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VERY EXCELLENT DOMINATING WEAKLY CONNECTED SET DOMINATING SETS

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ABSTRACT. A γ_{wcsd} set S of a connected graph G is a dominating weakly connected set dominating $(wcsd)$ set of G with minimum cardinality [2]. A connected graph G is a very excellent $wcsd$ if there is a γ_{wcsd} set S such that $\forall\, u \in V - S$ there exists a vertex $v \in S \ni$: $S - \{v\} \cup \{u\}$ is a γ_{wcsd} set of G [3] and S is called very excellent $wcsd$ set of G. In this paper we have obtained a very excellent $wcsd$ graphs from a very excellent $wcsd$ graphs with γ_{wcsd} level vertex [3] and also we have obtained the union of very excellent $wcsd$ graph is again a very excellent $wcsd$ graph under certain conditions.

1. INTRODUCTION

Sampath Kumar and Pushpa Latha have defined set domination in graphs. Hedetniemi et all have defined weakly connected domination in graphs [2,8]. We define the concept of weakly connected set dominating sets (wcs), Dominating weakly connected set dominating sets $(wcsd)$ and elucidate some results in our earlier paper [2]. We extend these to new class of very excellent $wcsd$- graphs [3].

2. PRELIMINARIES

Definition 2.1. *Let G be a connected graph. A sub set S of V is a set dominating set if $\forall T \subseteq V - S$ there exists $R \subseteq S$ such that $\prec T \cup R \succ$ is connected [1].*

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Definition 2.2. *Let G be a connected graph. A sub set S of V is a weakly connected set if the sub graph $\prec S \succ_w$ whose vertex set is $N[S]$ and whose edge set consists of those edges in $E(G)$ with at least one vertex and possibly both in S is connected [4].*

Definition 2.3. *Let G be a connected graph. A sub set S of V is a weakly connected set dominating (wcs) set if $\forall T \subseteq V - S$ there exists $R \subseteq S$ such that $\prec T \cup R \succ$ is weakly connected. A dominating weakly connected set dominating set is $wcsd$ set [2] and its minimal cardinality is $\gamma_{wcsd}(G)$*

Definition 2.4. *A connected graph G is a γ_{wcsd}-excellent if each vertex u of G is in some γ_{wcsd} set of G. [3]*

Definition 2.5. *A connected graph G is a γ_{wcsd}-flexible if to each vertex u of G, there is a γ_{wcsd} set not containing u. [3]*

Definition 2.6. *A vertex u in $V(G)$ is called γ_{wcsd} level vertex [3] of G if $\gamma_{wcsd}(G-u) = \gamma_{wcsd}(G)$.*

Definition 2.7. *A vertex u in $V(G)$ is called γ_{wcsd} non level vertex [3] of G if $\gamma_{wcsd}(G-u) = \gamma_{wcsd}(G) - 1$.*

3. Class of Very Excellent $wcsd$ Graphs

Theorem 3.1. *If G is a very excellent $wcsd$ graphs and u is level vertex of G, then the graph H obtained from G by attaching a path P_3 at 'u' is also very excellent $wcsd$ graph.*

Proof. Let G be a very excellent $wcsd$ graph. Then G has a very excellent γ_{wcsd} set S.

Let u be a γ_{wcsd} level vertex of G. Then $\gamma_{wcsd}(G-u) = \gamma_{wcsd}(G)$.

Let H be a graph obtained by attached a path $P_3 = w_1w_2w_3$ at u of G. Since S is γ_{wcsd} set of G, $S_H = S \cup \{w_2\}$ is a $wcsd$ set of H, we have

$$\gamma_{wcsd}(H) \leq \gamma_{wcsd}(G) + 1. \tag{3.1}$$

Then $S_H \cap \{w_1, w_2, w_3\} \neq \phi$ and $S_H \cap V(G)$ is a γ_{wcsd} set of G. Also, u is a γ_{wcsd} level vertex of G. Then,

$$\gamma_{wcsd}(H) - 1 = |S_H \cap V(G)| \geq \gamma_{wcsd}(G),$$

and this implies

$$\text{Implies } \gamma_{wcsd}(H) \geq \gamma_{wcsd}(G) + 1. \tag{3.2}$$

Therefore $\gamma_{wcsd}(H) = \gamma_{wcsd}(G) + 1$ by (3.1) and (3.2).

Let S_{VG} be a very excellent γ_{wcsd} set of G.

Claim: $S_{VH} = S_{VG} \cup \{w_2\}$ is a very excellent γ_{wcsd} set of H.

Since S_{VG} is a very excellent γ_{wcsd} set of G, $\forall u \in V(G) - S_{VG}$, $\exists v \in S_{VG}$, such that

$$S_{VG} - \{v\} \cup \{u\} \text{ is a } \gamma_{wcsd} \text{ set of } G. \tag{3.3}$$

Also,

$$S_{VG} - \{w_2\} \cup \{w_1\} \text{ and } S_{VG} - \{w_2\} \cup \{w_3\} \text{ are } \gamma_{wcsd} \text{ set of } G. \tag{3.4}$$

By (3.3) and (3.4), $\forall u \in V(G) - S_{VH}$, $\exists v \in S_{VH}$, such that $S_{VH} - \{v\} \cup \{u\}$ is a γ_{wcsd} set of H. This implies that S_{VH} be a very excellent γ_{wcsd} set of H. Thus H is a very excellent graph. □

Theorem 3.2. *A graph H is obtained from G by a path P_3 at a vertex of G is $wcsd$ very excellent if and only if G is $wcsd$ very excellent and there exist a very excellent γ_{wcsd} set S of G such that $u \in S$ and $S - \{u\}$ is a γ_{wcsd} set of $G - u$.*

Proof. Let G be the graph with vertex u. H is obtained from G by attaching a path $P_3 = w_3 w_2 w_1$ at a vertex u.

Assume that H is a very excellent graph. Let S_H be very excellent γ_{wcsd} set of H. If $w_1 \in S_H,\ w_2 \notin S_H$, then there exist $x \in S_H$ such that $S_H - x \cup \{w_2\}$ is a γ_{wcsd} set of H.

If $w_2 \in S_H,\ w_1 \notin S_H$, then there exist $y \in S_H$ such that $S_H - y \cup \{w_1\}$ is a γ_{wcsd} set of H.

Therefore there is no γ_{wcsd} set of H which contains both w_1 and w_2. This implies $x = w_1$ and $y = w_2$.

In order to dominate the vertex w_3, we realize that $S_H - w_2 \cup \{w_1\}$ contains either w_3 or u, and so $S_H \cap \{u,\ w_3\} \neq \phi$.

If $u \in S_H$, let $S = S_H$, and if $w_3 \in S_H$ let $S = S_H - w_3 \cup \{u\}$, then S is a very excellent γ_{wcsd} set of H containing both u and w_2.

Let $S_0 = S - w_2$. Then $S_0 - w_2$ is a γ_{wcsd} set of G. Given any vertex $v \in V(G)$ such that $v \in S_0$, then $v \in S$. Further, there exist $z_1 \in S$ such that $S - z_1 \cup \{v\}$ is a γ_{wcsd} set of H.

By our choice of S $z_1 \neq w_2$, $z_1 \in S_0$ implies $S_0 - z_1 \cup \{v\}$ is a γ_{wcsd} set of G, that S_0 is a very excellent γ_{wcsd} set of G, and that G is a very excellent graph. As $w_3 \notin S_1$, $\exists z_2 \in S$ such that $S - z_2 \cup \{w_3\}$ is a γ_{wcsd} set of H.

By our choice of S $z_2 \neq w_2$, as γ_{wcsd} set of H does not contain all 3 vertices $w_3,\ w_2$ and u, he have that $z_2 = u$, and $S - \{u\} \cup \{w_3\}$ is a γ_{wcsd} set of H. From here, $S_0 - u = S - \{u, w_2\} \cup \{w_3\}$ a minimal $wcsd$ set of $G - u$, and $S_0 - u$ is a γ_{wcsd} set of $G - u$.

Conversely, assume that G is a very excellent graph. Then, S is very excellent γ_{wcsd} set of G such that $u \in S$, and $S - u$ is a γ_{wcsd} set of $G - u$. Thus, $S \cup \{w_2\}$ is very excellent γ_{wcsd} set of H. □

Theorem 3.3. *Let $G_1,\ G_2$ be γ_{wcsd} very excellent graphs, and let $u_1,\ u_2$ be γ_{wcsd} level vertices of G_1 and G_2 respectively. Then, the graph H obtained from $G_1 \cup G_2$ by joining the vertices u_1, u_2 by an edge is γ_{wcsd} very excellent.*

Proof. Let G_1, G_2 be γ_{wcsd} very excellent graphs, and let S_1 and S_2 be very excellent γ_{wcsd} sets of G_1 and G_2, respectively.

Also, let u_1, u_2 be γ_{wcsd} level vertex of G_1 and G_2, respectively. Then,

$$\gamma_{wcsd}\ (G_1) = \gamma_{wcsd}(G_1 - u_1) \quad \text{and} \quad \gamma_{wcsd}\ (G_2) = \gamma_{wcsd}(G_2 - u_2). \tag{3.5}$$

Let S_H be γ_{wcsd} set of H. Then $S_H \cap V(G_1)$ is a γ_{wcsd} set of $G_1 - u_1$ and $S_H \cap V(G_2)$ is a γ_{wcsd} set of $G_2 - u_2$. Therefore,

$$|S_H| \geq \gamma_{wcsd}\,(G_1 - u_1) \quad \text{and} \quad \gamma_{wcsd}\,(G_2 - u_2)\,, \tag{3.6}$$

$$\begin{aligned} |S_H| &\geq \gamma_{wcsd}\,(G_1) + \gamma_{wcsd}\,(G_2) \quad \text{[by (3.5) and (3.6)]} \\ |S_H| &\geq |S_1| + |S_2| \\ |S_H| &= |S_1| + |S_2| \quad \text{and} \quad S_H = S_1 \cup S_2. \end{aligned}$$

Claim: S_H is a γ_{wcsd} very excellent set of H.

Let $w \in V(H) - S_H$. Then, $w \notin S_1 \cup S_2$, $w \notin S_1$ and $w \notin S_2$. Since S_1 is a very excellent γ_{wcsd} set of G_1, $\exists x \in S_1$ such that $S_1 - x \cup \{w\}$ is a γ_{wcsd} set of G_1. Therefore, $S_1 - x \cup \{w\} \cup S_2$ is γ_{wcsd} set of H and $S_1 \cup S_2 - x \cup \{w\}$ is a γ_{wcsd} set of

H. Thus, $S_1 \cup S_2$ is a very excellent γ_{wcsd} set of H, and further, H is a very excellent γ_{wcsd} graph. □

Theorem 3.4. *Let G_1 and G_2 be $wcsd$ very excellent graphs. Let $u_1 \in G_1$ and $u_2 \in G_2$ such that there exist a γ_{wcsd} very excellent sets S_1 of G_1 and S_2 of G_2 so that $u_1 \in S_1$ and $u_2 \in S_2$ and are $S_1 - u_1$ and $S_2 - u_2$ $wcsd$ sets of $G_1 - u_1$ and $G_2 - u_2$. If the graph H obtained from $G_1 \cup G_2$ by identifying the vertices u_1 and u_2 then H is $wcsd$ very excellent graph.*

Proof. Let G_1 and G_2 be $wcsd$ very excellent graphs, and S_1 and S_2 be γ_{wcsd} set of G_1 and G_2, respectively.

Also, let $u_1 \in S_1$ and $u_2 \in S_2$ be such that $S_1 - u_1$ and $S_2 - u_2$ is a $wcsd$ of $G_1 - u_1$ and $G_2 - u_2$, respectively.

Then, H is obtained from $G_1 \cup G_2$ by identifying the vertices u_1 and u_2 and let the vertex be 'u'. Any $A_1 \subseteq V(G_1) - u_1$ and $A_2 \subseteq V(G_2) - u_2$ $wcsd$ sets of $G_1 - N(u_1)$ and $G_2 - N(u_2)$. Then is $A_1 \cup \{u_1\}$ and $A_2 \cup \{u_2\}$ $wcsd$ set of G_1 and G_2, respectively.

Hence $|A_1| \geq \gamma_{wcsd}(G_1) - 1$ and $|A_2| \geq \gamma_{wcsd}(G_2) - 1$.

For any $wcsd$ set D of H,

$$\begin{aligned} G_1 \cap D &\text{ is } wcsd \text{ set of } G_1 \text{ and } G_1 - N(u_1), \\ G_2 \cap D &\text{ is } wcsd \text{ set of } G_2 \text{ and } G_2 - N(u_2). \end{aligned}$$

So for any $wcsd$ set D of H,

$$\begin{aligned} |D| &\geq \gamma_{wcsd}(G_1) + (\gamma_{wcsd}(G_2) - 1), \\ |D| &\geq \gamma_{wcsd}(G_2) + (\gamma_{wcsd}(G_1) - 1), \text{ and} \\ \gamma_{wcsd}(H) &\geq \gamma_{wcsd}(G_1) + \gamma_{wcsd}(G_2) - 1. \end{aligned}$$

Since S_1 and S_2 are γ_{wcsd} set of G_1 and G_2 respectively, $S_1 \cup S_2$ is a γ_{wcsd} set of H.

Claim: $S_1 \cup S_2$ $wcsd$ excellent set of H.

For any vertex $v \in H$ and $v \notin S_1 \cup S_2$, $v \neq u$. If $v \in G_1$, then there exists $w_1 \in S_1$ such that $S_1 - \{w_1\} \cup \{v\}$ is a γ_{wcsd} set of G_1. If $v \in G_2$, then there exists $w_2 \in S_2$ such that $S_2 - \{w_2\} \cup \{v\}$ is a γ_{wcsd} set of G_2.

So, $S_1 - \{w_1\} \cup S_2 - \{w_2\} \cup \{v\} = S_1 \cup S_2 \cup \{v\} - \{w_1, w_2\}$ is a set of H which contains v. Then $S_1 \cup S_2 \cup \{v\} - \{w_1, w_2\}$ is a $wcsd$ very excellent set of H, implying that H is a $wcsd$ very excellent graph. □

4. Conclusion

This paper has attempted to establish new class very excellent graphs with respect to the parameter dominating weakly connected set domination and enabled to study various properties of such graphs. The future scope of study is to make new class of very excellent graphs with respect to the parameter weakly connected point set domination.

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