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ANALYSIS OF MINIMUM VERTEX COVER ON TORI AND CENTRALLY CONNECTED TORUS NETWORKS

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ABSTRACT. The concept of vertex cover problem is motivated by the design of secure protocols for communication in interconnection networks. A tori and its variants are popular interconnection networks of massively parallel systems. The centrally connected torus networks have been introduced recently, building on the interesting properties of tori and enabling nodes clustering. In this paper, we solve the vertex cover problem in tori and centrally connected torus networks. Furthermore, we discuss the invertibility of these networks in terms of its bipartiteness.

1. INTRODUCTION

Combinatorial properties have been more important recently in the study of reliability and fault tolerance in interconnection networks. An interconnection network consists of hardware and software entities that are interconnected to facilitate efficient computation and communication. Since the interconnection networks can be represented as an undirected graph, where a processor is represented as a node and a communication link between processors as an edge between corresponding nodes, there are many graph theoretical parameters to describe the efficiency of a communication network. The minimum vertex cover

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problem in interconnection network consists of finding the minimum number of nodes which covers all the communication links.

Karp in 1931 proved that the vertex cover problem is NP complete for general graphs, since then this problem plays a vital role in the development of approximation algorithms and parameterized complexity. The minimum vertex cover problem is known to be APX approximable complete and NP hard in planar graphs [2, 4]. Graphs for which the domination and vertex covering numbers are equal are characterized by Yunjian Wu et. al. Lucas Jacques demonstrated a method to apply the vertex cover for solving shortest path problems. A tile assembly model has been proposed by Fan Wu et. al. to obtain a molecular solution for minimum vertex cover problem. Various graph characterizations have been done on covering relating to domination parameters and matching numbers and new graph parameters such as average covering number, strong covering number and inverse vertex cover are being defined [3, 5]. There are many variants of the vertex cover problem namely, the connected vertex cover, weighted vertex cover, capacitated vertex cover, inverse vertex cover and total vertex cover. However, in particular only recently has a systematic study of the minimum vertex cover and the inverse covering problems have been initiated especially for special classes of non-bipartite graphs.

The MVCP and the inverse covering problem has not been studied so far, especially for interconnection networks. Tori and CCTorus are used in the representation of interconnection networks. These graphs are excellent models for interconnection networks and are an integral part of any parallel processing or distributed system. The vertices in such graphs correspond to processing elements, memory modules, and the edges correspond to communication lines.

There is a very large literature devoted to vertex cover algorithms. In contrast there has been surprisingly little work done on covering problems in interconnection networks. The goal of this paper is to demonstrate that, the covering sets enhance the communication in interconnection networks. Motivated by the wide applications of the vertex cover problem [2] and in this paper we are concerned about the relation between vertex covering and edge covering parameters of Tori-CCTorus networks. We recall the notion of covering as follows.

Let G be a finite, simple undirected graph, $V(G)$ and $E(G)$ denote its vertex set and its edge set respectively. A vertex (edge) cover of a graph G is a set of vertices (edges) such that each edge (vertex) of G is incident with at least one vertex (edge) of the set. The vertex (edge) cover number of G denoted by $\beta(G)$ $(\beta'(G))$ is the minimum of the cardinalities of all vertex (edge) covers.

2. Invertible Graphs

While studying the vertex cover problem for several well-known interconnection networks, we compared the values of the covering number with a new parameter related to vertex covering which is called the inverse covering number. A set $S \subseteq V - D$ which is a vertex covering of G is called an inverse vertex covering of G with respect to the minimum vertex cover D. The inverse vertex covering number $\beta^{-1}(G)$ is the inverse covering set of smallest size of G. A graph G is said to be invertible if G admits an inverse vertex covering [1].

All graphs have vertex covers but not all graphs possess an inverse cover. Such graphs which possess inverse cover are called invertible graphs. The important property of invertible graphs is that we can find an independent covering set. Independent sets play a vital role in coding theory, map labeling, airline crew scheduling, geometric tiling and molecular biology. Most of the well-known parallel architectures are invertible graphs. For example, hypercube, butterfly and Benes are invertible graphs whereas some of the non-invertible networks include wrapped butterflies of odd dimension and odd Petersen graphs.

3. Tori network $T(n, m)$

The $T(n, m)$ Tori network is a hierarchical interconnection network. Its definition is as follows. A $T(n, m)$ Tori network is a super graph of the $G(n, m)$ grid with the additional edge set $\{\{(0, j), (n-1, j)\} | 0 \leq j \leq m-1\} \cup \{\{(i, 0), (i, m-1)\} | 0 \leq i \leq n-1\}$. $T(n)$ which is a 1-dimensional torus is simply a cycle or ring. Tori networks are bipartite if and only if all side lengths are even. Any torus has a Hamiltonian cycle. These networks are regular and vertex symmetric and are diametrically uniform graphs. The automorphism group consists of translations. The readers may refer to [3] in order to understand the structural properties of toroidal meshes. The tori mesh networks are applied in military communication, medical monitoring and security systems and so on.

Theorem 3.1. *Let G be a $T(n, m)$ Tori network. Then for $m \neq n$, $\beta(G) = m \times \lceil \frac{n}{2} \rceil$.*

Proof. Given that G is a $T(n,m)$ Tori network. We observe that, each $T(n,m)$ has m rows in which each row is a path of length n which are mutually disjoint.
Case (i): When n is odd.
To find a minimum vertex cover of G, we start from the left most top vertex and choose $\frac{n+1}{2}$ vertices alternatively in each row. Hence for m number of rows in G, we need to choose $\frac{m(n+1)}{2}$ vertices to cover all the edges of G.
Case (ii): When n is even.
Since each path is of even length n, we choose $\frac{n}{2}$ vertices from each row. Hence for m number of rows in G, we need to choose $\frac{mn}{2}$ vertices to cover all the edges of G. Hence combining both the cases, for all m and n, we have, $\beta(G) = m \times \lceil \frac{n}{2} \rceil$. □

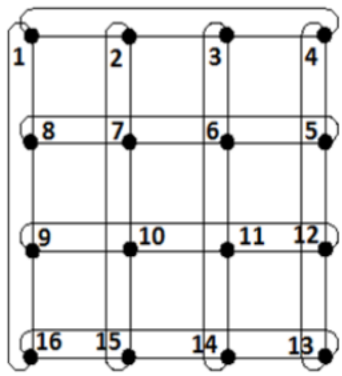

FIGURE 1. $T(4,4)$ Tori network

Theorem 3.2. *Let G be a $T(n,m)$ Tori network. Then for $m = n$,*

$$\beta(G) = \begin{cases} \left\lceil \frac{m^2}{2} \right\rceil + 2\left\lfloor \frac{m}{2} \right\rfloor & \text{if } n \text{ is odd} \\ \left\lceil \frac{m^2}{2} \right\rceil & \text{if } n \text{ is even} \end{cases}.$$

Proof.

Case (i): When m is odd.

To find a minimum vertex cover of G, we start from the left most top vertex and choose $\lceil \frac{m}{2} \rceil$ vertices alternatively in each row.

Then starting from row one, the alternate row (there are $\lceil \frac{m}{2} \rceil$ number of such rows) vertices are chosen. Thus, we first need to choose $\lceil \frac{m}{2} \rceil \times \lceil \frac{m}{2} \rceil$ vertices.

Then from the remaining $\lfloor \frac{m}{2} \rfloor$ rows, another $\lfloor \frac{m}{2} \rfloor$ vertices are chosen (that is, $\lfloor \frac{m}{2} \rfloor \times \lfloor \frac{m}{2} \rfloor$ number of vertices), so that all the straight edges in G are covered.

Finally, considering the horizontal curved edges and vertical curved edges of $T(n,m)$ which are not covered, we select $2\left\lfloor \frac{m}{2} \right\rfloor$ vertices.

Totally $\beta(G) = \left\lceil \frac{m}{2} \right\rceil \times \left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor \times \left\lfloor \frac{m}{2} \right\rfloor + 2\left\lfloor \frac{m}{2} \right\rfloor$.

Simplifying the expression we get, $\beta(G) = \left\lceil \frac{m^2}{2} \right\rceil + 2\left\lfloor \frac{m}{2} \right\rfloor$.

Case (ii): When m is even.

Since each path is of even length m, we choose $\frac{m}{2}$ vertices from each row.

Hence for m number of rows in G, we need to choose $\frac{m^2}{2}$ vertices to cover all the edges of G.

The vertices so chosen will cover all the curved edges.

Thus $\beta(G) = \frac{m^2}{2}$. □

It is well known that in a bipartite graph, the size of the minimum vertex cover is equal to the size of the maximum matching. A maximum matching which matches all the vertices of a graph is always a minimum edge cover. So we investigate the edge covering number of tori networks.

Theorem 3.3. *Let G be a $T(n,m)$ Tori network. Then for all m and n, $\beta'(G) = \left\lceil \frac{m \times n}{2} \right\rceil$.*

Proof. Given that G is a $T(n,m)$ Tori network. Case (i): If m and n both are even, then G contains a perfect matching. Since every perfect matching is also a minimum edge cover, we have, $\beta'(G) = \frac{|V|}{2} = \frac{m \times n}{2}$. Case (ii): If either m or n is odd, then G has a near perfect matching. Hence $\beta'(G) = \frac{|V|}{2} + 1 = \frac{m \times n}{2} + 1 = \left\lceil \frac{m \times n}{2} \right\rceil$. Combining both the cases, $\beta'(G) = \left\lceil \frac{m \times n}{2} \right\rceil$, for all m and n. □

Theorem 3.4. *Let G be a $T(n,m)$ Tori network. Then G is invertible for even values of m and n.*

Proof. Let G be a $T(n,m)$ Tori network. Since G do not contain any bridges and are bipartite if both m and n are even, G is invertible. □

4. Centrally Connected Torus networks $n \times n$ CCTorus

This is a new interconnection topology which has high throughput and better scalability than grid and tori networks [5]. It is a new modified version of the $T(n,n)$ Tori network structure. The diameter of a $n \times n$ Tori network is n whereas the $n \times n$ CCTorus has a shortest diameter which is $(n-1)$. The construction of $n \times n$ CCTorus is as follows:

(i) First we design a $T(n,n)$ Tori network.

(ii) Then we identity the center of that $T(n,n)$.

(a) If n is odd then there is only one center for a $n \times n$ CCTorus which is given as, Center node $c = \frac{n^2-1}{2}$. For example, node labelled 12 is the center of a 5×5 CCTorus network as shown in figure 2.

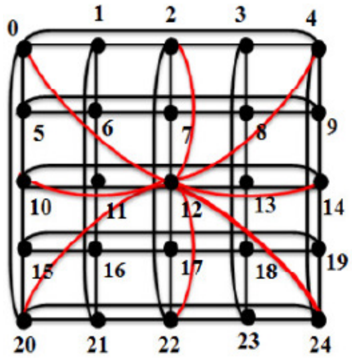

FIGURE 2. 5×5 CCTorus network

(b) There are four center nodes when n is even. The four centers are calculated using the formulas below. For instance in fig. 3, the nodes labelled 14, 15, 20 and 21 are the centers a, b, c and d respectively of a 6×6 CCTorus network.

Center node $a = \left(\frac{n}{2} - 1\right)(n+1)$.
Center node $b = \frac{n}{2}(n+1) - n$.
Center node $c = \frac{n}{2}(n+1) - 1$.
Center node $d = \frac{n}{2}(n+1)$.

(iii) We then connect the centers to all the four corner nodes and middle boundary nodes.

Theorem 4.1. *Let G be a $n \times n$ CCTorus network. Then*

$$\beta(G) = \begin{cases} \left\lceil \frac{n^2}{2} \right\rceil + 2\left\lfloor \frac{n}{2} \right\rfloor, & \text{if } n \text{ is odd} \\ \left\lceil \frac{n^2}{2} \right\rceil + 2, & \text{if } n \text{ is even} \end{cases}.$$

Proof. Given that G is a $n \times n$ CCTorus network. The proof of this theorem follows the same argument as in theorem 3.2. To find a minimum vertex cover

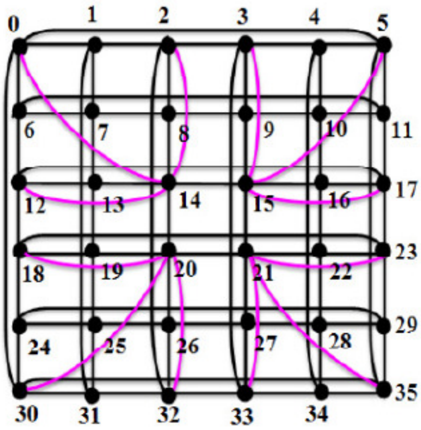

FIGURE 3. 6×6 CCTorus network

of G, we need to consider only the new edges which have been added to $T(n, n)$ Tori network while constructing $n \times n$ CCTorus network.

Case (i): When n is odd. All the extra edges that is, the red colored edges added to construct the $n \times n$ CCTorus are incident on the center node c. (See figure 2) By the proof as in case (i) of theorem 3.2 the center vertex is already contained in the minimum vertex cover set if n is odd. Hence covering number of $T(n, n)$ and $n \times n$ CCTorus networks are equal when n is odd.

Case (ii): When n is even. The extra edges that is, the pink colored edges added to construct the $n \times n$ CCTorus are incident on the center nodes a, b, c and d. (See fig. 3) By the proof as in case (ii) of theorem 3.2 the center vertices a and d are already contained in the minimum vertex cover set if n is even. Thus the extra 6 edges are covered by nodes a and d. Considering the remaining 6 uncovered edges incident on the other two center nodes namely b and c, we have $\beta(G) = \beta(T(n, n)) + 2$. □

Theorem 4.2. *A $n \times n$ CCTorus network is non-bipartite for all n.*

Proof. Given that G is a $n \times n$ CCTorus network. Since $T(n, n)$ Tori networks are bipartite if and only if n is even, $n \times n$ CCTorus networks should also be bipartite. But due to the extra edges that are present in $n \times n$ CCTorus network there exists at least one odd cycle, for all n. This implies that $n \times n$ CCTorus network is non-bipartite for all n. □

Theorem 4.3. *Let G be a $n \times n$ CCTorus network. Then G is not invertible for all n.*

Proof. Let G be a $n \times n$ CCTorus network. Since G is non-bipartite for all n, G is not invertible. □

5. Conclusion

The minimum vertex covering sets for any structural model of parallel computation is both useful for the construction of efficient algorithms for that structure. The exact values of the covering parameters of Tori and centrally connected tori networks are obtained. CCTorus not only inherits good symmetry from Torus, but also further reduces the average latency and increases average throughput as compared to other Torus like topologies. Moreover, it also provides fault tolerance because of many shortest paths available between node pairs. Furthermore, we have shown that the tori networks are invertible. The above results can also be applied to study the covering parameters of various other parallel architectures like pancake and pyramid networks.

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