ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **9** (2020), no.12, 11043–11062
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.12.86

# FINITE VOLUME METHOD IN A GENERAL MESH FOR CHEMOTAXIS MODEL

OUAFA SOUALHI[1] AND MOHAMED RHOUDAF

ABSTRACT. In this paper, we present the finite volume method DDFV applied to a model of (Patlak) Keller-Segel, this model consists of a coupled system of elliptic and parabolic equations with an additional cross-duffision term in the elliptic equation. The existence of a discrete solution is proved. Numerical simulations are performed to verify accuracy.

## 1. INTRODUCTION

Chemotaxis is the orientation of cells or organisms in response to the influence of chemical stimulus. This phenomena has a important role in several biological filds, like immunology, cancer growth, wound healing and embryogenesis (see [1]).

In the literature, there are several works presenting a numerical method to solve the classical Keller-Segel system, for example: Filbet proves the existence and singularity of a numerical solution to the finite volume scheme in [3] and the authors in [2] present the finite volume scheme for a Keller-Segel model with additional cross-diffusion. Moreover, [4] propose the numerical and theoretical study of Stochastic particle approximation for measure valued solutions of the 2D Keller-Segel system and the paper [5] concerned the numerical simulation of chemotactic using the mixed finite elements method. The authors in [6, 7] study

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 35B40, 35L70.

*Key words and phrases.* Keller-Segel, Finite volume, Chimotaxis.

the finite-element method for a simplified Keller-Segel system and finite difference schemes to a parabolic-elliptic system modelling chemotaxis was treated in [8].

Patlak in 1953 [9], Keller and Segel in 1970 [10], were created as a model to describe the evolution over time of the cell density $n(x,t)$ and the chemical signal concentration variable $S(x,t)$ the following model:

$$\begin{cases} \frac{\partial n}{\partial t} - div(\nabla n - n\nabla S) = 0, & \text{on } Q_T, \\ -div(\nabla S) - \mu n + S = 0, & \text{on } Q_T, \end{cases} \tag{1.1}$$

with $Q_T = \Omega \times (0,T)$, $T > 0$ is a fixed time, and $\Omega$ is an open bounded domain in $\mathbb{R}^d$ ;$d = 2$ or $3$, with smooth boundary $\partial\Omega$. The parameter $\mu > 0$ is the secretion rate at which the chemical substance is emitted by the cells. The nonlinear term $n\nabla S$ models the cell movement towards higher concentrations of the chemical signal.

Now, let us introduce the additional cell diffusion $\delta\Delta n$ in the equation of the chemical concentration of the system (1.1), we obtain the following system:

$$\begin{cases} \frac{\partial n}{\partial t} - div(\nabla n - n\nabla S) = 0, & \text{on } Q_T, \\ -div(\nabla S) - \delta\Delta n - \mu n + S = 0, & \text{on } Q_T. \end{cases} \tag{1.2}$$

The initial conditions on $\Omega$ are given by

$$n(x,0) = n^0(x), \text{ in } \Omega. \tag{1.3}$$

Therefore, the system (1.2) is supplemented by the following boundary conditions on $\partial\Omega \times (0,T)$

$$\nabla n.\nu = 0, \text{ in } \partial\Omega \times (0,T), \tag{1.4}$$

$$\nabla S.\nu = 0, \text{ in } \partial\Omega \times (0,T). \tag{1.5}$$

The vector $\nu$ is the normal unity vector, and the additional cell diffusion $\delta\Delta n$, with $\delta > 0$ is the additional diffusion constant, has an important role to determine the explosion time numerically.

The authors who has treated the system (1.2)-(1.5) by the finite volume method in [2], assume a condition of orthogonality on the mesh in the sense of Eymard et al. [11]. This excludes other types of meshes that do not satisfy this condition. For example in porous media most of the geological layers are quite deformed, and therefore the mesh used to study these problems in general does not satisfy the condition of orthogonality. Recently several schemes have been suggested to overcome this problem.

In this paper we are interested with the discrete duality finite volume method for the desritization of the Keller-Segel problem without orthogonality condition on the mesh. More precisely we prove the existence and the uniqueness of the approximate solution by using Brower's fixed point theorem. Then, Some numerical tests are also carried out to verify the validity of the numerical scheme proposed.

The DDFV (Discrete Duality Finite Volume ) method was presented by Hermeline [12], Domelevo, Omnes [13] and Andreianov, Boyer, Hubert [14], and it was extended to convection-diffusion by Coudière in [16]. In [15] and [14, 17] the authors are present the DDFV scheme apply to a nonlinear diffusion equation. Omens et al. study the DDFV approach apply to Hodge decomposition and div-curl problems on almost arbitrary two-dimensional meshes [18], and other work as : miscible fluid flows in porous media [19, 20], The authors in [21] present a DDFV schemes applying to a seawater problem.

This paper is organized as follows : In Section 2 we detail the DDFV formulation. The demonstrate of the existence and uniqueness of the DDFV solutions and number of numerical results obtained on different two-dimensional meshes are realized in section 3.

## 2. Discrete Duality finite volume schemes for modified Keller-Segel model

2.1. **Meshes and notations.** Let $\Omega$ be a polygonal open bounded connected subset of $\mathbb{R}^d$ with $d = 2$ or $3$, and $\partial\Omega = \overline{\Omega}\backslash\Omega$ its boundary .

Following Hermeline [12], Domelevo, Omnes [13] and Andreianov, Boyer, Hubert [14], we consider a DDFV mesh which is a triple $\mathcal{T} = (\mathfrak{M}, \mathfrak{M}^*, \mathfrak{D})$ described below.

The primal mesh $\mathfrak{M}$ is defined as the triplet $(\mathfrak{M}, \mathcal{E}, P)$ where:

- $\mathfrak{M}$ is a finite family of nonempty open disjoint subset $\mathcal{K}$ of $\Omega$ ($\mathcal{K}$ is the control volume primal) such that $\overline{\Omega} = \cup_{\mathcal{K}\in\mathfrak{M}}\overline{\mathcal{K}}$, with $\partial\mathcal{K} = \overline{\mathcal{K}}\backslash\mathcal{K}$ be the boundary of $\mathcal{K}$, let $m_{\mathcal{K}} = |\mathcal{K}| > 0$ is the measure of $\mathcal{K}$ and $d_{\mathcal{K}}$ the diameter of $\mathcal{K}$.
- $\mathcal{E}$,$\mathcal{E}_{int}$, $\mathcal{E}_{ext}$ and $\mathcal{E}_K$ are respectively the set of edges $\sigma$, the subset of the interior edges, the subset of exterior edges of the mesh and the subset of

the edges of $\mathcal{K}$, $m_\sigma$ is the measure of $\sigma$ and $\nu_{\mathcal{K},\sigma}$ is the unite vector normal to $\sigma$ outward to $\mathcal{K}$.

- $P = \{(x_\mathcal{K})_{\mathcal{K}\in\mathfrak{M}}; x_\mathcal{K} \in \mathcal{K}\}$ is the subset of the points of the mesh with $x_\mathcal{K}$ is the barycentre of $\mathcal{K}$ and $D_{\mathcal{K},\sigma}$ the cone with vertex $x_\mathcal{K}$ and basis $\mathcal{K}$.

Next, The dual mesh $\mathfrak{M}^*$ is defined as the triplet $(\mathfrak{M}^*, \mathcal{E}^*, P^*)$ where:

- $\mathfrak{M}^*$ is a finite family of nonempty open disjoint subset $\mathcal{K}^*$ of $\Omega$ ($\mathcal{K}^*$ is the control volume dual) such that $\overline{\Omega} = \cup_{\mathcal{K}^*\in\mathfrak{M}^*}\overline{\mathcal{K}^*}$, with $\partial\mathcal{K}^* = \overline{\mathcal{K}^*}\backslash\mathcal{K}^*$ be the boundary of $\mathcal{K}^*$, let $m_{\mathcal{K}^*} = |\mathcal{K}^*| > 0$ is the measure of $\mathcal{K}^*$ and $d_{\mathcal{K}^*}$ the diameter of $\mathcal{K}^*$.
- $\mathcal{E}^*$,$\mathcal{E}^*_{int}$, $\mathcal{E}^*_{ext}$ and $\mathcal{E}^*_{K^*}$ are respectively the set of edges $\sigma^*$, the subset of the interior edges, the subset of exterior edges of the mesh and the subset of the edges of $\mathcal{K}^*$, $m_{\sigma^*}$ is the measure of $\sigma^*$ and $\nu_{\mathcal{K}^*,\sigma^*}$ is the unite vector normal to $\sigma^*$ outward to $\mathcal{K}^*$.
- $P = \{(x_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*}; x_{\mathcal{K}^*} \in \mathcal{K}^*\}$ is the subset of the points of the mesh and $D_{\mathcal{K}^*,\sigma^*}$ the cone with vertex $x_{\mathcal{K}^*}$ and basis $\mathcal{K}^*$.

Finally, We denote by $\mathfrak{D}$ the sets of all diamonds $\mathcal{D}$, let:

- $\mathfrak{D}_\mathcal{K} = \{\mathcal{D} \in \mathfrak{D}$ such that $\sigma \in \mathcal{E}_\mathcal{K}\}$.
- $\mathfrak{D}_{\mathcal{K}^*} = \{\mathcal{D} \in \mathfrak{D}$ such that $\sigma^* \in \mathcal{E}^*_{\mathcal{K}^*}\}$.
- $\mathfrak{D}_{int} = \{\mathcal{D} \in \mathfrak{D}$ such that $\sigma \in \mathcal{E}_{int}\}$.
- $\mathfrak{D}_{ext} = \{\mathcal{D} \in \mathfrak{D}$ such that $\sigma \in \mathcal{E}_{ext}\}$.
- $\mathcal{M}_\mathcal{D} = \{\mathcal{K} \in \mathfrak{M}$ such that $\sigma \in \mathcal{E}_\mathcal{K}\}$.
- $\mathcal{M}^*_\mathcal{D} = \{\mathcal{K}^* \in \mathfrak{M}^*$ such that $\sigma^* \in \mathcal{E}^*_{\mathcal{K}^*}\}$.
- $m_\mathcal{D}$ measure of the diamond.
- For a diamond cell $\mathcal{D}$ recall that $(x_\mathcal{K}, x_{\mathcal{K}^*}, x_\mathcal{L}, x_{\mathcal{L}^*})$ are the vertices of $\mathcal{D}_{\sigma,\sigma^*}$.
- $\tau$ the unite vector parallel to $\sigma$, oriented from $\mathcal{K}^*$ to $\mathcal{L}^*$.
- $\tau^*$ the unite vector parallel to $\sigma^*$, oriented from $\mathcal{K}$ to $\mathcal{L}$.
- $\alpha_\mathcal{D}$ the angle between $\tau$ and $\tau^*$.
- $\nu_{\mathcal{K},\sigma} = -cos\alpha_\mathcal{D}\nu_{\sigma^*,\mathcal{K}^*} + sin\alpha_\mathcal{D}\tau_{\mathcal{K},\sigma}$.
- $d_\mathcal{D}$ the diameter of $\mathcal{D}_{\sigma,\sigma^*}$.

We consider the following property:

$$\frac{m_\sigma m_{\sigma^*}}{2m_\mathcal{D}} \leq \frac{mes(D_{\mathcal{K},\sigma})}{3}. \tag{2.1}$$

Finally, the size of the mesh: $size(\mathcal{T}) = \max_{\mathcal{D}\in\mathfrak{D}} d_\mathcal{D}$.

2.2. **Discrete operators and duality formula.** we define the spaces:

- $\mathbb{R}^{\mathcal{T}}$ is a linear space of scalar fields constant on the cells of $\overline{\mathfrak{M}}$ and $\overline{\mathfrak{M}^*}$.

$$\mathbb{R}^{\mathcal{T}} = \{u_{\mathcal{T}} = ((u_{\mathcal{K}})_{\mathcal{K}\in\overline{\mathfrak{M}}}, (u_{\mathcal{K}^*})_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}});\ \text{with } u_{\mathcal{K}} \in \mathbb{R},\ \text{for all } \mathcal{K} \in \overline{\mathfrak{M}}$$
$$\text{and } u_{\mathcal{K}^*} \in \mathbb{R},\ \text{for all } \mathcal{K}^* \in \overline{\mathfrak{M}^*}\}.$$

- $(\mathbb{R}^2)^{\mathcal{D}}$ is a linear space of vector fields constant on the cells of $\mathcal{D}$.

$$(\mathbb{R}^2)^{\mathcal{D}} = \{\xi_{\mathfrak{D}} = (\xi_{\mathcal{D}})_{\mathcal{D}\in\overline{\mathfrak{D}}};\ \text{with } \xi_{\mathcal{D}} \in \mathbb{R}^2,\ \text{for all } \mathcal{D} \in \mathfrak{D}\}.$$

Now, we recall the definition of the discrete gradient and the discrete divergence have been introduced respectively in [22] and [13]. We also introduce some trace operators and scalar products

**Definition 2.1.** *The discrete gradient is defined by:*

$$\begin{aligned}\nabla^{\mathfrak{D}} :& \mathbb{R}^{\mathcal{T}} \to (\mathbb{R}^2)^{\mathfrak{D}},\\ & u_{\mathcal{T}} \to \nabla^{\mathfrak{D}} u_{\mathcal{T}} = (\nabla^{\mathcal{D}} u_{\mathcal{T}})_{\mathcal{D}\in\mathfrak{D}}.\end{aligned}$$

*Such that for all $\mathcal{D} \in \mathfrak{D}$*

$$\begin{cases} \nabla^{\mathcal{D}} u_{\mathcal{T}}.\tau_{\mathcal{K}^*,\mathcal{L}^*} = \frac{u_{\mathcal{L}^*}-u_{\mathcal{K}^*}}{m_\sigma}, \\ \nabla^{\mathcal{D}} u_{\mathcal{T}}.\tau_{\mathcal{K},\mathcal{L}} = \frac{u_{\mathcal{L}}-u_{\mathcal{K}}}{m_{\sigma^*}}, \end{cases}$$

*equivalent to*

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{sin(\alpha_{\mathcal{D}})}\left[\frac{u_{\mathcal{L}}-u_{\mathcal{K}}}{m_{\sigma^*}}\nu_{\sigma,\mathcal{K}} + \frac{u_{\mathcal{L}^*}-u_{\mathcal{K}^*}}{m_\sigma}\nu_{\sigma^*,\mathcal{K}^*}\right],$$

*using the propriety $m_{\mathcal{D}} = \frac{1}{2}m_\sigma m_{\sigma^*} sin(\alpha_{\mathcal{D}})$, we have*

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{2m_{\mathcal{D}}}\left[(u_{\mathcal{L}}-u_{\mathcal{K}})m_\sigma\nu_{\sigma,\mathcal{K}} + (u_{\mathcal{L}^*}-u_{\mathcal{K}^*})m_{\sigma^*}\nu_{\sigma^*,\mathcal{K}^*}\right].$$

**Definition 2.2.** *The discrete divergence operator $div^{\mathcal{T}}$ is a mapping from $(\mathbb{R}^2)^{\mathfrak{D}}$ to $\mathbb{R}^{\mathcal{T}}$ defined for all $\xi_{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}$ by*

$$div^{\mathcal{T}}\xi_{\mathfrak{D}} = \left(div^{\mathfrak{M}}\xi_{\mathfrak{D}}, 0, div^{\mathfrak{M}^*}\xi_{\mathfrak{D}}, div^{\partial\mathfrak{M}^*}\xi_{\mathfrak{D}}\right),$$

*such that*

$$\begin{cases} div^{\mathfrak{M}}(\xi_{\mathfrak{D}}) = (div_{\mathcal{K}}(\xi_{\mathfrak{D}}))_{\mathcal{K}\in\mathfrak{M}}, \\ div^{\mathfrak{M}^*}(\xi_{\mathfrak{D}}) = (div_{\mathcal{K}^*}(\xi_{\mathfrak{D}}))_{\mathcal{K}^*\in\mathfrak{M}^*}, \\ div^{\partial\mathfrak{M}^*}(\xi_{\mathfrak{D}}) = (div_{\mathcal{K}^*}(\xi_{\mathfrak{D}}))_{\mathcal{K}^*\in\partial\mathfrak{M}^*}, \end{cases}$$

*with*

$$div_{\mathcal{K}}\xi_{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}}}\sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}}} m_\sigma \xi_{\mathcal{D}}.\nu_{\sigma,\mathcal{K}} \quad \textit{for all } \mathcal{K}\in\mathfrak{M},$$

$$div_{\mathcal{K}^*}\xi_{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}^*}}\sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}} m_{\sigma^*} \xi_{\mathcal{D}}.\nu_{\sigma^*,\mathcal{K}^*} \quad \textit{for all } \mathcal{K}^*\in\mathfrak{M}^*,$$

$$div_{\mathcal{K}^*}\xi_{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}^*}}\left(\sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}} m_{\sigma^*} \xi_{\mathcal{D}}.\nu_{\sigma^*,\mathcal{K}^*} + \sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}\cap\mathfrak{D}_{ext}} \frac{m_\sigma}{2}\xi_{\mathcal{D}}.\nu_{\sigma,\mathcal{K}}\right) \quad \textit{for all } \mathcal{K}^*\in\partial\mathfrak{M}^*.$$

Let us now define the scalar products $< .,. >_{\mathcal{T}}$ on $\mathbb{R}^{\mathcal{T}}$ and $< .,. >_{\mathcal{D}}$ on $(\mathbb{R}^2)^{\mathcal{D}}$ by

$$< v_{\mathcal{T}}, u_{\mathcal{T}} >_{\mathcal{T}} = \frac{1}{2}\left(\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} u_{\mathcal{K}} v_{\mathcal{K}} + \sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} u_{\mathcal{K}^*} v_{\mathcal{K}^*}\right), \text{ for all } u_{\mathcal{T}}, v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

$$< \xi_{\mathfrak{D}}, \varphi_{\mathfrak{D}} >_{\mathcal{D}} = \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} \xi_{\mathcal{D}}.\varphi_{\mathcal{D}}, \text{ for all } \xi_{\mathfrak{D}}, \varphi_{\mathfrak{D}} \in \left(\mathbb{R}^2\right)^{\mathcal{D}}.$$

The corresponding norms are denoted by $\|.\|_{p,\mathcal{T}}$ and $\|.\|_{p,\mathfrak{D}}$ for all $1 \leq p \leq +\infty$.

$$\|u_{\mathcal{T}}\|_{p,\mathcal{T}} = \left(\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}}|u_{\mathcal{K}}|^p + \frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*}|u_{\mathcal{K}^*}|^p\right)^{1/p},$$

for all $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ and for all $1 \leq p < +\infty$.

$$\|\xi_{\mathfrak{D}}\|_{p,\mathfrak{D}} = \left(\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}|\xi_{\mathcal{D}}|^p\right)^{1/p}, \text{ for all } \xi_{\mathfrak{D}} \in \left(\mathbb{R}^2\right)^{\mathfrak{D}} \text{ and for all } 1 \leq p < +\infty.$$

$$\|u_{\mathcal{T}}\|_{\infty,\mathcal{T}} = \max\left(\max_{\mathcal{K}\in\overline{\mathfrak{M}}} |u_{\mathcal{K}}|, \max_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} |u_{\mathcal{K}^*}|\right), \text{ for all } u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

$$\|\xi_{\mathfrak{D}}\|_{\infty,\mathcal{D}} = \max_{\mathcal{D}\in\mathfrak{D}} |\xi_{\mathcal{D}}|, \text{ for all } \xi_{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathcal{D}}.$$

**Definition 2.3** (Convection term). *Let* $divc^{\mathcal{T}} : (\mathbb{R}^2)^{\mathfrak{D}} \times \mathbb{R}^{\mathcal{T}} \to \mathbb{R}^{\mathcal{T}}$ *the convection operator defined for all* $\xi_{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}$ *and* $v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ *by*

$$divc^{\mathcal{T}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = \left(divc^{\mathfrak{M}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}), 0, divc^{\mathfrak{M}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}), divc^{\partial\mathfrak{M}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}})\right),$$

*such that*

$$divc^{\mathfrak{M}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = (divc_{\mathcal{K}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}))_{\mathcal{K}\in\mathfrak{M}},$$
$$divc^{\mathfrak{M}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = (divc_{\mathcal{K}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}))_{\mathcal{K}^*\in\mathfrak{M}^*},$$
$$divc^{\partial\mathfrak{M}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = (divc_{\mathcal{K}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}))_{\mathcal{K}^*\in\partial\mathfrak{M}^*},$$

*with*

$$divc_{\mathcal{K}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = \frac{1}{m_{\mathcal{K}}} \sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}}\\ \sigma=\mathcal{K}/\mathcal{L}}} m_\sigma \left[ (\xi_{\mathfrak{D}}.\nu_{\sigma,\mathcal{K}})^+ v_{\mathcal{K}} - (\xi_{\mathfrak{D}}.\nu_{\sigma,\mathcal{K}})^- v_{\mathcal{L}} \right], \text{ for all } \mathcal{K}\in\mathfrak{M},$$

$$divc_{\mathcal{K}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = \frac{1}{m_{\mathcal{K}^*}} \sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}\\ \sigma^*=\mathcal{K}^*/\mathcal{L}^*}} m_{\sigma^*} \left[ (\xi_{\mathfrak{D}}.\nu_{\sigma^*,\mathcal{K}^*})^+ v_{\mathcal{K}^*} - (\xi_{\mathfrak{D}}.\nu_{\sigma^*,\mathcal{K}^*})^- v_{\mathcal{L}^*} \right],$$

*for all* $\mathcal{K}^*\in\mathfrak{M}^*$,

$$divc_{\mathcal{K}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = \frac{1}{m_{\mathcal{K}^*}} \Big( \sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}\\ \sigma^*=\mathcal{K}^*/\mathcal{L}^*}} m_{\sigma^*} \left[ (\xi_{\mathfrak{D}}.\nu_{\sigma^*,\mathcal{K}^*})^+ v_{\mathcal{K}^*} - (\xi_{\mathfrak{D}}.\nu_{\sigma^*,\mathcal{K}^*})^- v_{\mathcal{L}^*} \right]$$
$$+ \sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}\cap\mathfrak{D}_{ext}\\ \sigma=\mathcal{K}/\mathcal{L}}} \frac{m_\sigma}{2} \left[ (\xi_{\mathfrak{D}}.\nu_{\sigma,\mathcal{K}})^+ v_{\mathcal{K}} - (\xi_{\mathfrak{D}}.\nu_{\sigma,\mathcal{K}})^- v_{\mathcal{L}} \right] \Big),$$

*for all* $\mathcal{K}^*\in\partial\mathfrak{M}^*$*, where* $x^+ = \max(x,0)$ *and* $x^- = \max(0,-x)$.

2.3. **The numerical scheme.** A DDFV scheme for the the discretization of the problem (1.2) is given by the following set of equations: for all $\mathcal{K}\in\mathfrak{M}$ and $\mathcal{K}^*\in\mathfrak{M}^*$,

$$n^0_{\mathcal{K}} = \frac{1}{m_{\mathcal{K}}} \int_{\mathcal{K}} n^0(x)dx \text{ and } n^0_{\mathcal{K}^*} = \frac{1}{m^*_{\mathcal{K}}} \int_{\mathcal{K}^*} n^0(x)dx.$$

At each time step $k$, the numerical solution will be given by $(n^k_{\mathcal{T}}, S^k_{\mathcal{T}})$. Then, the scheme for (1.2) writes for all $0\le k\le N_T-1$:

$$\text{(2.2)} \quad \begin{cases} \frac{n^{k+1}_{\mathcal{T}}-n^k_{\mathcal{T}}}{\Delta t} - div^{\mathcal{T}}(\nabla^{\mathcal{D}} n^{k+1}_{\mathcal{T}}) + divc^{\mathcal{T}}(n^k_{\mathcal{T}}\nabla^{\mathcal{D}} S^{k+1}_{\mathcal{T}}) = 0, \\ -div^{\mathcal{T}}(\nabla^{\mathcal{D}} S^{k+1}_{\mathcal{T}}) + S^{k+1}_{\mathcal{T}} = \delta div^{\mathcal{T}}(\nabla^{\mathcal{D}} n^k_{\mathcal{T}}) + \mu n^k_{\mathcal{T}}, \\ \nabla^{\mathcal{D}} n^k_{\mathcal{T}}.\nu = \nabla^{\mathcal{D}} S^k_{\mathcal{T}}.\nu = 0, \text{ for all } \mathcal{D}\in\mathfrak{D}_{ext}. \end{cases}$$

Such that $div^{\mathcal{T}}$, $\nabla^{\mathcal{D}}$ and $divc^{\mathcal{T}}$ are defined respectively by definition 2.2, definition 2.1 and definition 2.3.

## 3. The main results

### 3.1. Existence of DDFV solutions.

**Theorem 3.1.** *Let $\Omega$ be an open, bounded, connected, polygonal domain of $\mathbb{R}^2$ and $\mathcal{T}$ be a discretization of $\Omega\times(0,T)$. Let $n^0\in L^2(\Omega), n^0\geq 0$ in $\Omega$. Then, there exists a solution $\{(n^k_{\mathcal{T}}, S^k_{\mathcal{T}}), 0\leq k\leq N_T-1\}$ to (2.2) satisfying:*

$$\begin{cases} n^k_{\mathcal{K}}\geq 0 \text{ and } n^k_{\mathcal{K}^*}\geq 0 \text{ for all } \mathcal{K}\in\mathfrak{M} \text{ and } \mathcal{K}^*\in\mathfrak{M}^*, \text{ for all } 0\leq k\leq N_T-1,\\ \frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} n^k_{\mathcal{K}}+\frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} n^k_{\mathcal{K}^*}=\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} n^0_{\mathcal{K}}+\frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} n^0_{\mathcal{K}^*}\\ =\|n^0\|_{L^1(\Omega)}, \text{ for all } 0\leq k\leq N_T-1. \end{cases}$$

*Proof.* Let $k\in\{0,1,2,3,...,N_T-1\}$ and let $(n^k_{\mathcal{T}}, S^k_{\mathcal{T}})$ be a solution to (2.2), we introduce the set:

$$X_{\mathcal{T}}=\{v\in\mathbb{R}^{\mathcal{T}}; v\geq 0 \text{ in } \Omega, \|v\|_{L^1(\Omega)}\leq\|n^0\|_{L^1(\Omega)}\}.$$

Let using the fixed point theorem by solving a linearized problem. First we fixed $n^k_{\mathcal{T}}\in X_{\mathcal{T}}$ and we construct $\overline{S}\in X_{\mathcal{T}}$ using the following schemes

$$\begin{cases} -\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma \nabla_{\mathcal{D}}\overline{S}_{\mathcal{T}}.\nu_{\sigma,\mathcal{K}}+m_{\mathcal{K}}\overline{S}_{\mathcal{K}}=\\ \quad\delta\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma\nabla_{\mathcal{D}} n^k_{\mathcal{T}}.\nu_{\sigma,\mathcal{K}}+\mu m_{\mathcal{K}} n^k_{\mathcal{K}}, \text{ for all } \mathcal{K}\in\mathfrak{M},\\ -\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*}\nabla_{\mathcal{D}}\overline{S}_{\mathcal{T}}.\nu_{\sigma^*,\mathcal{K}^*}+m_{\mathcal{K}^*}\overline{S}_{\mathcal{K}^*}=\\ \quad\delta\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*}\nabla_{\mathcal{D}} n^k_{\mathcal{T}}.\nu_{\sigma^*,\mathcal{K}^*}+\mu m_{\mathcal{K}^*} n^k_{\mathcal{K}^*} \text{ for all } \mathcal{K}^*\in\mathfrak{M}^*. \end{cases}\tag{3.1}$$

Next, we compute $\overline{n}\in X_{\mathcal{T}}$ using the schemes

$$\begin{cases} m_{\mathcal{K}}\frac{\overline{n}_{\mathcal{K}}-n^k_{\mathcal{K}}}{\Delta t}-\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma\nabla^{\mathcal{D}}\overline{n}_{\mathcal{T}}.\nu_{\sigma,\mathcal{K}}\\ \quad+\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma\left(\overline{n}_{\mathcal{K}}(\nabla^{\mathcal{D}}\overline{S}_{\mathcal{T}}.\nu_{\sigma,\mathcal{K}})^+-\overline{n}_{\mathcal{L}}(\nabla^{\mathcal{D}}\overline{S}_{\mathcal{T}}.\nu_{\sigma,\mathcal{K}})^-\right)=0, \text{ for all } \mathcal{K}\in\mathfrak{M},\\ m_{\mathcal{K}^*}\frac{\overline{n}_{\mathcal{K}^*}-n^k_{\mathcal{K}^*}}{\Delta t}-\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*}\nabla^{\mathcal{D}}\overline{n}_{\mathcal{T}}.\nu_{\sigma^*,\mathcal{K}^*}\\ \quad+\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*}\left(\overline{n}_{\mathcal{K}^*}(\nabla^{\mathcal{D}}\overline{S}_{\mathcal{T}}.\nu_{\sigma^*,\mathcal{K}^*})^+-\overline{n}_{\mathcal{L}^*}(\nabla^{\mathcal{D}}\overline{S}_{\mathcal{T}}.\nu_{\sigma^*,\mathcal{K}^*})^-\right)=0,\\ \quad\text{for all } \mathcal{K}^*\in\mathfrak{M}^*. \end{cases}\tag{3.2}$$

**Step 1:** The system (3.1) can be written as $A\overline{S} = b$, where for all $\mathcal{K}, \mathcal{L} \in \mathfrak{M}$ such that $\sigma = \mathcal{K}|\mathcal{L}$ and for all $\mathcal{K}^*, \mathcal{L}^* \in \mathfrak{M}^*$ such that $\sigma^* = \mathcal{K}^*|\mathcal{L}^*$, $A$ is defined by:

$$A_{\mathcal{K},\mathcal{K}} = \sum_{\sigma\in\mathcal{K}} \frac{m_\sigma^2}{2m_\mathcal{D}} + m_\mathcal{K}, \quad \text{and} \quad A_{\mathcal{K}^*,\mathcal{K}^*} = \sum_{\sigma^*\in\mathcal{K}^*} \frac{m_\sigma^{*2}}{2m_\mathcal{D}} + m_{\mathcal{K}^*}.$$

$$\begin{cases} A_{\mathcal{K},\mathcal{L}} = -\frac{m_\sigma^2}{2m_\mathcal{D}}, \\ A_{\mathcal{K},\mathcal{K}^*} = -\frac{m_\sigma m_{\sigma^*}}{2m_\mathcal{D}} \nu_{\mathcal{K},\sigma}.\nu_{\sigma^*,\mathcal{K}^*}, \\ A_{\mathcal{K},\mathcal{L}^*} = \frac{m_\sigma m_{\sigma^*}}{2m_\mathcal{D}} \nu_{\sigma,\mathcal{K}}.\nu_{\sigma^*,\mathcal{K}^*}, \end{cases} \quad \text{and} \quad \begin{cases} A_{\mathcal{K}^*,\mathcal{L}^*} = -\frac{m_{\sigma^*}^2}{2m_\mathcal{D}}, \\ A_{\mathcal{K}^*,\mathcal{K}} = -\frac{m_\sigma m_{\sigma^*}}{2m_\mathcal{D}} \nu_{\sigma,\mathcal{K}}.\nu_{\sigma^*,\mathcal{K}^*}, \\ A_{\mathcal{K}^*,\mathcal{L}} = \frac{m_\sigma m_{\sigma^*}}{2m_\mathcal{D}} \nu_{\sigma,\mathcal{K}}.\nu_{\sigma^*,\mathcal{K}^*}, \end{cases}$$

and $b_\mathcal{K} = \delta \sum_{\sigma\in\mathcal{E}_\mathcal{K}} m_\sigma \nabla^\mathcal{D} n_\mathcal{T}^k.\nu_{\sigma,\mathcal{K}} + \mu m_\mathcal{K} n_\mathcal{K}^k$, $b_{\mathcal{K}^*} = \delta \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*} \nabla^\mathcal{D} n_\mathcal{T}^k.\nu_{\sigma^*,\mathcal{K}^*} + \mu m_{\mathcal{K}^*} n_{\mathcal{K}^*}^k$.
Since for all $\mathcal{K} \in \mathfrak{M}$ and $\mathcal{K}^* \in \mathfrak{M}^*$:

$$\begin{cases} |A_{\mathcal{K},\mathcal{K}}| - \sum_{\sigma\in\mathcal{E}_\mathcal{K}} [|A_{\mathcal{K},\mathcal{L}}| + |A_{\mathcal{K},\mathcal{L}^*}| + |A_{\mathcal{K},\mathcal{K}^*}|] = m_\mathcal{K} - 2\sum_{\sigma\in\mathcal{E}_\mathcal{K}} \frac{m_\sigma m_{\sigma^*}}{2m_\mathcal{D}} |cos(\alpha_\mathcal{D})|, \\ |A_{\mathcal{K}^*,\mathcal{K}^*}| - \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} [|A_{\mathcal{K}^*,\mathcal{L}^*}| + |A_{\mathcal{K}^*,\mathcal{L}}| + |A_{\mathcal{K}^*,\mathcal{K}}|] = m_{\mathcal{K}^*} - 2\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} \frac{m_{\sigma^*} m_\sigma}{2m_\mathcal{D}} |cos(\alpha_\mathcal{D})|. \end{cases}$$

Using the hypothesis (2.1) we have

$$\begin{cases} |A_{\mathcal{K},\mathcal{K}}| - \sum_{\sigma\in\mathcal{E}_\mathcal{K}} [|A_{\mathcal{K},\mathcal{L}}| + |A_{\mathcal{K},\mathcal{L}^*}| + |A_{\mathcal{K},\mathcal{K}^*}|] \geq 0, \\ |A_{\mathcal{K}^*,\mathcal{K}^*}| - \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} [|A_{\mathcal{K}^*,\mathcal{L}^*}| + |A_{\mathcal{K}^*,\mathcal{L}}| + |A_{\mathcal{K}^*,\mathcal{K}}|] \geq 0. \end{cases}$$

Then the matrix $A$ is strictly diagonally dominant with respect to the columns and hence, $A$ is invertible. This shows the unique solvability of (3.1).

Now, the system (3.2) equivalent to the system $B\overline{n} = C$, with:

$$B_{\mathcal{K},\mathcal{K}} = \sum_{\sigma\in\mathcal{E}_\mathcal{K}} \frac{m_\sigma^2}{2m_\mathcal{D}} + \frac{m_\mathcal{K}}{\Delta t} + \sum_{\sigma\in\mathcal{E}_\mathcal{K}} m_\sigma (\nabla^\mathcal{D}\overline{S}.\nu_{\sigma,\mathcal{K}})^+$$

and

$$B_{\mathcal{K}^*,\mathcal{K}^*} = \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} \frac{m_{\sigma^*}^2}{2m_\mathcal{D}} + \frac{m_{\mathcal{K}^*}}{\Delta t} + \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*} (\nabla^\mathcal{D}\overline{S}.\nu_{\sigma^*,\mathcal{K}^*})^+.$$

$$\begin{cases} B_{\mathcal{K},\mathcal{L}} = -\frac{m_\sigma^2}{2m_\mathcal{D}} - m_\sigma(\nabla^\mathcal{D}\overline{S}.\nu_{\sigma,\mathcal{K}})^-, \\ B_{\mathcal{K},\mathcal{K}^*} = \frac{m_\sigma m_{\sigma^*}}{2m_\mathcal{D}} \nu_{\sigma,\mathcal{K}}.\nu_{\sigma^*,\mathcal{K}^*}, \\ B_{\mathcal{K},\mathcal{L}^*} = -\frac{m_\sigma m_{\sigma^*}}{2m_\mathcal{D}} \nu_{\sigma,\mathcal{K}}.\nu_{\sigma^*,\mathcal{K}^*}, \end{cases} \quad \text{and} \quad \begin{cases} B_{\mathcal{K}^*,\mathcal{L}^*} = -\frac{m_{\sigma^*}^2}{2m_\mathcal{D}} - m_{\sigma^*}(\nabla^\mathcal{D}\overline{S}.\nu_{\sigma^*,\mathcal{K}^*})^-, \\ B_{\mathcal{K}^*,\mathcal{K}} = \frac{m_\sigma m_{\sigma^*}}{2m_\mathcal{D}} \nu_{\sigma,\mathcal{K}}.\nu_{\sigma^*,\mathcal{K}^*}, \\ B_{\mathcal{K}^*,\mathcal{L}} = -\frac{m_\sigma m_{\sigma^*}}{2m_\mathcal{D}} \nu_{\sigma,\mathcal{K}}.\nu_{\sigma^*,\mathcal{K}^*}, \end{cases}$$

and $C_\mathcal{K} = \frac{m_\mathcal{K} n_\mathcal{K}^k}{\Delta t}$, $C_{\mathcal{K}^*} = \frac{m_{\mathcal{K}^*} n_{\mathcal{K}^*}^k}{\Delta t}$.

Since for all $\mathcal{K} \in \mathfrak{M}$ and $\mathcal{K}^* \in \mathfrak{M}^*$:

$$\begin{cases} |B_{\mathcal{K},\mathcal{K}}| - \sum_{\sigma\in\mathcal{E}_\mathcal{K}} [|B_{\mathcal{K},\mathcal{L}}| + |B_{\mathcal{K},\mathcal{L}^*}| + |B_{\mathcal{K},\mathcal{K}^*}|] = \\ \quad \frac{m_\mathcal{K}}{\Delta t} + \sum_{\sigma\in\mathcal{E}_\mathcal{K}} m_\sigma |(\nabla^\mathcal{D}\overline{S}.\nu_{\sigma,\mathcal{K}})^+| - \sum_{\sigma\in\mathcal{E}_\mathcal{K}} m_\sigma |(\nabla^\mathcal{D}\overline{S}.\nu_{\sigma,\mathcal{K}})^-| - 2\sum_{\sigma\in\mathcal{E}_\mathcal{K}} \frac{m_\sigma m_{\sigma^*}}{2m_\mathcal{D}} |cos(\alpha_\mathcal{D})|. \\ |B_{\mathcal{K}^*,\mathcal{K}^*}| - \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} [|B_{\mathcal{K}^*,\mathcal{L}^*}| + |B_{\mathcal{K}^*,\mathcal{L}}| + |B_{\mathcal{K}^*,\mathcal{K}}|] = \frac{m_{\mathcal{K}^*}}{\Delta t} + \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*} |(\nabla^\mathcal{D}\overline{S}.\nu_{\sigma^*,\mathcal{K}^*})^+| \\ \quad - \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*} |(\nabla^\mathcal{D}\overline{S}.\nu_{\sigma^*,\mathcal{K}^*})^-| - 2\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} \frac{m_{\sigma^*} m_{\sigma}}{2m_\mathcal{D}} |cos(\alpha_\mathcal{D})|. \end{cases}$$

For all $\sigma \in \mathcal{E}_\mathcal{K}$ and $\sigma^* \in \mathcal{E}_{\mathcal{K}^*}$ we have

$$\begin{cases} \nabla^\mathcal{D}\overline{S}.\nu_{\sigma,\mathcal{K}} = -\nabla^\mathcal{D}\overline{S}.\nu_{\sigma,\mathcal{L}}, \\ \nabla^\mathcal{D}\overline{S}.\nu_{\sigma^*,\mathcal{K}^*} = -\nabla^\mathcal{D}\overline{S}.\nu_{\sigma^*,\mathcal{L}^*}. \end{cases}$$

Which yields

$$\begin{cases} (\nabla^\mathcal{D}\overline{S}.\nu_{\sigma,\mathcal{K}})^- = (\nabla^\mathcal{D}\overline{S}.\nu_{\sigma,\mathcal{L}})^+, \\ (\nabla^\mathcal{D}\overline{S}.\nu_{\sigma^*,\mathcal{K}^*})^- = (\nabla^\mathcal{D}\overline{S}.\nu_{\sigma^*,\mathcal{L}^*})^+. \end{cases}$$

That's give

$$\begin{cases} |B_{\mathcal{K},\mathcal{K}}| - \sum_{\sigma\in\mathcal{E}_\mathcal{K}} [|B_{\mathcal{K},\mathcal{L}}| + |B_{\mathcal{K},\mathcal{L}^*}| + |B_{\mathcal{K},\mathcal{K}^*}|] = \frac{m_\mathcal{K}}{\Delta t} - 2\sum_{\sigma\in\mathcal{E}_\mathcal{K}} \frac{m_\sigma m_{\sigma^*}}{2m_\mathcal{D}} |cos(\alpha_\mathcal{D})|, \\ |B_{\mathcal{K}^*,\mathcal{K}^*}| - \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} [|B_{\mathcal{K}^*,\mathcal{L}^*}| + |B_{\mathcal{K}^*,\mathcal{L}}| + |B_{\mathcal{K}^*,\mathcal{K}}|] = \frac{m_{\mathcal{K}^*}}{\Delta t} - 2\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} \frac{m_{\sigma^*} m_{\sigma}}{2m_\mathcal{D}} |cos(\alpha_\mathcal{D})|. \end{cases}$$

Using the hypothesis (2.1), we have

$$\begin{cases} |B_{\mathcal{K},\mathcal{K}}| - \sum_{\sigma\in\mathcal{E}_\mathcal{K}} [|B_{\mathcal{K},\mathcal{L}}| + |B_{\mathcal{K},\mathcal{L}^*}| + |B_{\mathcal{K},\mathcal{K}^*}|] \geq 0, \\ |B_{\mathcal{K}^*,\mathcal{K}^*}| - \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} [|B_{\mathcal{K}^*,\mathcal{L}^*}| + |B_{\mathcal{K}^*,\mathcal{L}}| + |B_{\mathcal{K}^*,\mathcal{K}}|] \geq 0. \end{cases}$$

Then the matrix $B$ is strictly diagonally dominant with respect to the columns and hence, $B$ is invertible. This shows the unique solvability of (3.2). Then $\overline{n}$ is nonnegative, implies that $\overline{n}$ satisfies (3.1).

In (3.2), summing the first equation over $\mathcal{K} \in \mathfrak{M}$ and the second equation over $\mathcal{K}^* \in \mathfrak{M}^*$, we obtain

$$\begin{cases} \sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} \overline{n}_{\mathcal{K}} = \sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} n^k_{\mathcal{K}}, \\ \sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} \overline{n}_{\mathcal{K}^*} = \sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} n^k_{\mathcal{K}^*}. \end{cases}$$

That's give

$$\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} \overline{n}_{\mathcal{K}} + \frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} \overline{n}_{\mathcal{K}^*} = \frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} n^k_{\mathcal{K}} + \frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} n^k_{\mathcal{K}^*} = \|n^0\|_{L^1(\Omega)}.$$

**Step 2:** Let $\mathfrak{H} : X_{\mathcal{T}} \to X_{\mathcal{T}}$ the operator define by the solution to (3.1) and (3.2) such that $\mathfrak{H}(n) = \overline{n}$, it must be shown that the operator $\mathfrak{H}$ is continuous to apply Brouwer fixed point theorem (i.e) we have to prove that $\overline{n}^\beta \to \overline{n}$ as $\beta \to \infty$ such that:

$$\begin{cases} (n^\beta)_{\beta\in\mathbb{N}} \subset X_{\mathcal{T}} \text{ be a sequence verified } n^\beta \to n \text{ as } \beta \to \infty \text{ in } X_{\mathcal{T}}, \\ \mathfrak{H}(n^\beta) = \overline{n}^\beta, \\ \mathfrak{H}(n) = \overline{n}. \end{cases}$$

It easy to show that $\overline{S}^\beta - \overline{S} \to 0$ in $X_{\mathcal{T}}$ as $\beta \to \infty$, since the map $n \to \overline{S}$ is linear on the finite dimensional space $X_{\mathcal{T}}$ and continuous. Later, using (3.2) and an adaptation of the proof of theorem 2.1 in [3] leads to:

$$\begin{cases} \sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} |\overline{n}^\beta_{\mathcal{K}} - \overline{n}_{\mathcal{K}}| \le 2\Delta t \left(\sum_{\mathcal{K}\in\mathfrak{M}} |\overline{n}_{\mathcal{K}}|^2\right)^{1/2} \left(\sum_{\mathcal{K}\in\mathfrak{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma |\nabla^{\mathcal{D}}(\overline{S}^\beta - \overline{S}).\nu_{\sigma,\mathcal{K}}|^2\right)^{1/2}, \\ \sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} |\overline{n}^\beta_{\mathcal{K}^*} - \overline{n}_{\mathcal{K}^*}| \le 2\Delta t \left(\sum_{\mathcal{K}^*\in\mathfrak{M}^*} |\overline{n}_{\mathcal{K}^*}|^2\right)^{1/2} \\ \qquad \cdot \left(\sum_{\mathcal{K}^*\in\mathfrak{M}^*}\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*} |\nabla^{\mathcal{D}}(\overline{S}^\beta - \overline{S}).\nu_{\sigma^*,\mathcal{K}^*}|^2\right)^{1/2}. \end{cases}$$

Let $c_1 > 0$ such that $2\Delta t \sum_{\mathcal{K}\in\mathfrak{M}} |\overline{n}_{\mathcal{K}}|^2 \leq c_1 \|n^0_{\mathcal{T}}\|_{L^2(\Omega)}$ and $2\Delta t \sum_{\mathcal{K}^*\in\mathfrak{M}^*} |\overline{n}_{\mathcal{K}^*}|^2 \leq c_1 \|n^0_{\mathcal{T}}\|_{L^2(\Omega)}$, then

$$\begin{cases} \sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} |\overline{n}^{\beta}_{\mathcal{K}} - \overline{n}_{\mathcal{K}}| \leq c_1 \|n^0_{\mathcal{T}}\|_{L^2(\Omega)} \left( \sum_{\mathcal{K}\in\mathfrak{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma |\nabla^{\mathcal{D}}(\overline{S}^{\beta} - \overline{S}).\nu_{\sigma,\mathcal{K}}|^2 \right)^{1/2}, \\ \sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} |\overline{n}^{\beta}_{\mathcal{K}^*} - \overline{n}_{\mathcal{K}^*}| \leq c_1 \|n^0_{\mathcal{T}}\|_{L^2(\Omega)} \left( \sum_{\mathcal{K}^*\in\mathfrak{M}^*} \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*} |\nabla^{\mathcal{D}}(\overline{S}^{\beta} - \overline{S}).\nu_{\sigma^*,\mathcal{K}^*}|^2 \right)^{1/2}. \end{cases}$$

Then

$$\begin{aligned} &\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} |\overline{n}^{\beta}_{\mathcal{K}} - \overline{n}_{\mathcal{K}}|^2 + \frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} |\overline{n}^{\beta}_{\mathcal{K}^*} - \overline{n}_{\mathcal{K}^*}|^2 \\ &\leq c_1^2 \|n^0_{\mathcal{T}}\|^2_{L^2(\Omega)} (\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma |\nabla^{\mathcal{D}}(\overline{S}^{\beta} - \overline{S}).\nu_{\sigma,\mathcal{K}}|^2 \\ &\quad + \frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*} |\nabla^{\mathcal{D}}(\overline{S}^{\beta} - \overline{S}).\nu_{\sigma^*,\mathcal{K}^*}|^2), \end{aligned}$$

using the poincare inequality, we have

$$\begin{aligned} &\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} |\overline{n}^{\beta}_{\mathcal{K}} - \overline{n}_{\mathcal{K}}|^2 + \frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} |\overline{n}^{\beta}_{\mathcal{K}^*} - \overline{n}_{\mathcal{K}^*}|^2 \leq \\ &c_1^2 \|n^0_{\mathcal{T}}\|^2_{L^2(\Omega)} \|\nabla^{\mathcal{D}}(\overline{S}^{\beta} - \overline{S})\|^2_{2,\mathcal{D}} \leq c_1)^2 \|n^0_{\mathcal{T}}\|^2_{L^2(\Omega)} \|\overline{S}^{\beta} - \overline{S}\|^2_{2,\mathcal{T}}. \end{aligned}$$

That's give

$$\|\overline{n}^{\beta}_{\mathcal{K}} - \overline{n}_{\mathcal{K}}\|^2_{2,\mathcal{T}} \leq c_1^2 \|n^0_{\mathcal{T}}\|^2_{L^2(\Omega)} \|\nabla^{\mathcal{D}}(\overline{S}^{\beta} - \overline{S})\|^2_{2,\mathcal{D}} \leq c_1^2 \|n^0_{\mathcal{T}}\|^2_{L^2(\Omega)} \|\overline{S}^{\beta} - \overline{S}\|^2_{2,\mathcal{T}}.$$

Since $\|n^0_{\mathcal{T}}\|_{L^2(\Omega)}$ is bounded and $\overline{S}^{\beta} - \overline{S} \to 0$ as $\beta \to \infty$, then $\overline{n}^{\beta} \to \overline{n}$ in $H^{\mathcal{T}}$ implies that $\mathfrak{H}$ is a continuous operator. Therefore using the Brouwer fixed point theorem the operator $\mathfrak{H}$ has a fixed point, hence the prove of theorem. □

3.2. **Numerical experiments.** In this section, we illustrate the behavior of the discrete duality finite volume scheme by applying it to the system (1.2) which describes the evolution over time of the cell density $n(x,t)$ and the chemical signal concentration variable $S(x,t)$, Some of the tests cases come from the paper [2] where a finite volume scheme is used, and our results compare very well to the ones in this reference.

In the tests 1, 2 and 3 the spatial domain is $\Omega = (-0.5; 0.5)^2$, the mesh of this domain is made of $1296$ triangles and $665$ nodes. Therefore, we use the sequence of general triangular meshes introduced in Section 2.

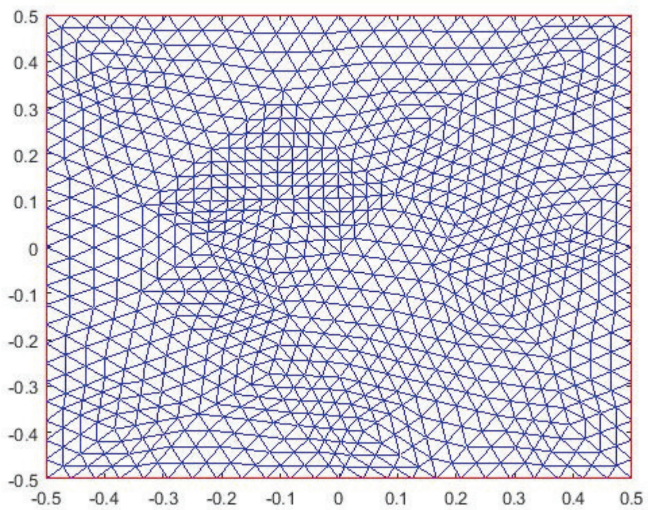



FIGURE 1. The mesh supported in the numerical tests with $h = 0.0471$

In all the tests we first solve the classical parabolic-elliptic Keller-Segel system, this is the case corresponds to $\delta = 0$, after we take $\delta = 10^{-2}$ and $\delta = 10^{-3}$, two initial nonsymmetry data are taken in the first two test and initial symmetry data in the third test.

3.2.1. *Test 1.* Firstly, we chose the nonsymmetric initial data on a square and we present the numerical solution of (1.2) for different values of $\delta$. in this subsection, $\mu = 1$, the time step is $\Delta t = 6.10^{-3}$, the number of triangles is $1296$ and the non-symmetric initial functions is given by: $n_{0,1}(x, y) = \frac{M}{2\pi\theta} exp\left(-\frac{(x-x_0)^2+(y-y_0)^2}{2\theta}\right)$,with the total mass is $M = 6\pi$, $\theta = 10^{-2}$ and $x_0 = y_0 = 0.1$.

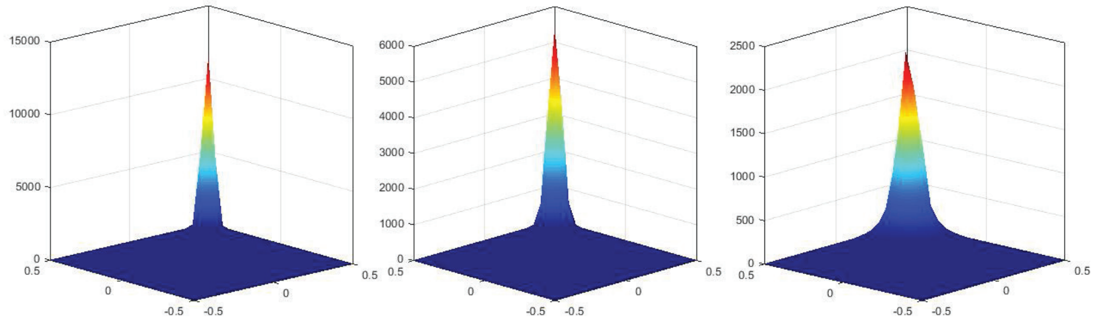
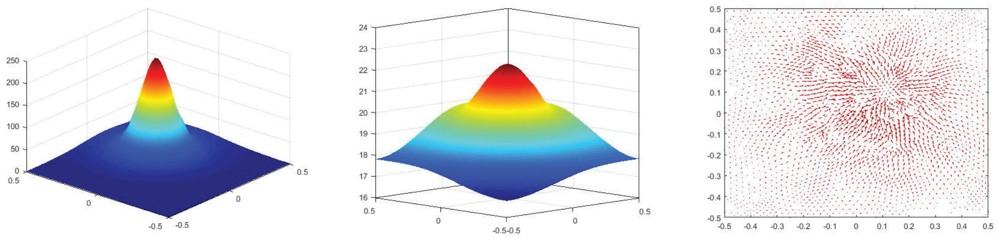
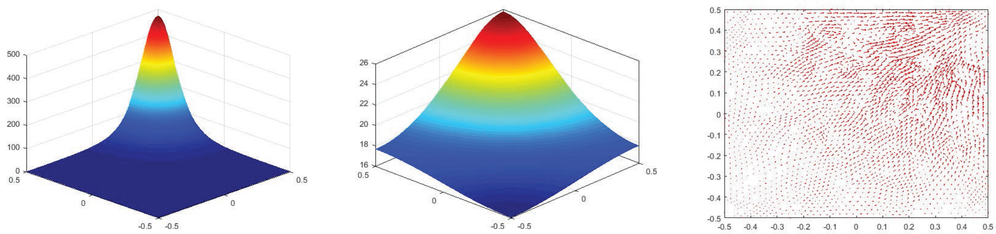


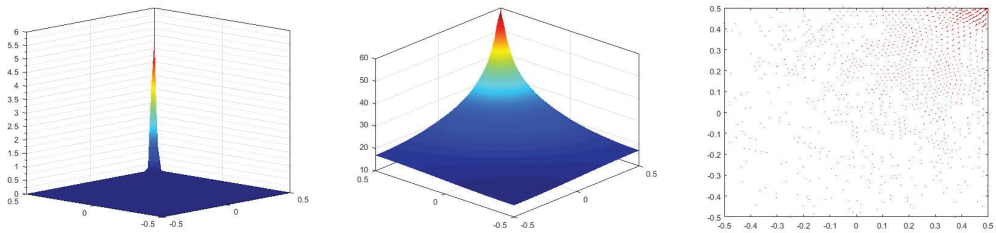

FIGURE 2. Cell density computed from symmetric initial data $n_{0,1}$ with $M = 6\pi$ and $\delta = 0$ (left), and $\delta = 10^{-3}$ (middle),and $\delta = 10^{-2}$ (right) at, $t = 6$.



(A) Cell density (left), chemical signal concentration (middle), chemical signal concentration gradient (right) at $t = 0.06$.



(B) Cell density (left), chemical signal concentration (middle), chemical signal concentration gradient (right) at, $t = 0.3$.



(C) Cell density (left) with $1 \times 10^5$, chemical signal concentration (middle), chemical signal concentration gradient (right) at, $t = 6$.

FIGURE 3. Cell density computed from nonsymmetric initial data $n_{0,1}$ with $M = 6\pi$ and $\delta = 0$ .

3.2.2. *Test 2.* Under the same conditions of test 1, and the nonsymmetric initial functions :

$$n_{0,2}(x,y) = \frac{4\pi}{2\pi\theta} exp\left(-\frac{(x-x_0)^2+(y-y_0)^2}{2\theta}\right) + \frac{2\pi}{2\pi\theta} exp\left(-\frac{(x-x_1)^2+(y-y_1)^2}{2\theta}\right),$$

with $\theta = 10^{-2}$, $x_0 = y_0 = 0.1$ and $x_1 = y_1 = -0.2$.

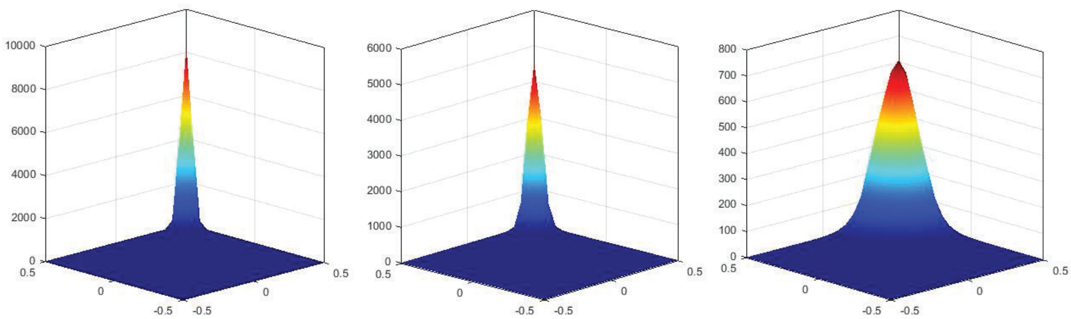
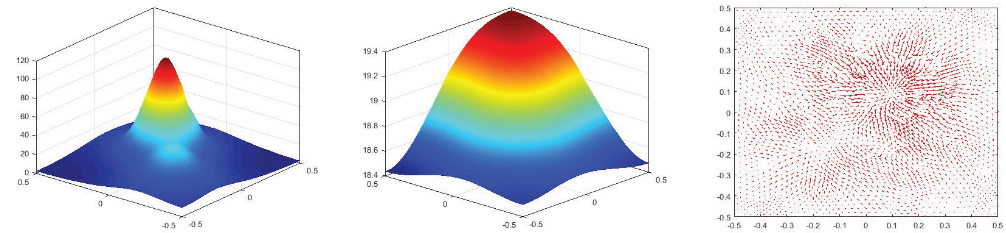
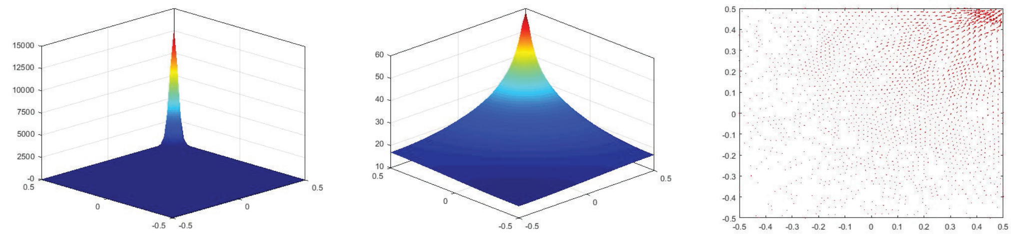


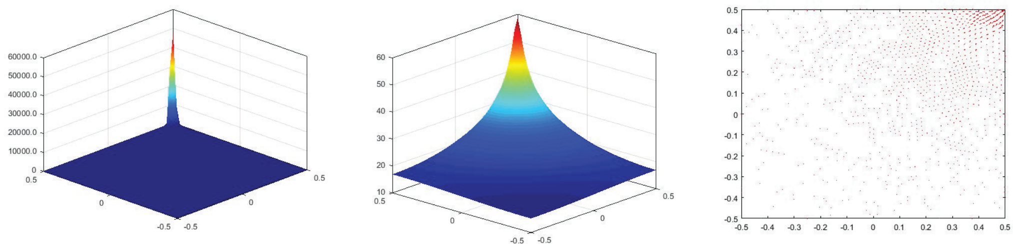

FIGURE 4. Cell density computed from nonsymmetric initial data $n_{0,2}$ with $M = 6\pi$ AND (left), (middle), (right) at, $t = 6$.



(A) Cell density (left), chemical signal concentration (middle), chemical signal concentration gradient (right) at $t = 0.3$.



(B) Cell density (left), chemical signal concentration (middle), chemical signal concentration gradient (right) at, $t = 1.2$.



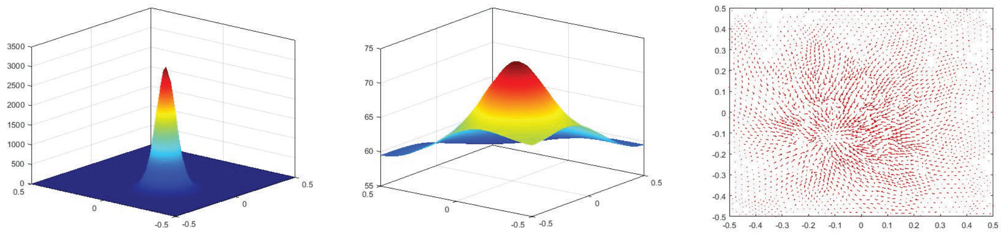
(C) Cell density (left), chemical signal concentration (middle), chemical signal concentration gradient (right) at, $t = 4.8$.

FIGURE 5. Cell density computed from nonsymmetric initial data $n_{0,2}$ with $M = 6\pi$ and $\delta = 0$ .

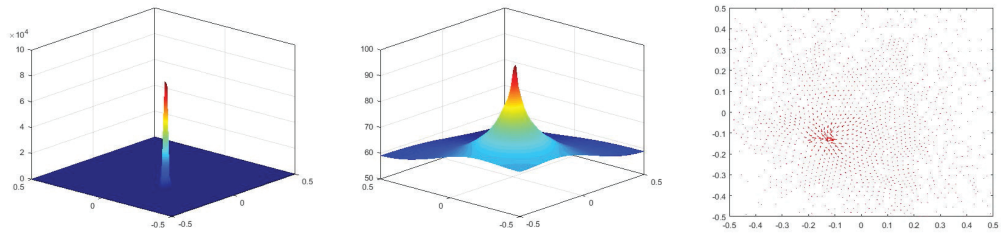
3.2.3. *Test 3.* Now, we present the numerical solution of (1.2) for different values of $\delta$. in this subsection, $\mu = 1$, $\Delta t = 6.10^{-3}$ and the symmetric initial functions is given by:

$$n_{0,3}(x,y) = \frac{M}{2\pi\theta} exp\left(-\frac{x^2+y^2}{2\theta}\right),$$

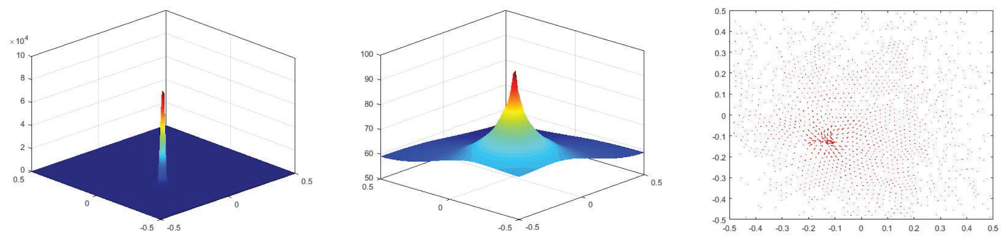
with the mass $M = 20\pi$ and $\theta = 10^{-2}$.



(A) Cell density (left), chemical signal concentration (middle), chemical signal concentration gradient (right) at $t = 0.3$.

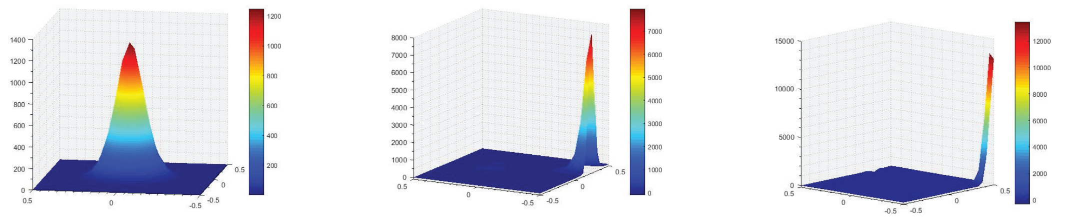


(B) Cell density (left), chemical signal concentration (middle), chemical signal concentration gradient (right) at, $t = 1.2$.



(C) Cell density (left), chemical signal concentration (middle), chemical signal concentration gradient (right) at, $t = 6$.

FIGURE 6. Cell density computed from symmetric initial data $n_{0,3}$ with $M = 20\pi$ and $\delta = 0$ .



(A) Initial datum $n_{0,3}$, $t = 0$ (B) Initial datum $n_{0,3}$, $t = 0.6$ (C) Initial datum $n_{0,3}$, $t = 6$

FIGURE 7. Cell density computed from symmetric initial data with $M = 20\pi$ and $\delta = 10^{-3}$ .

**Remark 3.1.** *We dispose a class of cell capable of producing a chemical signal concentration. The competition between cell and chemical signal concentration, non-linear terms, degradation, multiplication and displacement cause unexpected phenomena such as aggregation and blow-up.*

- *Indeed, our numerical results in the non-symmetric case (with $\delta = 0$) show that the blow-up occurs at the nearest corner of the point of inoculation (see figure 3 and 5) which is compatible with the cellular dynamics.*
- *We also remark that the numerical results obtained (figure 3 and 5 and 6) show that there exists an harmonization between the displacement of the cell density and the concentration.*
- *In the case of the initial datum is radially symmetric, the figure 6 shows that the blow-up in finite time of the classical Keller-Segel model without the additional diffusion term ($\delta = 0$) occurs in the center of the domain.*
- *Note that, in the case of the radially non symmetric initial datum and ($\delta = 0, \delta = 10^{-2}, \delta = 10^{-3}$), the cell density maximum decreases with increasing values of $\delta$ see figures 2 and 4.*
- *When we taking $\delta = 10^{-3}$ and the radially symmetric initial datum, we notice that the peak of cell density moves towards a corner, which is not the case when ($\delta = 0$), this due in the presence of the term of diffusion see figure 7.*

## 4. CONCLUSION

The role of the term of additional diffusion $\delta\Delta n$, in our problem, is that the solutions no longer blow up but have large gradients, which enables to determine the explosion time.

But the presence of this term results in major mathematical difficulties in the study of the convergence of the approximate solution because the resulting diffusion matrix is not positive definite symmetric therefore our scheme doesn't verify the principle of the maximum. And to overcome this difficulty, we plan to use the technique of logarithmic entropy in our perspectives.

## REFERENCES

[1] T. HILLEN, K. PAINTER: *A user's guide to PDE models for chemotaxis*, J. Math. Biol., **58** (2009), 183–217.

[2] M. BESSEMOULIN-CHATARD, A.JÜNGEL: *A finite volume scheme for a Keller-Segel model with additional cross-diffusion*, IMA Journal of Numerical Analysis, Oxford University Press (OUP), **34** (2014), 96–122.

[3] F. FILBET: *A finite volume scheme for the Patlak-Keller-Segel chemotaxis model*, Numer. Math. **104** (2006), 457–488.

[4] J. HAŜKOVEC, C. SCHMEISER: *Stochastic particle approximation for measure valued solutions of the 2D Keller-Segel system*, J. Stat. Phys., **135** (2009), 133–151.

[5] A. MARROCCO: *Numerical simulation of chemotactic bacteria aggregation via mixed finite elements*, Math. Mod. Numer. Anal., **4** (2003), 617–630.

[6] N. SAITO: *Conservative upwind finite-element method for a simplified Keller-Segel system modelling chemotaxis*, IMA J. Numer. Anal., **27** (2007), 332–365.

[7] N. SAITO: *Error analysis of a conservative finite-element approximation for the Keller-Segel model of chemotaxis*, Comm. Pure Appl. Anal., **11** (2012), 339–364.

[8] N. SAITO, T. SUZUKI: *Notes on finite difference schemes to a parabolic-elliptic system modelling chemotaxis*, Appl. Math. Comput., **171**(2005), 72–90.

[9] C. PATLAK: *Random walk with persistence and external bias*, Bull. Math. Biophys., **15** (1953), 311–338.

[10] E. KELLER, L. SEGEL: *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol., **26** (1970), 399–415.

[11] R. EYMARD, T. GALLOUET, R. HERBIN: *Finite volume methods, in: P. Ciarlet, J.-L. Lions (Eds.)*, Handbook of Numerical Analysis, vol. VII, North-Holland, 2000.

[12] F. HERMELINE: *A finite volume method for the approximation of diffusion operators on distorted meshes*, J. Comput. Phys., **160**(2) (2000), 481–499.

[13] A. DOMELEVO, P. OMNES: *A finite volume method for the Laplace equation on almost arbitrary two-dimensional grids*, M2AN Math. Model. Numer. Anal. **39**(6) (2005), 1203–1249.

[14] B. ANDREIANOV, F. BOYER, F. HUBERT: *Discrete duality finite volume schemes for Leray-Lions type elliptic problems on general 2D-meshes,* Num. Meth. for PDEs, **23** (2007), 145–195.

[15] Y. COUDIEÈRE, F. HUBERT: *A 3D discrete duality finite volume method for nonlinear elliptic equations*, SIAM J. Sci. Comput., **33** (2011), 1739–1764.

[16] Y. COUDIÈRE, G. MANZINI: *The discrete duality finite volume method for convection-diffusion problems*, SIAM J. Numer. Anal., **47** (2010), 4163–4192.

[17] F. BOYER, F. HUBERT: *Finite volume method for 2D linear and nonlinear elliptic problems with discontinuities*, SIAM J. Numer. Anal., **46** (2008), 3032–3070.

[18] S. DELCOURTE, K. DOMELEVO, P. OMNES: *A discrete duality finite volume approach to Hodge decomposition and div-curl problems on almost arbitrary two-dimensional meshes*, SIAM J. Numer. Anal., **45** (2007), 1142–1174.

[19] C. CHAINAIS-HILLAIRET, S. KRELL, A. MOUTON: *Study of discrete duality finite volume schemes for the peaceman model*, SIAM J. Sci. Comput., **35**(6), (2013), 2928–2952.

[20] C. CHAINAIS-HILLAIRET, S. KRELL, A. MOUTON: *Convergence analysis of a DDFV scheme for a system describing miscible fluid flows in porous media*, Numerical Methods for Partial Differential Equations, Wiley, **315**(3) (2015), 1098–2426.

[21] M. MANDARI, M. RHOUDAF, O. SOUALHI: *Numerical resolution of a degenerate elliptic-parabolic seawater intrusion problem using discrete duality finite volume method,* Journal of Mathematical Control Science and Applications, **6**(1) (2020), 25–46.

[22] Y. COUDIERE, J. P. VILA, P. VILLEDIEU: *Convergence rate of a finite volume scheme for a two-dimensional convection-diffusion problem*, M2AN Math. Model. Numer. Anal., **33** (1999), 493–516.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES
UNIVERSITY OF MOULAY ISMAIL
MEKNES, MOROCCO
*Email address*: ouafasoualhi@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES
UNIVERSITY OF MOULAY ISMAIL
MEKNES, MOROCCO
*Email address*: mohamedrhoudaf@gmail.com