Advances in Mathematics: Scientific Journal **9** (2020), no.12, 11147–11159
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.12.95

REGIONAL CONTROLLABILITY OF FRACTIONAL EVOLUTION SEMILINEAR SYSTEMS

SID AHMED OULD BEINANE

ABSTRACT. The objective of this article is to address the regional controllability problem for semi-linear systems involving Riemann-Liouville fractional derivatives. Firstly, we characterize a supposition to ensure the existence and uniqueness of mild solutions. Then, the necessary and sufficient conditions of the approximate regional controllability of the fractional evolution semi-linear systems are obtained and proved.

1. INTRODUCTION

Models closest to the problems in the real world can be expressed accurately through fractional differential equations, which involve generalization of integer order differential equations systems, Many applications have been found in the modeling and processes of systems in the fields if aerodynamics, physics, electrical science, viscoelastic [1, 6], control theory, electrochemistry [22], heat conduction [4], electricity mechanics and so forth. More details are provided in [2, 12, 19, 20, 25].

The interest of authors in this field lies in the application of this kind of construction in various fields and is also derived from the development of the theory of fractional calculus itself.

2020 *Mathematics Subject Classification.* 93D15, 93C10, 93B05.
Key words and phrases. Regional controllability, semilinear distributed systems, time fractional diffusion systems.

Controllability, first introduced by Kalman, [13], is useful for analyzing systems. Several authors have studied the concept of controllability in systems with infinite dimensions, using different types of methods to develop good system control. The control of semi-linear systems consisting of a linear part and nonlinear part is one of the most important results obtained in this area. For more details on these topics, see the works presented in [3, 5, 11, 15].

Recently, the probability density function and semigroup theory have been used to give a suitable definition of a mild solution for an evolution equation involving a Riemann–Liouville fractional derivative, which has created sufficient conditions to determine approximate controllability [17, 27, 29, 30].

The term "regional controllability" was studied for the first time by El Jai [7]; it is used to refer to control problems targeting a specific region ω from the whole domain Ω. This concept of the controllability of the distributed parameter system is logical, because it approaches real-world problems. Moreover, it can be applied to systems that cannot be controllable in the whole domain. This concept has been studied extensively and has yielded interesting results (see [7, 14, 23, 24, 28]).

The rest of this paper is presented as follows. Some preliminary results regarding the regional controllability problem and basic definitions, which will be used throughout the following sections, are introduced in the next section. In section 3, we present sufficient conditions for the existence and uniqueness of mild solutions for semi-linear fractional-order $\beta \in (0,1)$ systems, with the Riemann–Liouville fractional derivatives. In section 4, the regional controllability of time fractional semi-linear systems are presented, and necessary and sufficient conditions for regional approximate controllability results are given for fractional abstract Cauchy problems.

2. Preliminaries

Let Ω be an open bounded subset of $I\!R^n$ $(n = 1,2,3)$, and we consider the following fractional semi-linear time system:

$$(2.1) \qquad \begin{cases} D_t^{\beta} y(x,t) & = & Ay(x,t) + Bu(t) + Ny(x,t), & \Omega\times]0,T] \\ \lim\limits_{t\to 0^+} I_t^{\beta-1} y(x,0) & = & y_0(x), & \Omega \end{cases},$$

where I^{β} is the R.L fractional order integral defined in ([16]) by

$$I_t^{1-\beta}g(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}g(s)ds, \;\; \beta > 0 \tag{2.2}$$

and the R.L fractional order derivative D_t^{β} to time t is given in ([16]) by

$$D_t^{\beta}g(t) = \frac{d}{dt}I_t^{1-\beta}g(t), \;\; 0 < \beta \leq 1.$$

Next, $A: \; D(A) \subset L^2(\Omega) \; \rightarrow \; L^2(\Omega)$ is generated by the $S(t)_{t\leq 0}$ strongly continuous semigroup on $L^2(\Omega)$ (see [26], [8], [10]); $N \; : \; [0,T] \times L^2(\Omega) \; \rightarrow \; L^2(\Omega)$ is a nonlinear operator; and $B \; : \; I\!R^n \; \rightarrow \; L^2(\Omega)$, is a control operator, where $u(.) \in \widetilde{U} = \{u \in L^2(0,T; I\!R^p) \mid y_u(T) \in L^2(\Omega)\}$, where $\widetilde{U}$ is a Hilbert space.

In the following, we address definitions as follows.

Lemma 2.1. *(see [9], [10]) Let* $f \in L^2(0,T; L^2(\Omega)), 0 < \beta < 1$, *and* $g \in L^2(0,T; L^2(\Omega))$. *Then*

$$\begin{cases} D_t^{\beta}g(t) & = & Ag(t) + f(t) \qquad t \in [0,T] \\ \\ \lim\limits_{t\to 0^+} D_t^{\beta-1}g(x,0) & = & g_0(x) \in L^2(\Omega) \end{cases} . \tag{2.3}$$

The mild solution of system (2.3) satisfies

$$g(t) = t^{\beta-1}S_{\beta}(t)g_0 + \int_0^t (t-s)^{\beta-1}S_{\beta}(t-s)f(s)ds,$$

where

$$S_{\beta}(t) = \beta\int_0^{\infty} \alpha\varphi_{\beta}(\alpha)S(t^{\beta}\alpha)d\alpha.$$

Here, $\varphi_{\beta} = \frac{1}{\beta}\alpha^{-1-\frac{1}{\beta}}\xi_{\beta}(\alpha^{\frac{-1}{\beta}})$ where ψ_{β} is defined by

$$\xi_{\beta}(\alpha) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{\Gamma(n+1)}{n!}\alpha^{-\beta n-1)}\sin(n\pi\beta), \;\; \alpha > 0,$$

which is called the probability density function. From the arguments in [21], we see that $\xi_{\beta} \;\; (\alpha > 0)$ satisfies the following property:

$$\int_0^{\infty} \xi_{\beta}(\alpha) = 1$$

and

$$\widetilde{\xi_{\beta}}(\lambda) = \int_0^{\infty} e^{-\lambda\beta}\xi_{\beta}(\alpha)d\alpha = e^{-\lambda^{\beta}}, \quad \beta \in (0,1). \tag{2.4}$$

Proof. (see [10], [29]) Using Laplace transforms, we obtain the following:

$$\widetilde{g}(\lambda) = \int_0^\infty e^{-\lambda v} g(v)dv \quad \text{and} \quad \widetilde{f}(\lambda) = \int_0^\infty e^{-\lambda v} f(v)dv;$$

and the system (2.3) is equivalent to (see [18])

$$\lambda^\beta \widetilde{g}(\lambda) - g_0 - A\widetilde{g}(\lambda) = \widetilde{f}(\lambda).$$

Then,

$$\widetilde{g}(\lambda) = (\lambda^\beta I - A)^{-1}(g_0 - \widetilde{f}(\lambda)) = \int_0^\infty e^{-\lambda^\beta v} S(v)[g_0 - \widetilde{f}(\lambda)]dv.$$

Let $v = \tau^\beta$. We obtain

$$\widetilde{g}(\lambda) \;=\; \beta \int_0^\infty e^{-(\lambda\tau)^\beta} S(\tau^\beta)\tau^{\beta-1}[g_0 + \widetilde{f}(\lambda)]d\tau.$$

From (2.4), we obtain

$$e^{-(\lambda\tau)^\beta} = \int_0^\infty e^{-\lambda\tau\alpha} \xi_\beta(\alpha)d\alpha.$$

Then,

$$\begin{aligned}\widetilde{g}(\lambda) &= \beta \int_0^\infty \int_0^\infty e^{-\lambda\tau\alpha} \xi_\beta(\alpha) S(\tau^\beta)\tau^{(\beta-1)}[g_0 + \widetilde{f}(\lambda)]d\alpha d\tau \\ &= I_1(g_0) + I_2(f),\end{aligned}$$

where

$$I_1(g_0) = \beta \int_0^\infty \int_0^\infty e^{-\lambda\tau\alpha} \xi_\beta(\alpha)\phi(\tau^\beta)\tau^{(\beta-1)} d\alpha d\tau g_0$$

and

$$I_2(f) = \beta \int_0^\infty \int_0^\infty e^{-\lambda\tau\alpha} \xi_\beta(\alpha)\phi(\tau^\beta)\tau^{(\beta-1)} d\alpha d\tau \widetilde{f}(\lambda).$$

Suppose that $t = \tau\alpha$. Then, we obtain

$$\begin{aligned}I_1(g_0) &= \beta \int_0^\infty \int_0^\infty e^{-\lambda t} \xi_\beta(\alpha) S(\frac{t^\beta}{\alpha^\beta})\frac{t^{\beta-1}}{\alpha^\beta} d\alpha dt g_0 \\ &= \int_0^\infty e^{-\lambda t} \beta \int_0^\infty \xi_\beta(\alpha) S(t^\beta \alpha^{-\beta}) t^{\beta-1}\alpha^{-\beta} d\alpha dt g_0\end{aligned}$$

and

$$\begin{aligned}I_2(f) &= \beta \int_0^\infty \int_0^\infty \int_0^\infty e^{-\lambda\tau\alpha} \xi_\beta(\alpha) S(\tau^\beta)\tau^{(\beta-1)} e^{-\lambda v} f(v) dv d\alpha d\tau \\ &= \beta \int_0^\infty \int_0^\infty \int_0^\infty e^{-\lambda(t+v)} \xi_\beta(\alpha) S(\frac{t^\beta}{\alpha^\beta})\frac{t^{\beta-1}}{\alpha^\beta} f(v) dv d\alpha dt \\ &= \int_0^\infty e^{-\lambda(t)} \beta \int_0^t \int_0^\infty \xi_\beta(\alpha) S\big(\frac{(t-v)^\beta}{\alpha^\beta}\big)\frac{(t-v)^{\beta-1} f(v)}{\alpha^\beta} d\alpha dv dt.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\widetilde{g}(\lambda) &= \int_0^\infty e^{-\lambda t}\beta\left[\int_0^\infty \xi_\beta(\alpha)S(\frac{t^\beta}{\alpha^\beta})\frac{t^{\beta-1}}{\alpha^\beta}d\alpha g_0 dt\right.\\ &+ \left.\int_0^t\int_0^\infty \xi_\beta(\alpha)S\big(\frac{(t-v)^\beta}{\alpha^\beta}\big)\frac{(t-v)^{\beta-1}f(v)}{\alpha^\beta}d\alpha dv\right]dt.\end{aligned}$$

Now, using the invert Laplace transform, we find that

$$\begin{aligned}g(t) &= \int_0^\infty e^{-\lambda t}t^{\beta-1}\beta\int_0^\infty \frac{1}{\beta}\alpha^{-1-\frac{1}{\beta}}\xi_\beta(\alpha^{-\frac{1}{\beta}})\alpha S(t^\beta\alpha)d\alpha dt g_0\\ &+\int_0^\infty e^{-\lambda(t)}\beta\int_0^t\int_0^\infty \alpha\frac{1}{\beta}\alpha^{-1-\frac{1}{\beta}}\xi_\beta(\alpha^{-\frac{1}{\beta}})\frac{S((t-v)^\beta\alpha)f(v)}{(t-v)^{1-\beta}}d\alpha dv dt.\end{aligned}$$

Let $S_\beta(t)=\beta\int_0^\infty \alpha\varphi_\beta(\alpha)S(t^\beta\alpha)d\alpha$ and $\varphi_\beta=\frac{1}{\beta}\alpha^{-1-\frac{1}{\beta}}\xi_\beta(\alpha^{\frac{-1}{\beta}})$. Then, we obtain

$$g(t)=t^{\beta-1}S_\beta(t)g_0+\int_0^t(t-v)^{\beta-1}S_\beta(t-v)f(v)dv,$$

and the proof is complete. □

According to the above lemma (2.1), we give the following definition.

Definition 2.1. *For a state $y(.,u)\in L^2(\Omega)$, by lemma (2.1), $y(.,u)$ is called a mild solution of (2.1) and can be written as follows:*

$$\begin{aligned}y(t,u) &= t^{\beta-1}S_\beta(t)y_0+\int_0^t(t-v)^{\beta-1}S_\beta(t-v)Bu(v)dv\\ &+ \int_0^t(t-v)^{\beta-1}S_\beta(t-v)N(v,y(v))dv,\end{aligned}$$

For $\omega\subset\Omega$, an open, nonempty and positive Lebesgue measure, we consider the operator restriction:

$$\begin{aligned}\chi_\omega \;:\; L^2(\Omega) &\longrightarrow L^2(\omega)\\ y &\longrightarrow y_{|\omega}\end{aligned}$$

Definition 2.2. *Let $y_0\in L^2(\Omega)$ be the reachable set of systems (2.1) at terminal time T, which can be denoted by $\{\chi_\omega K_T\}(f)=\left\{\chi_\omega y(T,u): u(.)\in\widetilde{U}\right\}$. The system (2.1) is ω-approximately regionally controllable in the subregion ω if $\overline{\{\chi_\omega K_T\}(f)}=L^2(\omega)$.*

Lemma 2.2. *Due to Lemma 3.2 and Lemma 3.3 in ([30]):*
(1) For any fixed $t > 0$,The operator $S_\beta(t)$ is a linear and bounded operator for all $t > 0$, i.e., for any $y \in L^2(\Omega)$,

$$\|S_\beta(t)y\| \leq \frac{M}{\Gamma(\beta)}\|y\|. \tag{2.5}$$

(2) $S_\beta(t)$ $t > 0$ is strongly continuous.

3. Assumptions to guarantee the existence and uniqueness of mild solutions

In the following, we discuss the conditions which guarantee the existence and uniqueness of a fractional evolution semi-linear system involving Riemann–Liouville fractional derivatives.

Below, we present some hypotheses and list them as follows:

(1) (H_1) $S(t)$ is a C_0-semigroup and $S(t)$ is continuous in the uniform operator topology for $t > 0$.

(2) (H_2) There is a function $\psi(.) \in L^2((0,T], R^+)$, and $c > 0$ satisfies

$$\|N(t,z)\| \leq \psi(t) + ct^{1-\beta}\|z\|_{L^2(\Omega)}, \quad \forall\, (t \in (0,T] \text{ and } z \in L^2(\Omega)).$$

(3) (H_3) The function N satisfies

$$\|N(t,z_1) - N(t,z_2)\| \leq L\|z_1 - z_2\|_{L^2(\Omega)},$$

where $L > 0$ is a constant.

For our main result, we introduce

Theorem 3.1. *(see, [29]) If B^n is a contraction on Banach space Z, where n is a positive integer and B is an operator from Z to itself, then B has a unique fixed point on Z.*

Theorem 3.2. *Assume that hypotheses (H_1), (H_2) and (H_3) are satisfied. Then, for each control function $u(.) \in \widetilde{U} = \{u \in L^2(0,T; I\!R^p) \mid y_u(T) \in L^2(\Omega)\}$, the control system (2.1) has a unique mild solution on $L^2_{1-\beta}((0,T]; L^2(\Omega))$.*

Proof. Consider the operator Υ defined by

$$(\Upsilon y)(t) = t^{\beta-1}S_\beta(t)y_0 + \int_0^t (t-v)^{\beta-1}S_\beta(t-v)[Bu(v) + N(v,y(v))]dv.$$

First, under the assumptions of our theorem, it is not difficult to check that Υ maps $L^2_{1-\beta}((0,T];L^2(\Omega))$ into itself.

Next, we show that Υ^n is a contraction operator on $L^2_{1-\beta}((0,T];L^2(\Omega))$. In fact, $\forall t \in (0,T]$ and all $x, y \in L^2_{1-\beta}((0,T];L^2(\Omega))$ we have

$$\begin{aligned} & t^{1-\beta}\|(\Upsilon y)(t) - (\Upsilon y)(t)\| \\ = \ & t^{1-\beta}\|\int_0^t (t-v)^{\beta-1} S_\beta(t-v)[N(v,y(v)) - N(v,y(v))dv]\| \\ \leq \ & t^{1-\beta}\int_0^t (t-v)^{\beta-1}\|S_\beta(t-v)[N(v,x(v)) - N(v,y(v))]\|dv. \end{aligned}$$

Using (2.5), we obtain

$$t^{1-\beta}\|(\Upsilon y)(t) - (\Upsilon y)(t)\| \leq t^{1-\beta}\tfrac{M}{\Gamma(\beta)}\int_0^t (t-v)^{\beta-1}\|N(v,x(v)) - N(v,y(v))dv\|.$$

Then, by hypotheses (H_3),

(3.1)

$$\begin{aligned} t^{1-\beta}\|(\Upsilon y)(t) - (\Upsilon y)(t)\| & \leq t^{1-\beta}\tfrac{LM}{\Gamma(\beta)}\int_0^t (t-v)^{\beta-1}\|x(v)-y(v)\|dv \\ & \leq \tfrac{LM}{\Gamma(\beta)}\int_0^t t^{1-\beta}(t-v)^{\beta-1}v^{\beta-1}v^{1-\beta}\|x(v)-y(v)\|dv \\ & \leq \tfrac{\Gamma(\beta)LMt^\beta}{\Gamma(2\beta)}\|x-y\|_{L^2_{1-\beta}((0,T];L^2(\Omega))}. \end{aligned}$$

By induction on n, using (3.1), we can easily find that

$$t^{1-\beta}\|(\Upsilon^n y)(t) - (\Upsilon^n y)(t)\| \leq \tfrac{\Gamma(\beta)(LMt^\beta)^n}{\Gamma((n+1)\beta)}\|x-y\|_{L^2_{1-\beta}((0,T];L^2(\Omega))}.$$

Then,

$$\|(\Upsilon^n y) - (\Upsilon^n y)\|_{L^2_{1-\beta}((0,T];L^2(\Omega))} \leq \tfrac{\Gamma(\beta)(LMT^\beta)^n}{\Gamma((n+1)\beta)}\|x-y\|_{L^2_{1-\beta}((0,T];L^2(\Omega))}.$$

Moreover, let $E_{\beta,\gamma}(t) := \Sigma_{k=0}^{\infty}\dfrac{t^k}{\Gamma(\beta k+\gamma)}$—the Mittag–Leffler series, which is uniformly convergent for all $t \in (0,T]$. If $t = (LMT^\beta)$, then $\frac{\Gamma(\beta)(LMT^\beta)^n}{\Gamma((n+1)\beta)}$ is the general term $E_{\beta,1}(t)$; then, for a sufficiently large n, we can obtain

$$\frac{\Gamma(\beta)(LMT^\beta)^n}{\Gamma((n+1)\beta)} < 1.$$

Finally, according to Theorem 3.1 (see [29]), Υ^n is a contradiction; thus, Υ has a unique fixed point and satisfies the solution of (2.1). □

4. Regional controllability of the semi-linear system

In this section, we formulate and prove conditions for the approximate regional controllability of the semi-linear control system results with fractional evolution.

We define the bounded and linear operator $\Theta : L^2((0,T];L^2(\Omega)) \to L^2(\Omega)$ by

$$\Theta h = \int_0^T (t-v)^{\beta-1} S_\beta(t-v)h(v)dv, \qquad h(.) \in L^2((0,T];L^2(\Omega))$$

and $\Theta_\omega : L^2(\Omega) \to L^2(\omega)$, $\Theta_\omega h = \chi_\omega \Theta h$. In what follows, we assume that $S_\beta(t)y_0 \in Im\Theta_\omega$. We denote the Nemytskil operator corresponding to the nonlinear function N by

$$\Lambda_N : L^2_{1-\beta}((0,T];L^2(\Omega)) \;\; \to \;\; L^2(\Omega) \quad \Lambda_N(y)(t) = N(t, y(t)).$$

Then, the mild solution can be presented as

$$y(.,u) = t^{\beta-1} S_\beta(t)y_0 + \Theta Bu(t) + \Lambda_N(y(t)) \quad \forall \;\; t \in (0,T].$$

From definition (2.2), we know that if for any $y_0 \in L^2(\Omega)$ and $u(.) \in \widetilde{U}$, the system (2.1) is ω-approximately regionally controllable on $(0,T]$ if and only if $\overline{\{\chi_\omega K_T\}(f)} = L^2(\omega)$. Equivalently, if for any $\epsilon > 0$ and every desired state at time T denoted $y_d \in L^2(\omega)$,

$$\|y_d - \chi_\omega y(T,u)\| = \|y_d - T^{\beta-1}\chi_\omega S_\beta(T)y_0 - \Theta_\omega B u_\epsilon - \chi_\omega \Lambda_N(y_\epsilon)\|, \tag{4.1}$$

where $\chi_\omega y_\epsilon = \chi_\omega y(t;0,y_0,u_\epsilon)$, then system (2.1) is approximately regional controllable on $(0,T]$. Now, we can introduce the following suppositions:

(1) (H_3') There exists a constant L' such that

$$\|N(t,x) - N(t,y)\| \leq \;\; t^{1-\beta} L' \|x - y\|_{L^2_{1-\beta}((0,T];L^2(\Omega))}.$$

(2) (H_4) For all $\epsilon > 0$, $\exists \;\; u \in L^2((0,T];L^2(\Omega))$ satisfies

$$\|\Theta_\omega(\zeta) - \Theta_\omega Bu\|_{L^2(\omega)} < \epsilon \tag{4.2}$$

and

$$\|Bu(.)\|_{L^2((0,T];L^2(\Omega))} \leq q\|\zeta(.)\|_{L^2((0,T];L^2(\Omega))}, \tag{4.3}$$

where $\zeta(.) \in L^2((0,T]; L^2(\Omega))$ and q is constant which is independent of $\zeta(.)$ satisfies

$$\frac{L'Mq}{\Gamma(\beta)}\left(\sqrt{\frac{T}{2\beta-1}}\right)E_{(\beta,1)}(L'MT) < 1. \tag{4.4}$$

Given that (H_3') is stronger than (H_3), if (H_1), (H_2) and (H_3') hold, according to theorem 3.2, the control system (2.1) has a unique mild solution on $L^2_{1-\beta}((0,T]; L^2(\Omega))$.

Lemma 4.1. *(see, [29]) We suppose that N satisfies the conditions (H_2) and (H_3'). Then, the following inequalities are satisfied by the mild solution of system (2.1):*

$$\|\chi_\omega y(t;0,y_0,u)\|_{L^2_{1-\beta}((0,T];L^2(\omega))} \leq \nu E_{(\beta,1)}(cMT), \qquad \forall u(.) \in L^2((0,T]; L^2(\Omega)).$$

For any $u_1(.), u_2(.) \in L^2((0,T]; L^2(\Omega))$,

$$\|\chi_\omega y_1(.) - \chi_\omega y_2(.)\|_{L^2_{1-\beta}((0,T];L^2(\omega))} \leq \mu E_{(\beta,1)}(L'MT)\|Bu_1(.) - Bu_2(.)\|_{L^2(\Omega)},$$

where

$$\nu = \frac{M}{\Gamma(\beta)}\left[\|y_0\| + \left(\sqrt{\frac{1}{2\beta-1}}\right)(\|Bu\|_{L^2(\Omega)} + \|\psi(t)\|_{L^2(\Omega)})\sqrt{T}\right],$$

and $\mu = \frac{M}{\Gamma(\beta)}\left(\sqrt{\frac{1}{2\beta-1}}\right)\sqrt{T}$.

Proof. The proof is similar to lemma 4.1 in [29]. □

Lemma 4.2. *(see [30], [29]) Let $S(t)$ be a differentiable semigroup generated by A. Then, for $y \in L^2(\Omega)$, we have*

$$S_\beta(t)y \in D(A) \quad \forall t > 0,$$

$$S_\beta(t)S_\beta(z) = S_\beta(z)S_\beta(t) \quad \forall t, z > 0,$$

and

$$\frac{d\, S_\beta^2(t)y}{dt} = 2S_\beta(t)\frac{d\, S_\beta(t)y}{dt} \quad \forall t > 0.$$

Theorem 4.1. *Assume that hypotheses (H_2), (H_3') and (H_4) are satisfied. Then, system (2.1) is approximately regional controllable on $(0,T]$ if A generates a differentiable semigroup $S(t) \in L^2(\Omega)$.*

Proof. Let $y_d \in D(A)$. Since $\overline{D(A)} \in L^2(\Omega)$, we can prove by $D(A) \subset \{\chi_\omega K_T(N)\}$ the set of reachable states equivalently, if for any $\epsilon > 0$ and every desired state at time T denoted $y_d \in L^2(\omega)$ there is a control function $u_\epsilon(.) \in \widetilde{U}$, satisfies

$$\|y_d - \chi_\omega y(T, u_\epsilon)\|_{L^2(\omega)} = \|y_d - T^{\beta-1}\chi_\omega S_\beta(T)y_0 - \Theta_\omega Bu_\epsilon - \Theta_\omega \Lambda_N(y_\epsilon)\|_{L^2(\omega)}. \tag{4.5}$$

Firstly, we know that for any $y_0 \in L^2(\Omega)$, $T^{\beta-1}\chi_\omega S_\beta(T)y_0 \in D(A)$; therefore, for all $y_d \in D(A)$, there exists a function $\zeta(,) \in L^2(\omega)$ such that $\Theta_\omega \zeta = y_d - T^{\beta-1}\chi_\omega S_\beta(T)y_0$.

Then, (4.5) can be written

$$\|y_d - \chi_\omega y(T, u_\epsilon)\|_{L^2(\omega)} = \|\Theta_\omega \zeta - \Theta_\omega Bu_\epsilon - \Theta_\omega \Lambda_N(y_\epsilon)\|_{L^2(\omega)}$$

For any $\epsilon > 0$ and $u_1(.) \in \widetilde{U}$ by (4.2) in (H_4), there exists $u_2(.) \in \widetilde{U}$, such that

$$\|y_d - T^{\beta-1}\chi_\omega S_\beta(T)y_0 - \Theta_\omega \Lambda_N(y_1) - \Theta_\omega Bu_2\|_{L^2(\omega)} \leq \frac{\epsilon}{2^2}.$$

Since $\chi_\omega y(T, u_2) = T^{\beta-1}\chi_\omega S_\beta(T)y_0 - \Theta_\omega Bu_2 - \Theta_\omega \Lambda_N(y_2)$

$$\begin{aligned}
& \|y_d - T^{\beta-1}\chi_\omega S_\beta(T)y_0 - \Theta_\omega \Lambda_N(y_1) - \Theta_\omega Bu_2\|_{L^2(\omega)} \\
= \; & \|y_d - T^{\beta-1}\chi_\omega S_\beta(T)y_0 - \Theta_\omega \Lambda_N(y_1) - \chi_\omega y(T, u_2) + T^{\beta-1}\chi_\omega S_\beta(T)y_0 \\
& +\Theta_\omega \Lambda_N(y_2)\|_{L^2(\omega)} \\
= \; & \|y_d - \chi_\omega y(T, u_2) + \Theta_\omega[\Lambda_N(y_2) - \Lambda_N(y_1)]\|_{L^2(\omega)} \leq \tfrac{\epsilon}{2^2}.
\end{aligned} \tag{4.6}$$

Taking (4.6) and using (4.2) in (H_4) again, there exists $u_3(.) \in \widetilde{U}$ such that

$$\|\Theta_\omega[\Lambda_N(y_2) - \Lambda_N(y_1)] - \Theta_\omega Bu_3\|_{L^2(\omega)} \leq \frac{\epsilon}{2^3}.$$

Using (4.3) in (H_4), we obtain

$$\|Bu_3(.)\|_{L^2(\Omega)} \leq q\|\Lambda_N(y_2)(.) - \Lambda_N(y_1)(.)\|_{L^2((0,T];L^2(\Omega))}.$$

From Lemma (4.1), we obtain

$$\begin{aligned}
\|Bu_3(.)\|_{L^2(\Omega)} & \leq qL'T^{\beta-1}\|y_2(.) - y_1(.)\|_{L^2((0,T];L^2(\Omega))} \\
& = qL'\|y_2(.) - y_1(.)\|_{L^2_{\beta-1}((0,T];L^2(\Omega))} \\
& \leq qL'\mu E_{(\beta,1)}(L'MT)\|Bu_1(.) - Bu_2(.)\|_{L^2(\Omega)},
\end{aligned}$$

where $\mu = \frac{M}{\Gamma(\beta)}\left(\sqrt{\frac{1}{2\beta-1}}\right)\sqrt{T}$.

Let $w_1 = u_2 - u_3$, $w_1 \in \widetilde{U}$. Then, we obtain

$$\begin{aligned}
&\|y_d - T^{\beta-1}\chi_\omega S_\beta(T)y_0 - \Theta_\omega\Lambda_N(y_2) - \Theta_\omega Bw_1\| \\
&= \|y_d - T^{\beta-1}\chi_\omega S_\beta(T)y_0 - \Theta_\omega\Lambda_N(y_2) - \Theta_\omega Bu_2 + \Theta_\omega Bu_3\| \\
&= \|y_d - T^{\beta-1}\chi_\omega S_\beta(T)y_0 - \Theta_\omega\Lambda_N(y_1) + \Theta_\omega\Lambda_N(y_1) \\
&\quad -\Theta_\omega\Lambda_N(y_2) - \Theta_\omega Bu_2 + \Theta_\omega Bu_3\| \\
&\leq \|y_d - T^{\beta-1}\chi_\omega S_\beta(T)y_0 - \Theta_\omega\Lambda_N(y_1) - \Theta_\omega Bu_2\| \\
&\quad +\|\Theta_\omega Bu_3 - \Theta_\omega[\Lambda_N(y_2) - \Theta_\omega\Lambda_N(y_1)]\| \\
&\leq \left(\tfrac{1}{2^2} + \tfrac{1}{2^3}\right)\epsilon.
\end{aligned}$$

Thus, we can obtain a sequence $\{u_n\} \subset \widetilde{U}$ by induction as follows:

$$\|y_d - T^{\beta-1}\chi_\omega S_\beta(T)y_0 - \Theta_\omega\Lambda_N(y_n) - \Theta_\omega Bu_{n+1}\| \leq \left(\frac{1}{2^2} + \cdots + \frac{1}{2^n}\right)\epsilon, \|, \quad \forall t \in (0,T],$$

and

$$\|Bu_{n+1}(.) - Bu_n(.)\|_{L^2(\Omega)} \leq qL'\mu E_{(\beta,1)}(L'MT)\|Bu_{n+1}(.) - Bu_n(.)\|_{L^2(\Omega)}.$$

Using (4.4) in (H_4), it easy to determine that $\{Bu_n,\, n = 1,2,\cdots,\}$ is a Cauchy sequence on $L^2((0,T];L^2(\Omega))$. Then, $\{Bu_n,\, n = 1,2,\cdots\}$ has a subsequence Bu_K, (where K is positive integer number) which converges, i.e., $\forall\epsilon > 0$, and $Bu_n(.) \in L^2((0,T];L^2(\Omega))$ satisfies

$$\|\Theta_\omega Bu_{n+1}(.) - \Theta_\omega Bu_n(.)\|_{L^2(\omega)} \leq \frac{\epsilon}{2}.$$

Therefore, we obtain

$$\begin{aligned}
&\|y_d - T^{\beta-1}\chi_\omega S_\beta(T)y_0 - \Theta_\omega\Lambda_N(y_K) - \Theta_\omega Bu_K\|_{L^2(\omega)} \\
&= \|y_d - T^{\beta-1}\chi_\omega S_\beta(T)y_0 - \Theta_\omega\Lambda_N(y_K) - \Theta_\omega Bu_{K+1} \\
&\quad +\Theta_\omega Bu_{K+1} - \Theta_\omega Bu_K\|_{L^2(\omega)} \\
&\leq \|y_d - T^{\beta-1}\chi_\omega S_\beta(T)y_0 - \Theta_\omega\Lambda_N(y_K) - \Theta_\omega Bu_{K+1}\|_{L^2(\omega)} \\
&\quad +\|\Theta_\omega Bu_{K+1} - \Theta_\omega Bu_K\|_{L^2(\omega)} \\
&\leq \left(\tfrac{1}{2^2} + \cdots + \tfrac{1}{2^n}\right)\epsilon + \tfrac{\epsilon}{2} < \epsilon
\end{aligned}$$

Now, this proves $y_d \in D(A)$ and then, $y_d \in \{\chi_\omega K_T(N)\}$; thus, system (2.1) is approximately regionally controllable on $(0,T]$. □

REFERENCES

[1] R. L. BAGLEY, P. J. TORVIK: *A theoretical basis for the application of fractional calculus to viscoelasticity*, Journal of Rheology, **27**(3) (1983), 201–210.

[2] D. BALEANU, A. K. GOLMANKHANEH: *On electromagnetic field in fractional space*, Nonlinear Anal. Real Word Appl., **11** (2010), 288–292.

[3] A. E. BASHIROV, N. I. MAHMUDOV: *On concepts of controllability for deterministic and stochastic systems*, SIAM J. Control Optim., **37** (1999), 1808–1821.

[4] J. L. BATTAGLIA, J. CH. BATSALE, L. LE LAY, A. OUSTALOUP, O. COIS: *Heat flux estimation through inverted non-integer identification models,* International Journal of Thermal Sciences, **39**(3) (2000), 374–389.

[5] M. BENCHOHRA, A. OUAHAB: *Controllability results for functional semilinear differential inclusions in Frechet spaces*, Nonlinear Anal.: TMA, **61** (2005), 405–423.

[6] G. CATANIA, S. SORRENTINO: *Analytical modelling and experimental identification of viscoelastic mechanical systems,* Advances in Fractional Calculus, 403–416, Springer, Dordrecht (2007).

[7] A. EL JAI, A. J. PRITCHARD, M. C. SIMON, E. ZERRIK:*Regional controllability of distributed systems*, International journal of control, **62**(6) (1995), 1351–1365.

[8] K.-J. ENGEL, R. NAGEL: *A Short Course on Operator Semigroups, Springer*, New York, 2006.

[9] F. GE, Y. Q. CHEN, C. KOU: *Regional controllability of anomalous diffusion generated by the time fractional diffusion equations*, ASME IDETC/CIE 2015, Boston, August 2-5, Paper No. DETC2015-46697, V009T07A031; 8 pp(2015).

[10] F. GE, Y. Q. CHEN, C. KOU: *Regional controllability analysis of fractional diffusion equations with Riemann–Liouville time fractional derivatives*, Automatica, **76** (2017), 193–199.

[11] L. GÓRNIEWICZ, S. K. NTOUYAS, D. ÒREGAN: *Controllability of semilinear differential equations and inclusions via semigroup theory in Banach spaces*, Rep. Math. Phys.m **56** (2005), 437–470.

[12] E. HERNANDEZ, D. ÒREGAN, E. BALACHANDRAN: *On recent developments in the theory of abstract differential equations with fractional derivatives*, Nonlinear Anal., **73** (2010), 3462–3471.

[13] R. E. KALMAN: *Controllablity of linear dynamical systems*, Contrib. Diff. Equ., **1** (1963), 190–213.

[14] A. KAMAL, A. BOUTOULOUT, S. A. O. BEINANE: *Regional Controllability of Semi-Linear Distributed Parabolic Systems: Theory and Simulation*, Intelligent Control and Automation, **3**(2) (2012), 146–158.

[15] J. KLAMKA: *Constrained controllability of semilinear systems with delays*, Nonlinear Dynam., **56** (2009), 169–177.

[16] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO: *Theory and applications of fractional differential equations*, Elsevier Science Limited, 2006.

[17] S. KUMAR, N. SUKAVANAM: *Approximate controllability of fractional order semilinear systems with bounded delay*, J. Differential Equations, **252** (2012), 6163–6174.

[18] S.-D. LIN, C.-H.LU: *Laplace transform for solving some families of fractional differential equations and its applications*, Advances in Difference Equations, (2013), art.no.137.

[19] Z. H. LIU, J. F. HAN: *Integral boundary value problems for fractional order integrodifferential equations*, Dynam. Systems Appl., **21** (2012), 535–548.

[20] Z. H. LIU, J. H. SUN, I. SZ'ANT'O: *Monotone iterative technique for Riemann-Liouville fractional integro-differential equations with advanced arguments*, Results in Math., **63** (2013), 1277–1287.

[21] F. MAINARDI, P. PARADISI, R. GORENFLO: *Probability distributions generated by fractional diffusion equations*, (2007), arXiv preprint arXiv:0704.0320.

[22] K. B. OLDHAM: *Fractional differential equations in electrochemistry*. Advances in Engineering Software, **41**(1) (2010), 9–12.

[23] S. OULD BEINANE, A. KAMAL, A. BOUTOULOUT: *Regional gradient controllability of semi-linear parabolic systems*, International Review of Automatic Control, **6**(5) (2003), 641–653.

[24] M. OULD SIDI, S. . BEINANE: *Regional gradient optimal control problem governed by a distributed bilinear systems*, Telkomnika, **17**(4) (2019), 1957–1965.

[25] I. PODLUBNY: *Fractional Differential Equations*, Academic Press, San Diego, CA, 1999.

[26] M. RENARDY, R. C. ROGERS: *An Introduction to Partial Differential Equations*, Springer-Verlag, New York, 2004.

[27] R. SAKTHIVEL, Y. RENB, N. I. MAHMUDOVIC: *On the approximate controllability of semilinear fractional differential systems*, Computers and Mathematics with Applications, **62** (2011), 1451–1459.

[28] E. ZERRIK, A. KAMAL: *Output controllability for semi-linear distributed parabolic systems*, Journal of Dynamical and Control Systems, **13**(2) (2007), 289–306.

[29] Z. LIU, X. LI: *Approximate Controllability of Fractional Evolution Systems with Riemann–Liouville Fractional Derivatives*, SIAM Journal on Control and Optimization, **53**(4) (2015), 1920–1933.

[30] Y. ZHOU, F. JIAO: *Existence of mild solutions for fractional neutral evolution equations*, Comput. Math. Appl., **59** (2010), 1063–1077.

MATHEMATICS DEPARTMENT
COLLEGE OF SCIENCE, JOUF UNIVERSITY
SAKAKA P.O.BOX 2014, SAUDI ARABIA
Email address: beinane06@gmail.com, sabeinane@ju.edu.sa