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SOME PROPERTIES OF INVOLUTORY ADDITION CAYLEY GRAPH

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ABSTRACT. Let Γ be an abelian group and $X \subseteq \Gamma$. The addition Cayley graph is a graph whose vertex set is Γ and edge set is $E(G)=\{ab: \ a,b \in \Gamma, a+b \in X\}$. For a positive integer $n > 1$, the involutory addition Cayley graph $G^+(Z_n, I_v)$ is the graph whose vertex set is $Z_n=\{0,1,2,3,\ldots,n-1\}$ and edge set $E(G)=\{ab : a,b \in Z_n, a+b \in I_v\}$ where I_v $=\{a \in Z_n : a^2 \equiv 1(\mod n)\}$.In this paper, some properties of Involutory addition Cayley graph are discussed.

1. INTRODUCTION

A graph $G(V,E)$ is a mathematical object that is perceived as a set of vertices that connect any or all of the vertices and as a set of edges. If an edge connects two vertices in a graph G, they are said to be adjacent, otherwise it is said to be non- adjacent. We denote the number of vertices and edges of a graph G is $V(G)$ and $E(G)$ respectively. The order of G is defined as the cardinality of $V(G)$. We denote the cardinality of $V(G)$ as $|V|$ and $E(G)$as $|E|$ respectively. In a graph G, the degree of vertex v is denoted by $deg(v)$ and it is defined as the number of edges occurring with v. The minimum degree of a graph G is denoted by δ and Δ is denoted by the maximum degree.

The distance $d(u,v)$ is the minimum length of a $(u,v)-$ path between any two vertices $u,v \in G$ and the eccentricity of a vertex v of a connected graph G is e(v)

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$= \max\{d(u,v) : v \in V\}$. The diameter of a graph G is diam (G) $=\max\{e(v) : v \in V\}$.

All through the content, non-trivial, finite, undirected graphs without any loops or multiple edges are considered. For standard terminology and notation in graph theory we invoke Bondy and Murty [1]and Harary [2].

In literature, Cayley graphs are extensively discussed as they can be tackled to solve particular issues such as rearrangement and parallel CPUs design [3]. In 1878, Cayley introduced the Cayley graph for finite groups. Let Γ be finite group and X be a subset of Γ such that X does not contain identity of Γ. The Cayley graph Cay(Γ, X) relative to X is a graph with vertex set Γ and edge set $E(\Gamma, X)$=$\{xy/yx^{-1} or x^{-1}y \in X\}$.Clearly $G(\Gamma, X)$ is undirected graph without loops. Cayley graphs have been studied extensively in [4], [5] and [6].

Let Γ be abelian group and $X \subseteq \Gamma$. The addition Cayley graph is a graph whose vertex set is Γ and edge set is $\{xy : x, y \in \Gamma, x + y \in X\}$. Some properties of addition Cayley graphs have been discussed in [8].

Involutory Cayley graph is defined by Venkata Anusha et al. [7] and some structural properties are discussed.

For a positive integer n, the involutory Cayley graph $Cay(Z_n, I_v)$ is the graph whose vertex set is Z_n and any two vertices $a, b \in Z_n$ adjacent if and only if $a + b \in I_v$ where I_v denotes the set of all involutory elements in Z_n. The involutory Cayley graph is denoted by $G(Z_n, I_v)$.

Motivated by this, in this paper, the concept involutory addition Cayley graph is discussed and further some properties are studied.

2. Involutory Addition Cayley Graph

The Involutory addition Cayley graph is defined as follows:

Definition 2.1. *For a positive integer* $n > 1$*, the involutory addition Cayley graph* $G_n^+ = Cay^+(Z_n, I_v)$ *or* $G^+(Z_n, I_v)$ *is the graph whose vertex set is* $Z_n = \{0, 1, 2, \ldots n-1\}$ *and edge set* $E(G_n) = \{ab : a, b \in Z_n, a + b \in I_v\}$*, where* $I_v = \{a \in Z_n / a^2 \equiv 1(\bmod n)\}$.

3. Properties Of Involutory Addition Cayley Graph

In this section degree of vertices and number of edges of $G^+(Z_n, I_v)$ are discussed. Also the diameter for different values of n is obtained.

Theorem 3.1. *If n is even then the vertex degree of involutory addition cayley graph $G^+(Z_n, I_v)$ is $|I_v|$.*

Proof. Consider an involutory addition Cayley graph $G^+(Z_n, I_v)$ with vertex set Z_n and $I_v = \{x \in Z_n/x^2 \equiv 1(\mod n)\}$. Let n be even. Then the set I_v contains four or eight elements and all are odd. By the definition of an involutory addition Cayley graph, the degree of a vertex $a \in Z_n$ is considered as it is adjacent to a vertex $b \in Z_n$, $b \neq a$ such that $a + b \in I_v$. Hence the degree of a vertex v is $d(v)$=$|I_v|$. □

Theorem 3.2. *If $n = p^\alpha$, where p is odd prime and $\alpha \geq 1$ then the vertex degree of involutory addition cayley graph $G^+(Z_n, I_v)$ is*

$$d(v) = \begin{cases} |I_v| - 1, & \text{if } v = \left[\frac{n}{2}\right] \text{ or } \left[\frac{n}{2}\right] + 1, \\ |I_v|, & \text{if } v \neq \left[\frac{n}{2}\right] \text{ or } \left[\frac{n}{2}\right] + 1. \end{cases}$$

Proof. Let $n = p^\alpha$, where p is odd prime and $\alpha \geq 1$. And let v be any vertex of $G^+(Z_n, I_v)$. If $v \neq \left[\frac{n}{2}\right]$ and $v \neq \left[\frac{n}{2}\right]+1$.Then by the definition of involutory addition Cayley graph, v is adjacent to $|I_v|$ vertices. Therefore $d(v) = |I_v|$.If $v = \left[\frac{n}{2}\right]$ and $v = \left[\frac{n}{2}\right] + 1$. Then by the definition, these two vertices are not adjacent, since $\left[\frac{n}{2}\right] + \left[\frac{n}{2}\right] + 1 \equiv 0(\mod n)$ and $0 \notin I_v$. Therefore, $d(v) = |I_v| - 1$ if $v = \left[\frac{n}{2}\right] + 1$ or $\left[\frac{n}{2}\right]$. □

Theorem 3.3. *If n is odd and not a prime or prime power and $I_v = \{I_{v_1}, I_{v_2}, I_{v_3}, I_{v_4}\}$ is the set of involutory elements of Z_n. Then the vertex degree of the involutory addition Cayley graph $G^+(Z_n, I_v)$ is*

$$d(v) = \begin{cases} |I_v| - 1, & \text{if } v = \left[\frac{n}{2}\right], \left[\frac{n}{2}\right] + 1, \left(\frac{I_{v2}}{2}\right) \text{ and } I_{v_3} + \left(\frac{I_{v2}}{2}\right) \\ |I_v|, & \text{if } v \neq \left[\frac{n}{2}\right], \left[\frac{n}{2}\right] + 1, \left(\frac{I_{v2}}{2}\right) \text{ and } I_{v_3} + \left(\frac{I_{v2}}{2}\right). \end{cases}$$

Proof. Consider an involutory addition Cayley graph $G^+(Z_n, I_v)$ with vertex set Z_n and $I_v = \{x \in Z_n/x^2 \equiv 1(\mod n)\}$.

Let n be odd and not a prime or prime power. Then the set I_v contains four elements in which two are even and two are odd.

Let us denote $I_v = \{I_{v_1}, I_{v_2}, I_{v_3}, I_{v_4}\}$, where $\{I_{v_1}, I_{v_3}\}$ are odd and $\{I_{v_2}, I_{v_4}\}$ are even. Let $v \in Z_n$ and $v \neq \left[\frac{n}{2}\right]$ or $\left[\frac{n}{2}\right]+1$ or $\left(\frac{I_{v2}}{2}\right)$ or $I_{v_3}+\left(\frac{I_{v2}}{2}\right)$. Then by the definition of involutory addition Cayley graph, v is adjacent to $|I_v|$ vertices. Therefore $d(v) = |I_v|$. And let $v = \left[\frac{n}{2}\right]$, $\left[\frac{n}{2}\right]+1, \left(\frac{I_{v2}}{2}\right)$ and $I_{v_3}+\left(\frac{I_{v2}}{2}\right)$. Then by the definition, these four vertices are not adjacent, since $\left[\frac{n}{2}\right]+\left[\frac{n}{2}\right]+1 \equiv 0(\mod n)$ and $0 \notin I_v$ and also $\left(\frac{I_{v2}}{2}\right)+\left[I_{v_3}+\left(\frac{I_{v2}}{2}\right)\right] \equiv 0(\mod n)$ and $0 \notin I_v$. Therefore, $d(v) = |I_v| - 1$. □

Theorem 3.4. *[7] The involutory Cayley graph $G^+(Z_n, I_v)$ is $|I_v|-$ regular, moreover the number of edges in $G^+(Z_n, I_v)$ is $\frac{|Z_n||I_v|}{2}$.*

Theorem 3.5. *The total number of edges in the involutory addition Cayley graph*

$$G^+(Z_n, I_v) = \begin{cases} \frac{n}{2}|I_v|, & \textit{if} \quad n \textit{ is even}, \\ \left(\frac{n-1}{2}\right)|I_v|, & \textit{if} \quad n \textit{ is odd}. \end{cases}$$

Proof. Consider a graph $G^+(Z_n, I_v)$ with vertex set $V = Z_n = \{0, 1, 2, \dots n-1\}$
Case 1: Let n be even. Then by Theorem(3.1), $d(v) = |I_v|$ for any $v \in V$. As there are n vertices in $G^+(Z_n, I_v)$, the total number of edges are $\frac{n}{2}|I_v|$. Hence the total number of edges in $G^+(Z_n, I_v)$ is $\frac{n}{2}|I_v|$.
Case 2: Let n be odd. Then $n - I_v$ vertices are of degree $|I_v|$ and $|I_v|$ vertices are of degree $|I_v| - 1$. Thus $q = \frac{1}{2}[(n - I_v)(I_v) + (I_v - 1)I_v]$, where q denotes the total number of edges. This implies that, $I_v[n - I_v + I_v - 1] = 2q \Rightarrow I_v(n-1) = 2q$. Therefore $q = \frac{1}{2}(n-1)|I_v|$. Hence the total number of edges in $G^+(Z_n, I_v)$ is $q = \frac{1}{2}(n-1)|I_v|$. □

Theorem 3.6. *The involutory addition Cayley graph $G^+(Z_n, I_v)$ is $|I_v|$ - regular if n is even and $(|I_v|, |I_v| - 1)$ - semiregular if n is odd.*

Proof. Due to Theorem (3.1), if n is even, then the graph $G^+(Z_n, I_v)$ is $|I_v|$ - regular. If n is odd then the graph $G^+(Z_n, I_v)$ is $(|I_v|, |I_v| - 1)$ - semiregular. □

Theorem 3.7. *[7] If n is odd number, then the graph $G(Z_v, I_v)$ is not bipartite.*

Theorem 3.8. *The involutory addition Cayley graph $G^+(Z_n, I_v)$, where $n > 2$ is bipartite if and only if either n is even or $n = 3$.*

Proof. Consider a graph $G^+(Z_n, I_v)$ with vertex set $V = Z_n = \{0, 1, 2, \dots n-1\}$.

Necessity: If $n > 2$, Suppose the graph $G^+(Z_n, I_v)$, $n > 2$ is bipartite.
If possible, let n be odd and not a 3 multiple except 9. Then by the definition of

$G^+(Z_n, I_v)$, the vertex 0 is adjacent to the vertices 1 and $n-1$. Also the vertex $n-1$ is adjacent to the vertex 2, 2 is adjacent to $n-3$, $n-3$ is adjacent to 4 and the path is continuing up to $(\frac{n}{2}+1)$. Similarly vertex 1 is adjacent to $n-2$, $n-2$ is adjacent to 3, 3 is adjacent to $n-4$ and the path is continuing up to $\frac{n}{2}$. But the vertices 1 and $n-1$ are not connected by any path and therefore there is no cycle exists. This implies that $G^+(Z_n, I_v)$ is not bipartite, a contradiction to our assumption. Hence either n is even or $n = 3$, the graph $G^+(Z_n, I_v)$ is bipartite.

Sufficiency: Suppose n is even. Then the vertex set $V = \{n-1, n-2, n-3, \ldots, 2, 1, 0\}$ and I_v is a set of all involutory elements in Z_n. Therefore in this case I_v contains only odd numbers. Divide the vertex set into two partitions $V_1 = \{0, 2, 4, 6, \ldots, n-2\}$ and $V_2 = \{1, 3, 5, \ldots, n-1\}$.

Let u$\in I_v$ then u is adjacent to a vertex $v \in V_2$, since $u + v \in I_v$. That means, in V_1 each vertex is adjacent to a vertex of V_2 and not adjacent to a vertex of V_1 and vice versa. Hence the graph $G^+(Z_n, I_v)$ is bipartite.

If $n = 3$. Then $G^+(Z_n, I_v)$ is a tree and every tree is bipartite. Therefore, the graph $G^+(Z_n, I_v)$ is bipartite. □

Observation: If n is odd and $n > 3$ then the graph $G^+(Z_n, I_v)$ is not bipartite.

Theorem 3.9. *[7] For case $n = 4, 8$, the graph $G(Z_n, I_v)$ is complete bipartite.*

Observation: For case $n = 4, 8$, the graph $G^+(Z_n, I_v)$ is complete bipartite.

Theorem 3.10. *The diameter of involutory addition Cayley graph $G^+(Z_n, I_v)$ is $\frac{n}{2}$, if $n = 2p$, where p is a prime.*

Proof. Consider a graph $G^+(Z_n, I_v)$ with vertex set $V = Z_n = \{0, 1, 2, 3, \ldots n-1\}$. Let $n = 2p$, where p is a prime. Then the graph $G^+(Z_n, I_v)$ is bipartite with vertex partition $V = \{0, 2, 4, \ldots, n-2\} \cup \{1, 3, 5, \ldots n-1\}$ which has diameter $\frac{n}{2}$. Hence the diameter of the graph $G^+(Z_n, I_v)$ is $\frac{n}{2}$. □

Theorem 3.11. *If $n =p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_n^{\alpha_n}$, where $p_1, p_2, \ldots p_n$ are primes and $\alpha_1 \geq 1$, then the diameter of involutory addition Cayley graph $G^+(Z_n, I_v)$ is $n-1$.*

Proof. Consider a graph $G^+(Z_n, I_v)$ with vertex set $V = Z_n = \{0, 1, 2, 3, \ldots n-1\}$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_n^{\alpha_n}$, where each p_i is a prime and $\alpha_i \geq 1$. Then there exists two non-adjacent vertices $\left[\frac{n}{2}\right]$ and $\left[\frac{n}{2}\right] + 1$ in $G^+(Z_n, I_v)$, Since $\left[\frac{n}{2}\right] + \left[\frac{n}{2}\right] + 1 \equiv 0(\mod n)$ and $0 \notin I_v$.This implies that these two vertices has no common neighbor and also the number of edges between $\left[\frac{n}{2}\right]$ and $\left[\frac{n}{2}\right] + 1$ is $n-1$, which is nothing but the length of $\left[\frac{n}{2}\right]$ and $\left[\frac{n}{2}\right]+1$. Hence the diameter of $G^{+}(Z_n, I_v)$ is $n-1$ if $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_n^{\alpha_n}$, where p_i is a prime and $\alpha_i \geq 1$. □

Theorem 3.12. *If n is odd but not a prime or a prime power. Then the diameter of involutory addition Cayley graph $G^{+}(Z_n, I_v)$ is 4.*

Proof. Consider a graph $G^{+}(Z_n, I_v)$ with vertex set $V = Z_n = \{0, 1, 2, 3, \dots n-1\}$. and I_v is a set of all involutory elements in Z_n. Suppose n is odd but not a prime and prime power. In this case $|I_v| = 4,$. Then there exist a path $(0, I_{v_1}, I_{v_2}-1, I_{v_3}, 0)$ of length 4. Hence the diameter of $G^{+}(Z_n, I_v)$ is 4. □

4. Conclusion

Using Number theory, it is interesting to develop an arithmetic graph like involutory addition Cayley graph and studied some properties of it. This work gives the scope for the study of domination parameters, chromating number and topological indices of involutory addition Cayley graph and the authors have also studied this aspect.

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