$, has a local expression on the adapted chart $\left(y^i, x^a, z^a\right)$ of the domain $p^{-1}\left(U\right)$ of $\mathfrak{Q}^*$ by:*

$$\overline{X} = Y^i\left(y^i, x^a\right)\frac{\partial}{\partial y^i} + X^a\left(x^a\right)\frac{\partial}{\partial x^a} - z^b\frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial z^a}.$$

Let $\overline{L_{\mathfrak{F}}}$ be the Lie algebra of the complete lift of elements of $L_{\mathfrak{F}}$. If a vector field $X$ is tangent to the leaves of $\mathfrak{F}$ then its lift $\overline{X}$ is tangent to the leaves of $\mathfrak{F}^*$. Consequently, the transversal field $\tilde{\overline{X}} \in l_{\mathfrak{F}^*}$ definied by the lift $\overline{X}$ of $X$ depends only on the transversal field $\tilde{X}$. We denote by $\overline{l_{\mathfrak{F}}}$ the Lie algebra obtained from $l_{\mathfrak{F}}$.

**Proposition 4.2.** *Locally, a vector field $\overline{X} = Y^i\left(y^i, x^a\right)\frac{\partial}{\partial y^i} + X^a\left(x^a\right)\frac{\partial}{\partial x^a} - z^b\frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial z^a}$ belongs to $\overline{\mathfrak{A}_\Gamma}$ such that*

$$\frac{\partial X^a}{\partial y^i} = 0,$$

$$\underset{1\le a\le n}{X^a\frac{\partial \Gamma_\alpha^\beta}{\partial x^a}} - \underset{1\le a,b\le n}{z^b\frac{\partial X^a}{\partial x^b}\frac{\partial \Gamma_\alpha^\beta}{\partial x^a}} + \underset{1\le a\le n}{\Gamma_a^\beta\frac{\partial X^a}{\partial x^\alpha}} - \underset{1\le a,b\le n}{z^b\Gamma_a^\beta\frac{\partial^2 X^a}{\partial x^\alpha\partial x^b}} + \underset{1\le i\le m}{Y^i\frac{\partial \Gamma_\alpha^\beta}{\partial y^i}} - \underset{1\le i\le m}{\Gamma_\alpha^i\frac{\partial Y^\beta}{\partial y^i}} = 0,$$

*for $1 \le \alpha \le n$ and $1 \le \beta \le m$.*

*Proof.* This is an immediate consequence of the proposition 3.1 by replacing the elements of $\mathfrak{A}_\Gamma$ by those of $\overline{\mathfrak{A}_\Gamma}$. □

We immediately obtain the following corollary

**Corollary 4.1.** *All element of $\overline{\mathfrak{A}_\Gamma}$ is projetable. That is, we have $\frac{\partial X^a}{\partial y^i} = 0, 1 \le a \le n$ and $1 \le i \le m$.*

*Proof.* Indeed, according to the first equation of propositon 4.2 we obtain this corollary. □

**Remark 4.1.** *We remark that the Lie algebra $\overline{\mathfrak{A}_\Gamma}$ is of dimension superior or equal to 2. $\overline{\mathfrak{A}_\Gamma}$ can be of infinite dimension.*

**Proposition 4.3.** *If $M$ is compact without board manifold of dimension $2k+1$ with $k \in \mathbb{N}$, then the Lie algebra $\overline{\mathfrak{A}_\Gamma}$ of complete lift $\overline{X}$ of the vector fields $X$ is generated by the vector fields tangent to the leaves of the foliation $\mathfrak{F}^*$.*

*Proof.* We suppose that $M$ is a compact without board manifold then the nulity space of the curvature $R$ associated to the almost-product structure $\Gamma$ is null. And if the dimension of $M$ is equal to $2k+1$, according to lemma 3.1, the Lie algebra $\mathfrak{A}_\Gamma$ is null. So the Lie algebra $\overline{\mathfrak{A}_\Gamma}$ does not a projectable field. Hence the result. □

We adopt the following proposition:

**Proposition 4.4.** *Let an finite integer $p \geq 2$ be the dimension of Lie algebra $\overline{\mathfrak{A}_\Gamma}$, we have*

(1) *if the Lie algebra $\overline{\mathfrak{A}_\Gamma}$ of the complete lift $\overline{X}$ of the vector field $X$ is semisimple, we have $\mathfrak{H}^1\left(\overline{\mathfrak{A}_\Gamma}\right) = \mathfrak{H}^2\left(\overline{\mathfrak{A}_\Gamma}\right) = \{0\}$ and $\mathfrak{H}^p\left(\overline{\mathfrak{A}_\Gamma}\right)$ is $1$ dimension,*
(2) *if $\overline{\mathfrak{A}_\Gamma}$ isn't semisimple and $p$ is an pair integer we have $\mathfrak{H}^p\left(\overline{\mathfrak{A}_\Gamma}\right) = \{0\}$.*

*Proof.* Since $p$ is finite integer and $p \geq 2$ then the set $\mathfrak{H}^n\left(\overline{\mathfrak{A}_\Gamma}\right)$ with $1 \leq n \leq p$ are finites. So, we can calculate them using a program from the Maple Software. In every case, we will find these results. □

**Remark 4.2.** *In general this proposition 4.4 is still true for any Lie algebras of finite dimensional.*

On $\mathfrak{Q}^*$, we suppose that there is a $1-$form $w$ called canonic form defined by: $w_\theta(X) = \theta\left(\overline{p_*(X)}\right)$ for all $\theta \in \mathfrak{Q}^*$ and $X \in T_\theta\mathfrak{Q}^*$. Locally, $w = z^a dx^a, 1 \leq a \leq n$. Let $C$ be a canonic field on $\mathfrak{Q}^*$ then its local expression is $C = z^a \frac{\partial}{\partial z^a}, 1 \leq a \leq n$. We have

$$i_C dw = w, \quad L_C\theta = \theta, \quad L_C d\theta = d\theta,$$

where $i_C$ and $L_C$ indicate respectively the interior product and the Lie derivative in comparaison to $C$.

**Proposition 4.5.** *[12] A vector field $Z$ on $\mathfrak{Q}^*$ is tangent to leaves of $\mathfrak{F}^*$ if and only if $i_Z\theta = i_Z d\theta = 0$.*

Now let $\mathfrak{A}^w$ be the Lie algebra of vector fields of $\mathfrak{Q}^*$ which leaves $w$ invariant. A vector field $X = Y^i \frac{\partial}{\partial y^i} + X^a \frac{\partial}{\partial x^a} + Z^b \frac{\partial}{\partial z^b}$ $(1 \leq a, b \leq n, 1 \leq i \leq m)$ belongs to $\mathfrak{A}^w$, if

$$\frac{\partial X^a}{\partial y^i} z^a = 0; \quad \frac{\partial X^a}{\partial z^b} z^a = 0; \quad Z^b = -\frac{\partial X^b}{\partial x^a} z^b.$$

Locally $X = Y^i\left(y^j, x^a, z^b\right)\frac{\partial}{\partial y^i} + X^a\left(x^c\right)\frac{\partial}{\partial x^a} - z^b \frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial z^a}$ for $1 \leq a, b, c \leq n$ and $1 \leq i, j \leq m$. Let $\overline{\mathfrak{A}^w_\Gamma}$ be the Lie algebra of vector fields of $\overline{\mathfrak{A}_\Gamma}$ which leave $w$ invariant.

**Proposition 4.6.** *A vector field* $X = Y^i\left(y^j, x^a, z^b\right)\frac{\partial}{\partial y^i} + X^a\left(x^c\right)\frac{\partial}{\partial x^a} - z^b\frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial z^a}$ *belongs to* $\overline{\mathfrak{A}_\Gamma^w}$ *if it verifies:*

$$\underset{1\leq a\leq n}{X^a\frac{\partial\Gamma_\alpha^\beta}{\partial x^a}} - \underset{1\leq a,b\leq n}{z^b\frac{\partial X^a}{\partial x^b}\frac{\partial\Gamma_\alpha^\beta}{\partial x^a}} + \underset{1\leq a\leq n}{\Gamma_a^\beta\frac{\partial X^a}{\partial x^\alpha}} - \underset{1\leq a,b\leq n}{z^b\Gamma_a^\beta\frac{\partial^2 X^a}{\partial x^\alpha\partial x^b}} + \underset{1\leq i\leq m}{Y^i\left(y^j, x^a, z^b\right)\frac{\partial\Gamma_\alpha^\beta}{\partial y^i}}$$
$$- \underset{1\leq i\leq m}{\Gamma_\alpha^i\frac{\partial Y^\beta\left(y^j, x^a, z^b\right)}{\partial y^i}} = 0,$$

*for* $1\leq\alpha\leq n$ *and* $1\leq\beta\leq m$.

*Proof.* To demonstrate this proposition, we just need to adapt the previous theorem 4.2 to the vector fields of $\overline{\mathfrak{A}_\Gamma^w}$. □

**Theorem 4.1.** *We get*

(i) $\overline{\mathfrak{A}_\Gamma}\subset\overline{\mathfrak{A}_\Gamma^w}\subseteq\overline{\mathfrak{A}^w}$,

(ii) *the derived ideal of* $\overline{\mathfrak{A}_\Gamma^w}$ *coïncides to itself. That is,* $\left[\overline{\mathfrak{A}_\Gamma^w}, \overline{\mathfrak{A}_\Gamma^w}\right] = \overline{\mathfrak{A}_\Gamma^w}$,

(iii) *in addition,* $\overline{\mathfrak{A}^w}$ *is the normalizer of* $\overline{\mathfrak{A}_\Gamma^w}$ *on* $(\mathfrak{Q}^*, \mathfrak{F}^*)$.

We assume $P = \overline{\mathfrak{A}_\Gamma^w}\cap Kerw$. $\overline{Z}\in P$ if and only if $\overline{Z} = Y^i\left(y^j, x^a, z^b\right)\frac{\partial}{\partial y^i} + X^a\left(x^c\right)\frac{\partial}{\partial x^a} - z^b\frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial z^a}$ and $w\left(\overline{Z}\right) = 0$. Then $X^a = 0$ and $\overline{Z} = Y^i\left(y^j, x^a, z^b\right)\frac{\partial}{\partial y^i} + X^a\left(x^c\right)\frac{\partial}{\partial x^a} - z^b\frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial z^a}$. So $\overline{Z} = Y^i\left(y^j, x^a, z^b\right)\frac{\partial}{\partial y^i}\in P$. That is, $P$ is the Lie algebra of the vector fields tangent to leaves of $\mathfrak{F}^*$.
We can easily verify that $P$ is an ideal of $\overline{\mathfrak{A}_\Gamma^w}$. Indeed, let $X\in P$ and $Y\in\overline{\mathfrak{A}_\Gamma^w}$. It is obvious that $[X, Y]\in\overline{\mathfrak{A}_\Gamma^w}$ and we immediately obtain that $w\left([X, Y]\right) = 0$. So $[X, Y]\in P$. And $\overline{\mathfrak{A}_\Gamma^w}/_P\subset\overline{l_\mathfrak{F}}$ and that the application $\varphi : \overline{\mathfrak{A}_\Gamma^w}\to\overline{l_\mathfrak{F}}$ is thus defined a surjection according to [12]. We have the following exact sequence

$$0\to P\to\overline{\mathfrak{A}_\Gamma^w}\to\overline{L_\mathfrak{F}}\to 0.$$

**Theorem 4.2.** *All application* $A\longmapsto[B, A], B\in\overline{\mathfrak{A}_\Gamma^w}$ *of* $P$ *to itself is a derivation of* $P$ *which isn't necessarily inner.*

*Proof.* We adapt the theory of Ton Van Duc in [12]. Indeed, let $D$ be a derivation of $P$ and $B\in\overline{\mathfrak{A}_\Gamma^w}$. According to the theorem 4.1, we have $\left[\overline{\mathfrak{A}_\Gamma^w}, \overline{\mathfrak{A}_\Gamma^w}\right] = \overline{\mathfrak{A}_\Gamma^w}$. We can write $B$ by $B = [B_1, B_2]$ where $B_1, B_2\in\overline{\mathfrak{A}_\Gamma^w}$. We get $DB = [DB_1, B_2] + [B_1, DB_2]\in\overline{\mathfrak{A}_\Gamma^w}$. Then $D|_P$ is a derivation of $\overline{\mathfrak{A}_\Gamma^w}$. And there is $B\in\overline{\mathfrak{A}_\Gamma^w}$ such that $D|_P$ is the application $A\longmapsto[B, A]$ for all $A\in P$. Indeed, let be $A\in P$ and $B'\in\overline{\mathfrak{A}_\Gamma^w}$, we have $D\left[B', A\right] = \left[DB', A\right] + \left[B', DA\right] = \left[L_B B', A\right] + \left[B', DA\right] = L_B\left[B', A\right] =$

$\left[L_B B', A\right] + \left[B', L_B A\right]$. So we get $\left[B', (D - L_B) A\right] = 0$. Thus $DA = L_B A = [B, A]$ for $A \in P$. It's obvious that this derivation is not inner. It suffices to take into account the elements of $P$ and $\overline{\mathfrak{A}_\Gamma^w}$. Hence the result. □

**Example 6.** *Let $\overline{\mathfrak{A}_\Gamma}$ be the Lie algebra generated by the vector fields $X$ and $Y$ of $\overline{L_{\mathfrak{F}}}$ such that $[X, Y] = X$. We know that $\overline{\mathfrak{A}_\Gamma}$ isn't semisimple. By definition, $\mathfrak{H}^1\left(\overline{\mathfrak{A}_\Gamma}\right) \cong \overline{\mathfrak{A}_\Gamma}/_{\left[\overline{\mathfrak{A}_\Gamma}, \overline{\mathfrak{A}_\Gamma}\right]}$ where $\left[\overline{\mathfrak{A}_\Gamma}, \overline{\mathfrak{A}_\Gamma}\right] = \langle X\rangle$. So we have $\mathfrak{H}^1\left(\overline{\mathfrak{A}_\Gamma}\right) \cong \langle Y\rangle$. Next let $C$ be a $2-$cochain of $\overline{\mathfrak{A}_\Gamma}$. So we obtain that $\partial C(X, Y) = -C([X, Y])$ by definition and thus $\partial^2 C(X, Y) = \partial(\partial C(X, Y)) = -\partial C([X, Y]) = C([[X, Y], X]) = 0$, according to Jacobi's identity and we easily find that $\partial^3 C(X, Y) = \partial(\partial^2 C(X, Y)) = 0$. Thus $\mathfrak{H}^2\left(\overline{\mathfrak{A}_\Gamma}\right) \cong \{0\}$.*

**Example 7.** *Next, we consider that the Lie algebra $\overline{\mathfrak{A}_\Gamma}$ is generated by the vector fields $X, Y$ and $Z$ of $\overline{L_{\mathfrak{F}}}$ such that $[X, Z] = X;\ [Y, Z] = Y$ and $[X, Y] = Z$. $\overline{\mathfrak{A}_\Gamma}$ is semisimple and, we have $\left[\overline{\mathfrak{A}_\Gamma}, \overline{\mathfrak{A}_\Gamma}\right] = \overline{\mathfrak{A}_\Gamma}$, so $\mathfrak{H}^1\left(\overline{\mathfrak{A}_\Gamma}\right) \cong \{0\}$. In addition, let $\phi \in Ker\partial^3$ where $\phi : \overline{\mathfrak{A}_\Gamma} \times \overline{\mathfrak{A}_\Gamma} \to \overline{\mathfrak{A}_\Gamma}$. We have $\phi([X, Y], Z) + \phi([Y, Z], X) + \phi([Z, X], Y) = \phi(Z, Z) + \phi([Y, Z], X) + \phi(Y, X) = \phi(Y, X) - \phi(X, Y) = 0$. Then $Ker\partial^3 = C^2\left(\overline{\mathfrak{A}_\Gamma}, \overline{\mathfrak{A}_\Gamma}\right)$ where $C^2\left(\overline{\mathfrak{A}_\Gamma}, \overline{\mathfrak{A}_\Gamma}\right)$ is the set of $2-$cochains on $\overline{\mathfrak{A}_\Gamma}$. So we obtain $\mathfrak{H}^2\left(\overline{\mathfrak{A}_\Gamma}, \overline{\mathfrak{A}_\Gamma}\right) = C^2\left(\overline{\mathfrak{A}_\Gamma}, \overline{\mathfrak{A}_\Gamma}\right)/_{Im\partial^2}$. Finally let $\psi \in Ker\partial^4$ where $\psi$ is an anti-symmetric linear application $\psi : \overline{\mathfrak{A}_\Gamma} \times \overline{\mathfrak{A}_\Gamma} \times \overline{\mathfrak{A}_\Gamma} \to \overline{\mathfrak{A}_\Gamma}$. We have $\psi(X, Y, Z) = K([X, Y], Z)$ with $K$ is the Killing form on $\overline{\mathfrak{A}_\Gamma}$. Thus we have $\dim\left(\mathfrak{H}^3\left(\overline{\mathfrak{A}_\Gamma}\right)\right) = 1$.*

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