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CONTRACTION PRINCIPLE IN RECTANGULAR M-METRIC SPACES WITH A BINARY RELATION

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ABSTRACT. The present work brings the new concept of proving Banach contraction principle (BCP) in rectangular M-metric spaces using relation-theoretic conception. The results are new and extend the opportunity to obtain new results in rectangular M-metric spaces. With the help of some examples validity of the result is also prove here.

1. INTRODUCTION

In the literature of fixed point theory, the famous contraction principle named as Banach contraction principle has been established and studied in various spaces known as M-metric space, rectangular metric space, partial rectangular metric space etc, which were initiated by Asadi et.al. [2], Branciari [3], Shukla S. [6] respectively. Ran and Reurings [5] in 2003 proved similar version of Banach contraction principle in partially ordered metric spaces. Definitions of these spaces are as follows:

Definition 1.1. *[3] A rectangular metric space is an ordered pair (X, ρ) where $X \neq \emptyset$. Let $\rho : X \times X \to \mathbb{R}$ be a map such that $\forall\, r, s \in X$,*

(i) $0 \leq \rho(r, s)$ *if and only if* $r = s$*;*
(ii) $\rho(r, s) = \rho(s, r)$*;*

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(iii) $\rho(r,s) \leq \rho(r,w) + \rho(w,t) + \rho(t,s)$ $\forall$ $r,s \in X$ *and* $\forall$ *distinct point* $w,t \in X - \{r,s\}$ *(this is known as rectangular property).*

Motivated by the work of Branciari [3], Shukla [6] in 2014 introduced partial rectangular metric space which is defined as follows.

Definition 1.2. *[6] A partial rectangular metric space is an ordered pair* (X,η) *where* $X \neq \emptyset$*. Let* $\eta : X \times X \to \mathbb{R}^+$ *be map such that* $\forall$ $r,s \in X$*,*

(i) $\theta \preceq \eta(r,s)$*;*
(ii) $r = s$ *if and only if* $\eta(r,r) = \eta(r,s) = \eta(s,s)$*;*
(iii) $\eta(r,r) \leq \eta(r,s)$*;*
(iv) $\eta(r,s) = \eta(s,r)$*;*
(v) $\eta(r,s) \leq \eta(r,w) + \eta(w,t) + \eta(t,s) - \eta(w,w) - \eta(t,t)$ $\forall$ $r,s \in X$ *and* $\forall$ *distinct point* $w,t \in X - \{r,s\}$ *(this is known as rectangular property)*

After that M-metric space is defined by Asadi et al. [2] as given below.

Notation 1.

(i) $m_{r,s} = min\{m(r,r), m(s,s)\}$
(ii) $M_{r,s} = max\{m(r,r), m(s,s)\}$

Definition 1.3. *[2] A* M*- metric space is an ordered pair* (X,m) *where* $X \neq \emptyset$*. Let* $m : X \times X \to [0,\infty)$ *be a map such that* $\forall$ $r,s,t \in X$*,*

(i) $m(r,r) = m(s,s) = m(r,s)$ *if and only if* $r = s$*;*
(ii) $m_{r,s} \leq m(r,s)$*;*
(iii) $m(r,s) = m(s,r)$*;*
(iv) $(m(r,s) - m_{r,s}) \leq (m(r,t) - m_{r,t}) + (m(t,s) - m_{t,s})$.

Inspiring by various authors, in 2018, N. özgur et.al. [4] defined rectangular M-metric spaces which is the generalization of M- metric space as follows:

Notation 2.

(i) $m_{r_{s,t}} = min\{m_r(s,s), m_r(t,t)\}$
(ii) $M_{r_{s,t}} = max\{m_r(s,s), m_r(t,t)\}$

Definition 1.4. *[4] A rectangular* M*- metric space is an ordered pair* (X,m_r) *where* $X \neq \emptyset$*. Let* $m_r : X \times X \to [0,\infty)$ *be a map such that* $\forall$ $s,t, \in X$ *and* $\forall$ $u,v \in X - \{s,t\}$

(i) $m_r(s,s) = m_r(t,t) = m_r(s,t) \Longleftrightarrow s = t$;

(ii) $m_{r_{s,t}} \leq m_r(s,t)$;

(iii) $m_r(s,t) = m_r(t,s)$;

(iv) $(m_r(s,t) - m_{r_{s,t}}) \leq (m_r(s,u) - m_{r_{s,u}}) + (m_r(u,v) - m_{r_{u,v}}) + (m_r(v,t) - m_{r_{v,t}})$.

Remark 1.1. *[4] In rectangular M-metric space (X, m_r), we have*

(i) $0 \leq M_{r_{s,t}} + m_{r_{s,t}} = m_r(s,s) + m_r(t,t)$

(ii) $0 \leq M_{r_{s,t}} - m_{r_{s,t}} = m_r(s,s) - m_r(t,t)$ *for every* $s, t \in X$

(iii) $(M_{r_{s,t}} - m_{r_{s,t}}) \leq (M_{r_{s,u}} - m_{r_{s,u}}) + (M_{r_{u,v}} - m_{r_{u,v}}) + (M_{r_{v,t}} - m_{r_{v,t}})$

2. Convergence conditions in rectangular M-metric spaces

Let (X, m_r) be a rectangular M- metric space. In that case,

(i) A sequence $\{s_n\} \in X$ is said to converge to a point s if and only if $\lim_{n\to\infty}(m_r(s_n, s) - m_{r_{s_n,s}}) = 0$.

(ii) A sequence $\{s_n\} \in X$ is said to m_r-Cauchy sequence if and only if $\lim_{n\to\infty}(m_r(s_n, s_m) - m_{r_{s_n,s_m}}$ and $\lim_{n\to\infty}(M_r(s_n, s_m) - m_{r_{s_n,s_m}})$ exist and finite.

(iii) A rectangular M-metric space is $m_r$complete if every m_r-Cauchy sequence $\{s_n\}$ converges to a point s with $\lim_{n\to\infty}(m_r(s_n, s) - m_{r_{s_n,s}}) = 0$ and $\lim_{n\to\infty}(M_r(s_n, s) - m_{r_{s_n,s}}) = 0$.

(iv) If in a rectangular M- metric space (X, m_r), $s_n \longrightarrow s$ and $t_n \longrightarrow t$ as $n \longrightarrow \infty$. In that case $m_r(s,t) = m_{r_{s,t}}$. Further if $m_r(s,s) = m_r(t,t)$ implies that $s = t$.

Remark 2.1. *[4] Let (X, m_r) is a rectangular M- metric space and $\{s_n\}$ be a sequence in X. Then there exists $k \in [0,1)$ with $m_r(s_{n+1}, s_n) \leq k m_r(s_n, s_{n-1})$ for all $n \in \mathbb{N}$, we have*

(i) $\lim_{n\to\infty} m_r(s_n, s_{n-1}) = 0$;

(ii) $\lim_{n\to\infty} m_r(s_n, s_n) = 0$;

(iii) $\lim_{n,m\to\infty} m_{r_{s_n,s_m}} = 0$;

(iv) $\{s_n\}$ *is m_r-Cauchy sequence.*

Alam A. et.al. [1] established contraction principle on a complete metric space equipped with a binary relation and gave a new approach to prove contraction principles. Now we recall some terminologies of relations with characteristics given by Alam A. et.al. [1]

Definition 2.1. *Let $X \neq \emptyset$ and $\mathcal{T}$ is a binary relation on X. Let H be a self map on X.*

(1) *s and t are $\mathcal{T}$-comparative if $(s,t) \in \mathcal{T}$ or $(t,s) \in \mathcal{T}$. This is shown by $[s,t] \in \mathcal{T}$.*
(2) *A sequence $\{s_n\} \in X$ is said to be $\mathcal{T}$-preserving if $(s_n, s_{n+1}) \in \mathcal{T}$ for all $n \in \mathbb{N}$.*
(3) *A mapping $H : X \to X$ is called H-closed if for every $(s,t) \in \mathcal{T}$, we have $(Hs, Ht) \in \mathcal{T}$.*
(4) *In $\mathcal{T}$ a path of length k (where $k \in \mathbb{N}$) for $s,t \in X$, from s to t is a finite sequence $\{w_j\}_{j=0}^{k} \subset X$ satisfied the given:*
    - *$w_0 = s$ and $w_r = t$;*
    - *$(w_j, w_{j+1}) \in \mathcal{T}$ for each $0 \leq j \leq k-1$.*
(5) *Let $E \subset X$ is said to be $\mathcal{T}$-directed if for each $s,t \in E$, $\exists\ w \in X$ with $(s,w) \in \mathcal{T}$ and $(t,w) \in \mathcal{T}$.*

In present work, same from [1] we adopt the given symbols:

- $F(H) = \{u \in X : Hu = u\}$;
- $X(H, \mathcal{T}) = \{u \in X : (u, Hu) \in \mathcal{T}\}$;
- $\mathcal{W}(u, v, \mathcal{T})$ =the set of all paths in $\mathcal{T}$ from u to v.

Proof of the following propositions is same as in [1], so we omitted the proof .

Proposition 2.1. *Let (X, m_r) be a complete rectangular M- metric space, $\mathcal{T}$ be relation on X. Then for a mapping $H : X \to X$ and $t \in [0,1)$, the given contractive conditions are identical:*

(i) *$m_r(Hu, Hv) \leq t m_r(u,v)$ for all $u, v \in X$ such that $(u,v) \in \mathcal{T}$;*
(ii) *$m_r(Hu, Hv) \leq t m_r(u,v)$ for all $u, v \in X$ such that $[u,v] \in \mathcal{T}$.*

Proposition 2.2. *Let if $\mathcal{T}$ is H-closed, so $\mathcal{T}^s$ where $\mathcal{T}^s = \mathcal{T} \bigcup \mathcal{T}^{-1}$ is also H-closed.*

Definition 2.2. *Let (X, m_r) be a complete rectangular M- metric space, $\mathcal{T}$ be binary relation on X is said to be d-self closed if for every $\mathcal{T}$- preserving sequence $\{u_n\}$ with $u_n \longrightarrow u \in X$ as $n \longrightarrow \infty$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $[u_{n_k}, u] \in \mathcal{T}$ for all $k \in \mathbb{N}$.*

3. Main Results

Theorem 3.1. *Let (X, m_r) be a complete rectangular M- metric space and $\mathcal{T}$ be binary relation on X. Let $H : X \to X$ be a self mapping which satisfy the given conditions:*

(i) $X(H, \mathcal{T}) \neq \emptyset$;
(ii) $\mathcal{T}$ *is H-closed;*
(iii) *One of the two conditions is hold from:- H is continuous or $\mathcal{T}$ is d-self closed;*
(iv) *There exists $t \in [0, 1)$ such that $m_r(Hu, Hv) \leq tm_r(u, v)$ for all $u, v \in X$ such that $(u, v) \in \mathcal{T}$ then H has a fixed point.*
(v) *Further, if $\mathcal{W}(u, v, \mathcal{T}^S) \neq \emptyset$ for each $u, v \in X$, so H has a unique fixed point.*

Proof. Let $u_0 \in X(H, \mathcal{T})$ arbitrarily i.e., $(u_0, Hu_0) \in \mathcal{T}$. Define a sequence $\{u_n\}$ by $u_n = Hu_{n-1}$ for all $n \in \mathbb{N}$. First of all we show that the sequence $\{u_n\}$ is $\mathcal{T}$- preserving. Since $(u_0, u_1) \in \mathcal{T}$, so from assumption (ii), we have $(Hu_0, Hu_1) = (u_1, u_2) \in \mathcal{T}$. Repetition of this argument gives, $(u_{n-1}, u_n) \in \mathcal{T}$ for all $n \in \mathbb{N}$. Hence $\{u_n\}$ is $\mathcal{T}$-preserving. Applying the condition (iv) to $\{u_n\}$ which is $\mathcal{T}$-preserving, implies that for all $n \in \mathbb{N}$,

$$\begin{aligned}
m_r(u_n, u_{n+1}) &= m_r(Hu_{n-1}, Hu_n) \\
&\leq tm_r(u_{n-1}, u_n) \\
&= tm_r(Hu_{n-2}, Hu_{n-1}) \\
&\leq t^2 m_r(u_{n-2}, u_{n-1}) \\
&\ \ \vdots \\
&\leq t^n m_r(u_0, u_1) = t^n m_r(u_0, Hu_0).
\end{aligned}$$

Assume that $u_n \neq u_{n+1} \forall n \in \mathbb{N}$. Otherwise, let if $u_n = u_{n+1} \forall n \in \mathbb{N}$ so $u^* = u_n$ will become fixed point of H. Also suppose that u_0 is not a periodic point of H. Indeed, let if $u_0 = u_n$, so for any $n \geq 2$,

$$\begin{aligned}
m_r(u_0, Hu_0) &= m_r(u_n, Hu_n) \\
m_r(u_0, u_1) &= m_r(u_n, u_{n+1}) \\
&\leq t^n m_r(u_0, Hu_0)
\end{aligned}$$

this shows that $m_r(u_0, Hu_0) = 0$ that is u_0 become the fixed point of H. Therefore suppose that $u_n \neq u_m$ for all distinct $n \in \mathbb{N}$.

Now take two cases for the sequence $\{u_n\}$. If $m = 2p+1$ with $p \geq 1$.

$$\begin{aligned}
& m_r(Hu_n, Hu_{n+m}) \\
& \quad \leq m_r(Hu_n, Hu_{n+1}) + m_r(Hu_{n+1}, Hu_{n+2}) + \ldots + m_r(Hu_{n+2p}, Hu_{n+2p+1}) \\
& \leq t^n m_r(u_0, Hu_0) + t^{n+1} m_r(u_0, Hu_0) + \ldots + t^{n+2p} m_r(u_0, Hu_0) \\
& \leq t^n m_r(u_0, Hu_0)[1 + t + t^2 + \ldots + t^{2p}] \\
& \leq \frac{t^n}{t-1} m_r(u_0, Hu_0)
\end{aligned} \tag{3.1}$$

If $m = 2p$ with $p \geq 2$.

$$\begin{aligned}
& m_r(Hu_n, Hu_{n+m}) \\
& \leq m_r(Hu_n, Hu_{n+2}) + m_r(Hu_{n+2}, Hu_{n+3}) + \ldots + m_r(Hu_{n+2p-1}, Hu_{n+2p}) \\
& \leq t^n m_r(u_0, Hu_2) + t^{n+2} m_r(u_0, Hu_0) + \ldots + t^{n+2p-1} m_r(u_0, Hu_0) \\
& \leq t^n m_r(u_0, Hu_2) + \frac{t^{n+2}}{1-t} m_r(u_0, Hu_0) \\
& \leq t^n m_r(u_0, Hu_2) + \frac{t^n}{1-t} m_r(u_0, Hu_0)
\end{aligned} \tag{3.2}$$

Hence, we deduce from (3.1) and (3.2)

$$m_r(Hu_n, Hu_{n+m}) \;\leq\; t^n m_r(u_0, Hu_2) + \frac{t^n}{1-t} m_r(u_0, Hu_0) \forall n, m \geq 0. \tag{3.3}$$

which implies that R.H.S.$\to$ 0 as $n \to \infty$, and since $m_r(Hu_n, Hu_{n+m}) - m_{r_{Hu_n, Hu_{n+m}}}$ $\leq m_r(Hu_n, Hu_{n+m})$, we obtain that the sequence $\{u_n\}$ is an m_r-Cauchy sequence in (X, m_r). Hence there exists some $u^* \in X$ such that $\lim_{n \longrightarrow \infty} m_r(Hu_n, u*) = \lim_{n,m \longrightarrow \infty} m_r(Hu_n, Hu_m) = m_r(u^*, u^*)$.

In the view of (3.3), we get

$$m_r(u^*, u^*) = \lim_{n \longrightarrow \infty} m_r(Hu_n, u^*) = \lim_{n,m \longrightarrow \infty} m_r(Hu_n, Hu_m) = 0. \tag{3.4}$$

Now to prove that u^* is a fixed point of H. From condition (iii), consider the two options one by one.

Option 1. Assume continuity of H. This implies that $u_{n+1} = Hu_n \longrightarrow Hu^*$ as $n \longrightarrow \infty$. From the uniqueness of limit we have $u^* = Hu^*$, that is u^* is a fixed point H.

Option 2. Assume d-self-closedness of $\mathcal{T}$. Since $\{u_n\}$ is $\mathcal{T}$-preserving sequence and $u_n \longrightarrow u$ as $n \longrightarrow \infty$ therefore there exists subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$[u_{n_k}, u] \in \mathcal{T} \quad \text{for all} \quad k \in \mathbb{N}.$$

By [iv] and proposition(2.1), we obtain

$$\begin{aligned} m_r(u^*, Hu^*) &\leq m_r(u^*, u_{n_k}) + m_r(u_{n_k}, u_{n_{k+1}}) + m_r(u_{n_{k+1}}, u^*) \\ &\leq m_r(u^*, u_{n_k}) + m_r(u_{n_k}, Hu_{n_k}) + m_r(Hu_{n_k}, Hu^*) \\ &\leq m_r(u^*, u_{n_k}) + m_r(u_{n_k}, Hu_{n_k}) + tm_r(u_{n_k}, u^*) \end{aligned}$$

Taking $n \longrightarrow \infty$ and using (3.4) we get, $m_r(u^*, Hu^*) = 0$. that is $Hu^* = u^*$.

Now to show that fixed point is unique, let $u, v \in F(H)$ that is, $H(u) = u$ and $H(v) = v$. From (v) there exists a path say $\{w_0, w_1, \ldots, w_k\}$ of some finite length k in $\mathcal{T}^s$ such that $w_0 = u, w_k = v, [w_i, w_{i+1}] \in \mathcal{T}$ for each $0 \leq j \leq k-1$. Since $\mathcal{T}$ is H closed so from proposition(2.2), we obtain $\mathcal{T}^s$ is also H-closed and so $[H^n w_j, H^n w_{j+1}] \in \mathcal{T}$ for $j = 0, 1, \ldots, k-1$ and for all $n \geq 0$,

$$\begin{aligned} m_r(u, v) &= m_r(H^n w_0, H^n w_k) \\ &\leq \sum_{j=0}^{k-1} m_r(H^n w_i, H^n w_{i+1}) \\ &\vdots \\ &\leq k^n \sum_{j=0}^{k-1} m_r(w_i, w_{i+1}) \\ &\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \end{aligned}$$

This implies that $u = v$. Thus fixed point of H is unique. □

Corollary 3.1. *Theorem 3.1 is still valid if condition(v) is changed by one of the below conditions(keeping the rest same):*

(v1) *$\mathcal{T}$ is complete;*
(v2) *X is $\mathcal{T}^s$-directed.*

Proof. Proof remains the same as in Alam A. et.al. [1]. □

Corollary 3.2. *[4] Let (X, m_r) be the complete rectangular M-metric space. let $H : X \rightarrow X$ be a self map. If there exists $t \in (0, 1)$ such that*

$$m_r(Hu, Hv) \leq tm_r(u, v) \ \forall \ u, v \in X.$$

this implies that fixed point of H is unique in X and for any $u \in X$, the monotonous sequence $\{H^n u\}$ converges to the fixed point of X.

Proof. For this, assign relation $\mathcal{T}$ on X by universal relation that is $\mathcal{T} = X \times X$, so all the conditions of theorem 3.1 are hold and H has a unique fixed point of X. □

Example 1. *Let $X = \{1,2,3,4\}$ and function $m_r : X \times X \to [0,\infty)$ assigned as*

$$m_r(1,1) = m_r(2,2) = m_r(3,3) = 1 \text{ and } m_r(4,4) = 8$$
$$m_r(1,2) = m_r(2,1) = m_r(1,3) = m_r(3,1) = m_r(1,4) = m_r(4,1) = 4$$
$$m_r(2,3) = m_r(3,2) = 5$$
$$m_r(2,4) = m_r(4,2) = 6$$
$$m_r(3,4) = m_r(4,3) = 7$$

Clearly, (X, m_r) is a complete rectangular M- metric space.

Now consider a binary relation $\mathcal{T}$ on X as follows

$$\mathcal{T} = \{(1,1),(2,4),(1,3),(1,4),(4,3)\}$$

and the self map H on X represented by

$$H(u) = \begin{cases} 1, & \text{if } 1 \le u \le 3; \\ 3, & \text{if } u = 4. \end{cases}$$

Then, $\mathcal{T}$ is H-closed. Here H is not continuous. Consider $\mathcal{T}$-preserving sequence $\{u_n\}$ with $(u_n, u_{n+1}) \in \mathcal{T}$ forall $n \in \mathbb{N}_{\not\vdash}$. Notice that, here (u_n, u_{n+1}) does not belong to $\{(1,4)\}$ so that $(x_n, x_{n+1}) \in \{(1,1),(2,4),(1,3),(4,3)\}$ $\forall n \in \mathbb{N}_{\not\vdash}$, which shows that $\{u_n\} \subset \{1,1\}$. Thus $[x_n, x] \in \mathcal{R}$ and $\mathcal{T}$ is d-self closed. Assumption (iv) of theorem 1 holds for, $k = \frac{5}{6}$. And the fixed point of H is 1.

Example 2. *Consider (X, m_r) be the complete rectangular M-metric space where $X = \mathbb{R}^+$ and $m_r(u,v) = \frac{u+v}{3}$ $\forall\ u,v \in X$. Assign a binary relation on X by $\mathcal{R} = \{(u,v) \in \mathbb{R}^2 : u \ge v, u \in \mathbb{Q}\}$. Take a self map H on X assigned by*

$$H(u) = \tfrac{u}{3}$$

It is easy to see that $\mathcal{T}$ is H-closed as well as continuous.

Let $u, v \in X$ with $(u,v) \in \mathcal{T}$, we get $m_r(Hu, Hv) = m_r(\frac{u}{3}, \frac{v}{3}) = \frac{1}{3}(\frac{u}{3} + \frac{v}{3}) = \frac{u+v}{9} \le \frac{3}{4}\frac{u+v}{3}$, that is, H follows assumption (iv) of theorem 3.1 for $k = \frac{3}{4}$. Thus all the conditions (i-iv) of theorem 3.1 are satisfied and H has a fixed point in X. Also, assumption (v) holds here and therefore 0 is the unique fixed point of H.

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