Advances in Mathematics: Scientific Journal **9** (2020), no.12, 11063–11073
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.12.87

WEAK ROMAN DOMINATION EXCELLENT GRAPHS

P. ROUSHINI LEELY PUSHPAM[1] AND N. SRILAKSHMI

ABSTRACT. A *Roman dominating function* (RDF) on a graph G is a function f : $V(G) \to \{0,1,2\}$ such that every vertex with label 0 has a neighbor with label 2. A vertex u with $f(u) = 0$ is said to be *undefended* if it is not adjacent to a vertex with $f(v) > 0$. The function $f : V(G) \to \{0,1,2\}$ is a *weak Roman dominating function*(WRDF) if each vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) > 0$ such that the function $f' : V(G) \to \{0,1,2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u,v\}$, has no undefended vertex. A graph G is said to be γ_r-excellent, if for each vertex $x \in V$ there is a γ_r-function f on G with $f(x) \neq 0$. In this paper, we initiate a study of γ_r-excellent graphs.

1. INTRODUCTION

A subset S of vertices of G is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. Cockayne *et al.* [1] defined a *Roman dominating function* (RDF) in a graph G to be a function f: $V(G) \to \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $w(f) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function of a graph G is called the *Roman domination number* of G and denoted by $\gamma_R(G)$. Roman domination in graphs has been studied in [6–8, 15].

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 05C69, 05C76.
Key words and phrases. Weak Roman dominating function, weak Roman domination number, excellent graphs.

Henning *et al.* [5] defined a *weak Roman dominating function* as follows: For a graph G, let f: $V(G) \rightarrow \{0,1,2\}$ be a function. A vertex u with $f(u) = 0$ is said to be *undefended* with respect to f if it is not adjacent to a vertex v with the positive weight. A function f: $V(G) \rightarrow \{0,1,2\}$ is said to be a *weak Roman dominationg function* (WRDF) if each vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) > 0$ such that the function f': $V(G) \rightarrow \{0,1,2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u,v\}$, has no undefended vertex. We say that v *defends* u. The weight $w(f)$ of f is defined to be $\sum_{u \in V} f(u)$. The minimum weight of a weak Roman dominating function of a graph G is called the *weak Roman domination number* of G and denoted by $\gamma_r(G)$. A WRDF with weight $\gamma_r(G)$ is called a $\gamma_r(G)$-function. This concept of weak Roman domination as suggested by Henning et al. [5] is an attractive alternative for Roman domination as it further reduces the weight of the Roman dominating function. Weak Roman domination in graphs has been studied in [9–14]. A weak Roman dominating function f can also be written as $f = (V_0, V_1, V_2)$ where $V_i = \{v/f(v) = i\}$, $i = 0, 1, 2$. Notice that in a WRDF, every vertex in V_0 is dominated by a vertex in $V_1 \cup V_2$, while in an RDF every vertex in V_0 is dominated by at least one vertex in V_2. Furthermore, in a WRDF every vertex in V_0 can be defended without creating an undefended vertex. For a vertex v with $f(v) > 0$, we define the dependent set of v with respect to f, denoted by $DG(v)$ to be the set of all vertices in $N(v)$ which are defended by v alone.

G. Fricke *et al.* [2] in 2002 began the study of graphs which are excellent with respect to various graph parameters. A graph G is γ-excellent if each of its vertex belongs to some γ-set of G. This concept was extended by Vladimir Samodivikin [16] to Roman domination. He defined a graph to be γ_R-excellent if for each vertex $x \in V$ there is a γ_R-function h_x on G with $h_x(x) \neq 0$. Motivated by this concept, we further extend this concept to weak Roman domination as follows. We call a graph to be γ_r-excellent if for each vertex $x \in V$ there is a γ_r-function f on G with $f(x) \neq 0$. In this paper, we initiate a study of γ_r-excellent graphs.

2. NOTATION

For notation and graph theoretic terminology, we in general follow [3,4]. Throughout this paper, we consider only simple and connected graphs. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by n. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex *v* in a graph *G* is the number of edges that are incident to the vertex *v* and is denoted by *deg*(*v*). The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$. A vertex of degree zero is called an *isolated* vertex, while a vertex of degree one is called a *leaf* vertex or a *pendant* vertex of *G*. An edge incident to a leaf is called a *pendant edge*. A set S of vertices is called *independent* if no two vertices in S are adjacent. A simple graph in which every pair of distinct vertices are adjacent is called a complete graph. A clique of a simple graph G is a subset S of V such that $G[S]$ is complete. The *vertex clique cover number* θ_0 is the smallest number of complete subgraphs of G whose union includes all the vertices of G. For two positive integers r, s, the *complete bipartite* graph $K_{r,s}$ is the graph with partition $V(G) = V_1 \cup V_2$ such that $|V_1| = r$, $|V_2| = s$ and such that $G[V_i]$ has no edges for $i = 1, 2$, and every two vertices belonging to different partition sets are adjacent to each other. The *corona* of two graphs G_1 and G_2, is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the *ith* vertex of G_1 is adjacent to every vertex in the *ith* copy of G_2.

3. Properties of γ_r-excellent graphs

In this section, we investigate graphs that are γ_r-excellent. We observe that all vertex transitive graphs are γ_r-excellent. In all the discussions that follow, we assume that $\mathcal{F}$ to be the set of all γ_r-functions $f = (V_0, V_1, V_2)$ of a graph G.

Theorem 3.1. *For a graph G if there exists a γ_r-function $f = (V_0, V_1, V_2)$ such that the following holds.*

i) $V_2 = \emptyset$ *and for every* $x \in V_1$, $D_G(x) \cup \{x\}$ *induces a clique.*
ii) $\bigcup_{x \in V(G)} (D_G(x) \cup \{x\}) = V(G)$,

then G is γ_r-excellent.

Proof. Suppose that there is a $f \in \mathcal{F}$ such that the conditions hold. Let $x \in V_1$, then $D_G(x) \cup \{x\}$ induces a clique. Hence, we can find a γ_r-function in $\mathcal{F}$ which will assign positive weights namely, 1 to every member of $D_G(x) \cup \{x\}$. By condition(ii), every $x \in V(G)$ will receive a positive weight by some $f \in \mathcal{F}$. Hence G is γ_r-excellent. $\square$

Enqiang Zhu and Zehui Shao [17] has proved that for any connected graph G, $\gamma_r(G) \leq \frac{2n}{3}$. In view of this, we prove the following theorem.

Theorem 3.2. *For any graph G, $\gamma_r(G) = \frac{2n}{3}$ if and only if $V(G)$ can be partitioned into sets $V_1, V_2, \dots, V_k$ such that each $V_i, 1 \leq i \leq k$ induces a P_3 and any two P_3's are joined only at their central vertices.*

Proof. Suppose that $\gamma_r(G) = \frac{2n}{3}$. Then for every set of three vertices any γ_r-function of G will assign a total weight of 2. Let a, b, c be the three such vertices. Then, a, b, c cannot form a clique. For, otherwise a, b, c will receive a total weight 1. Hence, a, b, c induces a P_3. Hence $V(G)$ can be partitioned into sets $V_1, V_2, \dots, V_k$ such that each V_i induces a P_3. Suppose that an end vertex of a P_3 and an end vertex of another P_3 are adjacent, then these two P_3's will form a P_6, which will have a total weight of 3 assigned by any γ_r-function of G which implies that $\gamma_r(G) < \frac{2n}{3}$, a contradiction. Similarly, if an end vertex of one P_3 and a central vertex of another P_3 are adjacent, any γ_r-function of G will assign a total weight of 3, a contradiction. Hence, any two P_3's are connected only at their central vertices.

Converse is straight forward. □

Theorem 3.3. *For any graph G, $\gamma_r(G) = \frac{2n}{3}$ if and only if $G = H \circ 2K_1$ where H is a connected graph.*

Proof. By Theorem 3.2, $V(G)$ can be partitioned into sets $V_1, V_2, \dots, V_k$ such that each V_i induces a P_3 and any two P_3's are joined only at their central vertices. Hence, the end vertices of all the P_3's are all leaf vertices in G. Hence $G = H \circ 2K_1$. Conversely, if $G = H \circ 2K_1$, clearly $\gamma_r(G) = \frac{2n}{3}$. □

Corollary 3.1. *If for a graph G, $\gamma_r(G) = \frac{2n}{3}$, then G is γ_r-excellent and $\gamma(G) = \frac{n}{3}$.*

Theorem 3.4. *For any graph G, $\gamma_r(G) \leq \theta_0(G)$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_r-function. For every $v \in V_2$ at least two cliques are accounted for. For every $v \in V_1$ at least one clique is accounted for. Therefore $\theta_0(G) \geq 2|V_2| + |V_1|$. Hence, $\theta_0(G) \geq \gamma_r(G)$. □

Theorem 3.5. *For a graph G, if $\gamma_r(G) = \theta_0(G)$, then G is γ_r-excellent.*

Proof. Let f be a $\gamma_r(G)$-function. Suppose that $\gamma_r(G) = \theta_0(G)$. Then equality holds in the proof of Theorem 3.4, only if corresponding to each member of V_2 exactly two cliques are accounted for and corresponding to each member of V_1, one clique is taken into account. Hence, there is a γ_r-function f in G such that $V_2 = \emptyset$ and $D_G(v) \cup \{x\}$ induces a clique for every $v \in V_1$. Further $\bigcup_{x \in V(G)}(D_G(x) \cup \{x\}) = V(G)$. Hence by Theorem 3.1, G is γ_r-excellent. □

Theorem 3.6. *For a non complete graph G, $\gamma_r(G) = 2$ if and only if the following holds.*

i) $\Delta(G) = n - 1$.
ii) There are two vertices x and y in G such that $deg(x) = n - 2$ and $N(x) \setminus N(y)$ induces a clique.
iii) $V(G)$ can be partitioned into two sets such that each induces a clique.

Proof. Suppose that $\gamma_r(G) = 2$. If $\Delta(G) = n - 1$, we are through. Suppose that $\Delta(G) = n - 2$. Let $deg(x) = n - 2$ and y be a vertex non adjacent to x. Now, since $\gamma_r(G) = 2$, some $f \in \mathcal{F}$ will assign 1 to x and 1 to y. Now, $N(y) \subseteq N(x)$ and hence y will defend all its neighbors and x has to defend each member in $N(x) \setminus N(y)$. Thus, $N(x) \setminus N(y)$ induces a clique. Suppose that $\Delta(G) \leq n - 3$. Then any $f \in \mathcal{F}$ will assign 1 respectively to two vertices say x and y. Hence, x and y have to defend each member in $N(x)$ and $N(y)$ respectively and $N[x] \cup N[y] = V(G)$. Thus, $N(x)$ and $N(y)$ separately induces a clique. Hence, condition (iii) holds.

Conversely, suppose that one of the conditions hold. Then if $\Delta(G) = n - 1$, define $f : V(G) \to \{0, 1, 2\}$ by

$$f(v) = \begin{cases} 2, & \text{if } deg(v) = n - 1, \\ 0, & 0 \text{ otherwise.} \end{cases}$$

If condition (ii) holds, then define $f : V(G) \to \{0, 1, 2\}$ by $f(x) = f(y) = 1$ and $f(v) = 0$ for every $v \in V(G) \setminus \{x, y\}$.

If condition(iii) holds, then assign a weight 1 to each of the cliques. In all the cases, we have $\gamma_r(G) = 2$. □

In the following theorem, we characterize 2-γ_r-excellent graphs.

Theorem 3.7. *A graph G is 2-γ_r-excellent if and only if $V(G)$ can be partitioned into two sets each of which induces a clique.*

Proof. Suppose that G is 2-γ_r-excellent. If $\Delta(G) = n - 1$, let $deg(x) = n - 1$. If $D_G(x)$ induces more than two cliques, then no $f \in \mathcal{F}$ will assign a positive weight to a member of $D_G(x)$, a contradiction. Hence, $D_G(x)$ induces two cliques. If

$\Delta(G) = n-2$, let $deg(x) = n-2$ and y be not adjacent to x. By Theorem 3.6, $N(x) \setminus N(y)$ induces a clique. Let $f \in \mathcal{F}$ be such that $f(x) = 1$ and $f(y) = 1$. Then clearly y defends each member of $N(y)$. If $N(x) \setminus N(y) \neq \emptyset$, then $N(y) \setminus N(x)$ induces a clique. For, otherwise no $f \in \mathcal{F}$ will assign a positive weight to a member of $N(y) \setminus N(x)$, a contradiction. Hence, $N(y) \setminus N(x)$ induces a clique. If $N(x) \setminus N(y) = \emptyset$, then both x and y are adjacent to every vertex in $V(G) \setminus \{x, y\}$. In this case $|V(G)| = 4$. For otherwise, no $f \in \mathcal{F}$ will assign a positive weight to a vertex in $V(G) \setminus \{x, y\}$. In both the cases $V(G)$ is partitioned into two sets each of which induces a clique. If $\Delta(G) \leq n-3$, clearly the condition implies the requirement.

Converse is straightforward. □

4. SOME STANDARD GRAPHS

In this section, we investigate paths, cycles and complete bipartite graphs that are γ_r-excellent.

Theorem 4.1. *Paths P_n are γ_r-excellent if and only if $n \equiv 1, 3, 5\,(mod\ 7)$.*

Proof. Suppose that P_n is γ_r-excellent. Let $P_n = (v_1, v_2, \ldots, v_n)$. Suppose that $n \equiv 2(\bmod 7)$. Here $\gamma_r(P_n) = \frac{3n+1}{7}$. Then, we claim that no function $f \in \mathcal{F}$ will assign a positive weight to v_5. Suppose that there is a $f \in \mathcal{F}$ such that $f(v_5) > 0$. Hence, $f(v_5) = 1$. Now, f has to assign a total weight of 2 to the vertices v_1, v_2, v_3, v_4 and the remaining vertices need at least a total weight of $\lceil \frac{3(n-4)}{7} \rceil = \frac{3(n-6)}{7}$. Hence, the total weight of the vertices P_n is $\frac{3n+8}{7}$, which is a contradiction to the fact that $\gamma_r(P_n) = \frac{3n+1}{7}$. If $n \equiv 4(\bmod 7)$, no function in $\mathcal{F}$ will assign a positive weight to v_5. Suppose for some $f \in \mathcal{F}$, $f(v_5) = 1$, then $\sum_{i=1}^{4} f(v_i) = 2$ and for the path, $Q = (v_5, v_6, \ldots, v_n)$ which is of order $0(\bmod 7)$, the assignment is unique. Hence, as before we get a contradiction. When $n \equiv 6(\bmod 7)$, no function f will assign a positive weight to v_7. Suppose that for some $f \in \mathcal{F}$, $f(v_7) = 1$, $\sum_{i=1}^{6} f(v_i) = 3$ and for the path, $Q = (v_7, v_8, \ldots, v_n)$ which is of order $0(\bmod 7)$, the assignment is unique. Hence, as before we get a contradiction. If $n \equiv 0(\bmod 7)$, then P_n has a unique γ_r-function and the vertex v_3 is not assigned a positive weight, a contradiction.

Conversely, suppose that $n \equiv 1, 3, 5 (\mathrm{mod}\, 7)$. We give below three functions, $f_i \in \mathcal{F}, i = 1, 2, 3$ where each v_i in P_n is assigned a positive weight. When $n \equiv 1, 3, 5 (\mathrm{mod}\, 7)$, define $f_i : V(G) \to \{0, 1, 2\}, i = 1, 2, 3$ by

$$f_1(v_i) = \begin{cases} 1, & \text{if } i = n \ and \ i \equiv 1, 4, 6 (\mathrm{mod}\, 7), \\ 0, & 0 \text{ otherwise}, \end{cases}$$

$$f_2(v_i) = \begin{cases} 1, & \text{if } i = 1 \ and \ i \equiv 0, 3, 5 (\mathrm{mod}\, 7), \\ 0, & 0 \text{ otherwise}, \end{cases}$$

and

$$f_3(v_i) = \begin{cases} 1, & \text{if } i \equiv 2, 4, 6 (\mathrm{mod}\, 7) \ and \ i = n, \\ 0, & 0 \text{ otherwise}. \end{cases}$$

We see that each v_i is assigned a positive weight by some $f_i, i = 1, 2, 3$ in P_n. □

Theorem 4.2. *Cycles C_n are γ_r-excellent.*

Proof. Cycles C_n are vertex-transitive graphs and hence they are γ_r-excellent. □

Next, we investigate complete bipartite graphs.

Theorem 4.3. *Let $G = K_{r,s}$, $r \leq s$ be a complete bipartite graph. Then G is γ_r-excellent if and only if G is neither $K_{1,s}$ nor $K_{2,s}$, $s \geq 3$.*

Proof. Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be the partite sets of G with $|X| = r, |Y| = s$. Let $G \neq K_{1,s}, K_{2,s}, s \geq 3$. If $G = P_2, P_3$ or C_4, then clearly, G is γ_r-excellent. Suppose that $r + s \geq 4$. If $G = K_{3,3}$, then $\gamma_r(G) = 3$. Let $f_i : V(G) \to \{0, 1, 2\}, i = 1, 2$ be such that

$$f_1(v) = \begin{cases} 1, & \text{if } v \in X, \\ 0, & 0 \text{ if } v \in Y, \end{cases}$$

and

$$f_2(v) = \begin{cases} 1, & \text{if } v \in Y, \\ 0, & 0 \text{ if } v \in X. \end{cases}$$

Then, $f_1, f_2 \in \mathcal{F}$ and we see that G is γ_r-excellent.

If $G = K_{3,s}, s \geq 4$, then $\gamma_r(G) = 3$. Define $f : V(G) \to \{0, 1, 2\}$ by

$$f(v) = \begin{cases} 1, & \text{if } v \in X, \\ 0, & 0 \text{ if } v \in Y. \end{cases}$$

Then f is a γ_r-function of G. Further, define $f_i : V(G) \to \{0, 1, 2\}, 1 \leq i \leq |Y|$ such that

$$f_i(v) = \begin{cases} 1, & \text{if } v \in \{x_1, x_2, y_i\}, \\ 0, & 0 \text{ otherwise.} \end{cases}$$

Clearly, y_i defends x_3, $1 \leq i \leq |Y|$ and x_2 defends each vertex in $Y - \{y_i\}$. Hence, f_i is a γ_r function of G. Thus, G is γ_r-excellent. For all other cases, we see that $\gamma_r(G) = 4$ and for each $x \in X$, $y \in Y$, there is a $f \in \mathcal{F}$ such that $f(x) = 2$ and $f(y) = 2$. Thus, G is γ_r-excellent.

Conversely, suppose that $G = K_{1,s}$ or $K_{2,s}$, $s \geq 3$, then $\gamma_r(G) = 2$ and clearly, G is not γ_r-excellent. □

5. Split Graphs

A split graph is a graph G whose vertices can be partitioned into two sets X and Y where Y is an independent set and X induces a clique. Further, the subgraph induced by the edges between X and Y shall be denoted by $G[X, Y]$. A path is called a *maximal path* if no vertex can be added to it to make it longer.

In this section, we characterize split graphs that are γ_r-excellent.

Theorem 5.1. *Let G be a split graph with $|X| = r$, $|Y| = s$. Then, G is γ_r-excellent if and only if the following holds.*

i) $deg(x) \leq r + 1$ for every $x \in X$.
ii) If $deg(x) = r + 1$ for some $x \in X$, then $deg(v) \geq r + 1$ for every $v \in X \setminus \{x\}$.
iii) A maximal path in $G[X, Y]$ is of order at most 7. If a maximal path is of order 7, then both ends of the path are in X.

Proof. Let G be a γ_r-excellent graph. Suppose that there is a vertex x in X such that $deg(x) \geq r + 2$. Let x_1, x_2, x_3 be the neighbors of x in Y. If $deg(x_i) = 1$ for $i = 1, 2, 3$, then clearly $f(x) = 2$ for every $f \in \mathcal{F}$. Then no $f \in \mathcal{F}$ will assign a positive weight to the vertices x_1, x_2, x_3. Hence, G is not γ_r-excellent, a contradiction. Suppose that $deg(x_i) = 1$, for $i = 1, 2$ and $deg(x_3) > 1$. Since G is

γ_r-excellent, there is a γ_r-function, say f which assigns 1 to x_3. But, f will assign a total weight of 2 to the vertices x, x_1, x_2. Now, the vertices x, x_1, x_2, x_3 can be reassigned with a total weight of 2, which is a contradiction to the minimum of weight of $f(V)$. Suppose that $deg(x_1) = 1, deg(x_2) > 1$ and $deg(x_3) > 1$. Since G is γ_r-excellent there is a $f \in \mathcal{F}$ such that $f(x_2) = 1$. Without loss of generality, let $f(x_1) = 0$ and $f(x) = 1$. Now, $f(x_3) = 0$. For, otherwise, as earlier, we get a contradiction to the minimality of $f(V)$. Now, there is a vertex $z \in N(x_3)$ such that $f(z) = 1$ and z either protects or defends another vertex $x_4 \neq x_3$. Hence, no γ_r-function in $\mathcal{F}$ will assign a positive weight to x_3. Hence, G is not γ_r-excellent, a contradiction. Similarly, if $deg(x_i) > 1$, $1 \leq i \leq 3$, then there is a $f \in \mathcal{F}$ such that $f(x)+f(x_1)+f(x_2) = 2$ and $f(x_3) = 0$. As discussed earlier, we get a contradiction. To prove condition (ii), suppose that $deg(x) = r$ for some $x \in X$. We claim that $deg(v) \geq r$ for every $v \in X \setminus \{x\}$. Suppose to the contrary that $deg(v) = r - 1$ for some $v \in X \setminus \{x\}$, then clearly no $f \in \mathcal{F}$ will assign a positive weight to v. Hence G is not γ_r-excellent, a contradiction.

To prove (iii) suppose that $G[X, Y]$ contains a maximal path $P_k, k \geq 8$. If k is even, then the path P_k will have one of its ends in Y and of degree 1 and the other end in X of degree r. If k is odd, then the path P_k will have both of its ends either in Y and of degree 1 or in X of degree r. If k is even, then any $f \in \mathcal{F}$ will assign a total weight of $\frac{k}{2} - 1$ to the vertices of the path. If k is odd, any $f \in \mathcal{F}$ will assign a total weight of $\frac{k-3}{2}$ or $\frac{k-2}{2}$ to the vertices of the path according as the two end vertices of the path in P_k is in X or in Y respectively. Further 1 is assigned to each vertex of the path in X and such an assignment is unique. Hence all the vertices of the path in Y will be assigned zero by every $f \in \mathcal{F}$. Hence G is not γ_r-excellent, a contradiction. If $G[X, Y]$ contains a P_7, then both ends of P_7 are either in X or in Y. Then any $f \in \mathcal{F}$ will assign a weight 3 to the vertices of P_7 and the vertices of P_7 in Y will not receive a positive weight by any $f \in \mathcal{F}$. Hence, G is not γ_r-excellent, a contradiction. Hence, $G[X, Y]$ does not contain a $P_k, k \geq 8$ and a P_7 with both ends in Y. Thus, condition (iii) is proved.

Conversely suppose that the the conditions hold. Suppose that $deg(x) \leq r$ for every $x \in X$. Then, every $y \in Y$ along with its neighbors induce a clique in G. Further all the vertices of degree $r - 1$ induce a clique. Hence any $f \in \mathcal{F}$ will assign a weight 1 to each clique. Thus, $\gamma_r(G)$ in this case will be either $|Y| + 1$ or $|Y|$ according as there is a vertex in X of degree $r - 1$ or not. Hence there is a $f \in \mathcal{F}$ such that $V_2 = \emptyset$ and for each $x \in V_1$, $D_G(x)$ induces a clique and $\bigcup_{x \in V(G)}(D_G(x) \cup \{x\}) = V(G)$. Hence by Theorem 3.1, G is γ_r-excellent.

Suppose that $deg(x) = r+1$ for some $x \in X$. Then by the given condition $deg(v) \geq r$ for every $v \in V \setminus \{x\}$. Let $Y_1 = \{y \in Y$: each member in $N(y)$ is of degree $r\}$. Now, for each $y \in Y$, $N[Y]$ induces a clique. Now, suppose that $deg(x) = r+1$ for some $x \in X$. Then $deg(v) = r$ or $r+1$ for every $v \in V \setminus \{x\}$. Now, consider a maximal path P_k in $G[X, Y]$. By condition (iii), $k \leq 7$. If k is even, then the vertices of P_k will be assigned a total weight of 2 or 3 by any $f \in \mathcal{F}$, according as $k = 4$ or 6. If k is odd and the end vertices of P_k are in X, then the vertices of P_k will be assigned a total weight of 1 or 2 or 3 by any $f \in \mathcal{F}$ according as $k = 3$ or 5 or 7. If k is odd and the end vertices of P_k are in Y, then the vertices of P_k will be assigned a total weight of 2 or 3 by any $f \in \mathcal{F}$, according as $k = 3$ or 5. Hence 2 or 3 cliques are taken into account as $k = 4$ or 6. If k is odd and the end vertices of P_k are in X, then 1 or 2 or 3 cliques are taken into account according as $k = 3$ or 5 or 7. If k is odd and the end veritices of P_k are in Y, then 2 or 3 cliques are taken into account according as $k = 3$ or 5. In all the cases, we see that all the vertices which are not in the cliques induced by $N[Y]$ when $y \in Y_1$, lie in a clique. Hence there exists a $f \in \mathcal{F}$ such that $V_2 = \emptyset$ and for each $x \in V_1$, $D_G(x)$ induces a clique and $\bigcup_{x \in V(G)}(D_G(x) \cup \{x\}) = V(G)$. Hence, by Theorem 3.1, G is γ_r-excellent. □

REFERENCES

[1] E. J. COCKAYNE, P. A. DREYER, S. M. HEDETNIEMI: *Roman domination in graphs*, Discrete Math., **78** (2004), 11–22.

[2] G. FRICKE, T. W. HAYNES, S. T. HEDETNIEMI, R. LASKAR: *Excellent trees*, Bull. Int. Combin. Appl., **34** (2002), 27–38.

[3] T. W. HAYNES, S. T. HEDETNIEMI, P. J. SLATER, (EDS): *Domination in Graphs, Advanced Topics*, Marcel Dekker, New York, 1998.

[4] T. W. HAYNES, S. T. HEDETNIEMI, P. J. SLATER, (EDS): *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[5] S. T. HEDETNIEMI, M. A. HENNING: *Defending the Roman Empire - A new strategy*, Discrete Math., **266** (2003), 239–251.

[6] M. A. HENNING: *A characterization of Roman trees*, Discuss Math. Graph Theory, **22** (2002), 325–334.

[7] M. A. HENNING: *Defending the Roman Empire from multiple attacks*, Discrete Math., **271** (2003), 101–115.

[8] C. S. REVELLE: *Can you protect the Roman Empire?*, John Hopkins Magazine, **2**(70) (1997).

[9] B. MAHAVIR, P. ROUSHINI LEELY PUSHPAM, M. KAMALAM: *An algorithm to recognize weak Roman domination stable trees under vertex deletion*, Discrete Mathematics, Algorithms and Applications, 2050049, (2020).

[10] P. ROUSHINI LEELY PUSHPAM, M. KAMALAM: *Efficient weak Roman domination in graphs*, International Journal of Pure and Applied Mathematics, **101**(5) (2015), 701–710.

[11] P. ROUSHINI LEELY PUSHPAM, M. KAMALAM: *Efficient weak Roman domination in Myscielski graphs*, International Journal of Pure and Engineering Mathematics, **3**(2) (2015), 93–100.

[12] P. ROUSHINI LEELY PUSHPAM, M. KAMALAM: *Stability of weak Roman domination upon vertex deletion*, Asian Journal of Mathematics and Computer Research, **25**(2) (2018), 97–105.

[13] P. ROUSHINI LEELY PUSHPAM, M. KAMALAM: *Effects of vertex deletion on the weak Roman domination number of a graph*, AKCE International Journal of Graphs and Combinatorics, **16**(2) (2019), 204–212.

[14] P. ROUSHINI LEELY PUSHPAM, T. N. M. MALINI MAI: *Weak Roman domination in graphs*, Discuss. Math. Graph Theory, **31** (2011), 115–128.

[15] I. STEWART: *Defend the Roman Empire*, Scientific American, **281** (1991), 136–139.

[16] V. SAMODIVKIN: *Roman domination excellent graphs: trees*, Communications in Combinatorics and Optimization, **3** (2018), 1–24.

[17] E. ZHU, Z. SHAO: *Extremal problems on weak Roman domination number*, Inf. Process. Lett, **138** (2018), 12–18.

DEPARTMENT OF MATHEMATICS
D.B. JAIN COLLEGE
(AFFILIATED TO UNIVERSITY OF MADRAS), CHENNAI - 600 097
TAMIL NADU, INDIA
Email address: roushinip@yahoo.com

DEPARTMENT OF MATHEMATICS
D.B. JAIN COLLEGE
(AFFILIATED TO UNIVERSITY OF MADRAS), CHENNAI - 600 097
TAMIL NADU, INDIA
Email address: srilakshmi_murali@yahoo.com