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A STUDY ON PYTHAGOREAN FUZZY GRAPH COLORING

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ABSTRACT. A new concept of coloring of Pythagorean fuzzy graphs with Pythagorean fuzzy color is introduced. The node, arc and total coloring of Pythagorean fuzzy graph using Pythagorean fuzzy colors are discussed along with Pythagorean fuzzy chromatic numbers.

1. INTRODUCTION

Fuzzy coloring has various application in scheduling theory, assignment problem, timetabling problem, traffic flow problem, allocation problem etc. The concept of coloring the fuzzy graph was originated by Eslahchi and Onaghe [1] while the method of coloring the fuzzy graph with crisp nodes and fuzzy arcs was introduced by Munoz et.al [3]. They also proposed the idea of fuzzy chromatic number and discussed two different ways of coloring the fuzzy graphs. The coloring method of fuzzy graphs with fuzzy node and fuzzy arc was introduced by Samanta et.al [7] and they also defined the term fuzzy colors and strength cut graph. The idea of fuzzy node coloring was extended to total fuzzy coloring by Lavanya and Sattanathan [2].

The coloring concept of Intuitionistic fuzzy graphs was established by Rifayathali et.al [6] and they defined chromatic excellence in Intuitionistic fuzzy graphs. Hesitancy fuzzy graph coloring was developed by Prasanna et.al [5] where they

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colored strong and complete hesitancy fuzzy graph. The idea of Pythagorean fuzzy graph was originated by Naz et.al [4] and they developed a set of operational laws for it. Futher, the total degree and degree of nodes in Pythagorean fuzzy graph was discovered and their properties were discussed.

2. Preliminaries

A family $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ of fuzzy sets on V is called a $k-$ *fuzzy coloring* of the fuzzy graph $G = (V, \sigma, \mu)$ if (i) $\max(\Gamma) = \sigma(h_i); \;\; h_i \in V$, (ii) $\min(\gamma_s, \gamma_t) = 0$ and (iii) For every adjacent node h_i, h_j of G, $\min(\gamma_s(h_i), \gamma_s(h_j)) = 0; \;\; 1 \leq s \leq k$. The least value of k for which G has a $k-$ fuzzy coloring is called the *fuzzy chromatic number* of G and it is denoted by $\chi^f(G)$.

Let $C = \{c_1, c_2, \ldots, c_i\}, i \geq 1$ be a collection of basic colors. The color $c_i = (c_i, f(c_i))$ is called the *fuzzy color* corresponding to the basic color c_i where $f : C \to [0,1]$ and $f(c_i)$ is the degree of membership of (c_i). Two nodes h_i and h_j in $G = (V, E)$ are called *adjacent* if $\frac{1}{2}(\min(\sigma(h_i), \sigma(h_j))) \leq \mu(h_i, h_j); \;\; i, j = 1, 2, \ldots, n$. If h_i and h_j are adjacent, then the arc (h_i, h_j) is considered as strong or otherwise weak. Two arcs (h_i, h_j) and (h_j, h_k) are called *incident* if $2 * \min(\mu(h_i, h_j), \mu(h_j, h_k)) \leq \sigma(h_j)$ for $i, j, k = 1, 2, \ldots, n$. The neighborhood of a node h_i in $G = (V, E)$ is the subgraph of G induced by all nodes adjacent to h_i. The *strength of the arc* (h_i, h_j) is given by $I_{(h_i,h_j)} = \frac{\mu(h_i,h_j)}{\min(\sigma(h_i),\sigma(h_j))}$ and the *strength of the node* h_i is given by $I_{h_i} = \max(\Theta_{h_i}, \sigma(h_i))$ where $\Theta_{h_i} = \max(I_{(h_i,h_j)}/(h_i, h_j) \; are \; the \; arcs \; in \; G)$.

Let $G' = (V', E')$ be a *Pythagorean Fuzzy Graph* with Pythagorean Fuzzy Set V' on Z and Pythagorean Fuzzy Relation E' on $Z * Z$ such that $\mu_{E'}(h_i, h_j) \leq \min(\mu_{V'}(h_i), \mu_{V'}(h_j))$, $\upsilon_{E'}(h_i, h_j) \geq \max(\upsilon_{V'}(h_i), \upsilon_{V'}(h_j))$ and $0 \leq \mu^2_{E'}(h_i, h_j) + \upsilon^2_{E'}(h_i, h_j) \leq 1$ for all $(h_i, h_j) \in V' * V'$. Here $\mu_{V'} : V' \to [0,1]$ and $\mu_{E'} : V' * V' \to [0,1]$ represent the degree of membership of the nodes and arcs, $\upsilon_{V'} : V' \to [0,1]$ and $\upsilon_{E'} : V' * V' \to [0,1]$ represent the degree of non membership of the nodes and arcs.

3. Pythagorean Fuzzy Coloring Graph

Two nodes h_i and h_j in a Pythagorean Fuzzy Graph $G' = (V', E')$ are called *adjacent* if $\frac{1}{2}[\min(\mu_{V'}(h_i), \mu_{V'}(h_j))] \leq \mu_{E'}(h_i, h_j)$ and $2[\max(\upsilon_{V'}(h_i), \upsilon_{V'}(h_j))] \geq$

$v_{E'}(h_i, h_j)$ for all $i, j = 1, 2, \ldots, n$. If arc (h_i, h_j) satisfies the above condition then (h_i, h_j) is considered as strong or otherwise weak.

Two arcs (h_i, h_j) and (h_j, h_k) in $G' = (V', E')$ are called *incident* if $2[\min(\mu_{E'}(h_i, h_j), \mu_{E'}(h_j, h_k))] \leq \mu_{V'}(h_j)$ and $\frac{1}{2}[\max(v_{E'}(h_i, h_j), v_{E'}(h_j, h_k))] \geq v_{V'}(h_j)$ for all $i, j, k = 1, 2, \ldots, n$. If node h_j satisfies the above condition then h_j is considered as strong or otherwise weak.

The *strength of the arc* (h_i, h_j) in G' is represented as $I_{(h_i,h_j)}$ and given as $I_{(h_i,h_j)} = (I_{\mu(h_i,h_j)}, I_{v(h_i,h_j)} = \left(\frac{\mu_{E'(h_i,h_j)}}{\min(\mu_{V'(h_i)},\mu_{V'(h_j)})}, \frac{v_{E'(h_i,h_j)}}{\max(v_{V'(h_i)},v_{V'(h_j)})}\right)$. Here $I_{\mu(h_i,h_j)}$ is the $\mu-$ strength of the arc and $I_{v(h_i,h_j)}$ is the $v-$ strength of the arc . The *strength of the node* $h_i \in G'$ is given as $I_{h_i} = (I_{\mu(h_i)}, I_{v(h_i)})$ where $I_{\mu(h_i)} = \max(\Theta_{\mu(h_i)}, \mu_{V'}(h_i))$ and $I_{v(h_i)} = \min(\Theta_{v(h_i)}, v_{V'}(h_i))$ such that $\Theta_{\mu(h_i)} = \max(I_{\mu(h_i,h_j)})$ and $\Theta_{v(h_i)} = \min\left(\frac{1}{I_{v(h_i,h_j)}}\right)$ for $i, j = 1, 2, \ldots, n$.

Let $A = \{a_1, a_2, \ldots, a_k\}, k \geq 1$ be a collection of basic colors. The color $a_k = (a_k, \mu^*(a_k), v^*(a_k))$ is called the *Pythagorean fuzzy color* corresponding to the basic color a_k where $\mu^*(a_k) : A \to [0, 1]$ and $v^*(a_k) : A \to [0, 1]$. Here $\mu^*(a_k)$ and $v^*(a_k)$ represent the degree of membership and non membership of a_k respectively.

The assignment of a Pythagorean fuzzy color to the nodes of $G' = (V', E')$ in such a way that the nodes connected by strong arc are assigned distint colors and nodes connected by week arc are assigned same color with distint membership and non membership values such that $0 \leq {\mu^*}^2(a_k)+{v^*}^2(a_k) \leq 1$ is called as *Pythagorean fuzzy node coloring*.

The assignment of a fuzzy color to the arc of $G' = (V', E')$ in such a way that the arc connected by strong node are assigned distint colors and arc connected by week node are assigned same color with distint membership and non membership values such that $0 \leq {\mu^*}^2(a_k) + {v^*}^2(a_k) \leq 1$ is called as *Pythagorean fuzzy arc coloring*.

The assignment of a fuzzy color to the nodes and arc of $G' = (V', E')$ in such a way that nodes connected by strong arc, arc connected by strong nodes and strong node incident with strong arc get distinct color and The nodes connected by weak arc, arc connected by weak nodes and weak node incident with weak arc get same color with distint membership and non membership values such that $0 \leq {\mu^*}^2(a_k) + {v^*}^2(a_k) \leq 1$ is called as *Pythagorean fuzzy total coloring*. The least number of Pythagorean fuzzy colors used to color a Pythagorean fuzzy graph is called a Pythagorean fuzzy chromatic number and it is denoted as $\chi^P(G)$.

4. Procedure for Pythagorean fuzzy node coloring

Let $G' = (V', E')$ be a Pythagorean fuzzy graph with vertices $\{h_1, h_2, \ldots, h_n\}$ and arcs $\{(h_i, h_j)\}$ for all $i, j = 1, 2, \ldots, n$. Let the collection of basic colors be denoted as $\{a_1, a_2, \ldots, a_k\}$. The node coloring of Pythagorean fuzzy graph is based on its node adjacency and strength of the arc.

Step 1: Consider the arc (h_i, h_j) and analysis whether it's strong or weak. By definition, we know that if $\frac{1}{2}[\min(\mu_{V'}(h_i), \mu_{V'}(h_j))] \leq \mu_{E'}(h_i, h_j)$ and $2[\max(v_{V'}(h_i), v_{V'}(h_j))] \geq v_{E'}(h_i, h_j)$ then the arc is strong. If not, the arc is considered to be weak.

Step 2: If the arc (h_i, h_j) is strong then the nodes h_i and h_j takes distint colors. Say, for strong arc (h_i, h_j) , the node h_i will take the color $(a_1, 1, 0)$ and h_j will take the color $(a_2, 1, 0)$.

Step 3: If the arc (h_i, h_j) is weak then the nodes gets the same color with distint membership and non membership values. Say, for weak arc (h_i, h_j), the node h_i will get the color $(a_k, 1, 0)$ and h_j will get the color $(a_k, \mu^*(a_k), v^*(a_k))$. This is calculated as $(a_k, \mu^*(a_k), v^*(a_k)) = \left(a_k, 1 - \mu \text{ strength } of(h_i, h_j), 1 - \frac{1}{v \text{ strength } of(h_i,h_j)}\right)$
$= \left(a_k, 1 - I_{\mu(h_i,h_j)}, 1 - \frac{1}{I_{v(h_i,h_j)}}\right) = \left(a_k, 1 - \frac{\mu_{E'}(h_i,h_j)}{\min(\mu_{V'}(h_i),\mu_{V'}(h_j))}, 1 - \frac{1}{\frac{v_{E'}(h_i,h_j)}{\max(v_{V'}(h_i),v_{V'}(h_j))}}\right)$.

Step 4: If the node h_i has both strong and weak neighborhood then node h_i takes the color distinct to the strong node by taking the same color of the adjacent weak node.

Step 5: If the node h_i has both strong and weak nodes with same color then node h_i takes distinct color to that of adjacent node.

Step 6: If all the neighborhood of h_i are weak and not colored then h_i takes the color $(a_1, 1, 0)$ and the colors of its neighborhood are calculated as step 3.

Step 7: If the neighborhood of h_j are weak and colored then h_j takes any one of its adjacent node color with membership value $\mu^*(a_k)$ and non membership value

$v^*(a_k)$ that are calculated as follows

$$\mu^*(a_k) = \max\left[1-\mu \text{ strength } of(h_i,h_j), 1-\mu \text{ strength } of(h_j,h_k)\right]$$
$$= \max[1 - I_{\mu(h_i,h_j)}, 1 - I_{\mu(h_j,h_k)}]$$
$$= \max\left[1-\left(\frac{\mu_{E'}(h_i,h_j)}{\min(\mu_{V'}(h_i),\mu_{V'}(h_j))}\right), 1-\left(\frac{\mu_{E'}(h_j,h_k)}{\min(\mu_{V'}(h_j),\mu_{V'}(h_k))}\right)\right]$$

and

$$v^*(a_k) = \min\left[1-\frac{1}{v \text{ strength } of(h_i,h_j)}, 1-\frac{1}{v \text{ strength } of(h_j,h_k)}\right]$$
$$= \min\left[1-\frac{1}{I_{v(h_i,h_j)}}, 1-\frac{1}{I_{v(h_j,h_k)}}\right]$$
$$= \min\left[1-\frac{1}{\frac{v_{E'}(h_i,h_j)}{\max(v_{V'}(h_i),v_{V'}(h_j))}}, 1-\frac{1}{\frac{v_{E'}(h_j,h_k)}{\max(v_{V'}(h_j),v_{V'}(h_k))}}\right]; \quad i,.k = \{1,2,\ldots,n\}.$$

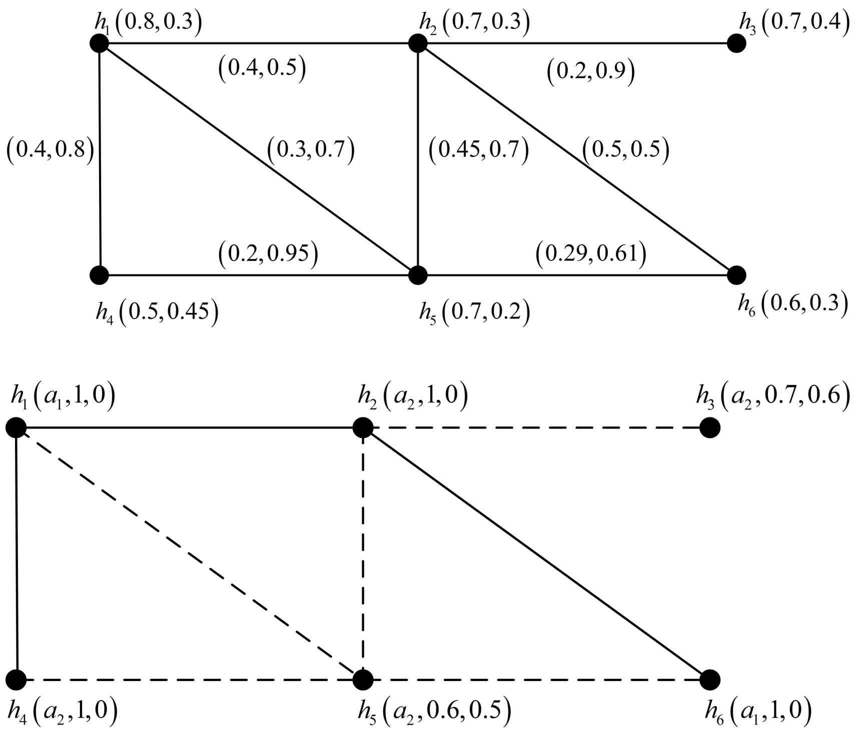

FIGURE 1. Pythagorean fuzzy graph and its node coloring with $\chi^P(G) = 2$

5. Procedure for Pythagorean fuzzy arc coloring

Let $G' = (V', E')$ be a Pythagorean fuzzy graph with vertices $\{h_1, h_2, \ldots, h_n\}$ and arcs $\{(h_1, h_2)(h_2, h_3), \ldots, (h_{n-1}, h_n)\}$. Let the collection of basic colors be denoted as $\{a_1, a_2, \ldots, a_k\}$. The arc coloring of Pythagorean fuzzy graph is based on the strength of node.

Step 1: Consider the node h_j and analysis whether it's strong or weak. By definition, we know that if $2[\min(\mu_{E'}(h_i, h_j), \mu_{E'}(h_j, h_k))] \leq \mu_{V'}(h_j)$ and $\frac{1}{2}\max[v_{E'}(h_i, h_j), v_{E'}(h_j, h_k)] \geq v_{V'}(h_j)$ then node h_j is strong or else the node is weak.

Step 2: If the node h_j is strong then the incident arcs of h_j takes distint colors. Say, for strong node h_j, the node (h_i, h_j) will take the color $(a_1, 1, 0)$ and (h_j, h_k) will take the color $(a_2, 1, 0)$

Step 3: If the node h_j is weak and the incident edges are not colored then one of the incident arcs of h_j takes the color $(a_k, 1, 0)$ and the other incident arcs takes the same color with distint membership and non membership value. These values are considered as $(a_k, \mu^*(a_k), v^*(a_k))$ and calculated as follows

$$\begin{aligned}
\mu^*(a_k) &= 1 - (\mu \text{ strength } of \ \ h_j) = 1 - (I_{\mu(h_j)}) \\
&= 1 - \max(\Theta_{\mu(h_j)}, \mu_{V'}(h_j)) \\
&= 1 - \max[\max(I_{\mu(h_i,h_j)}), \mu_{V'}(h_j)]; \\
&= 1 - \max\left[\max\left(\frac{\mu_{E'}(h_i, h_j)}{\min(\mu_{V'}(h_i)), \mu_{V'}(h_j)}\right), \mu_{V'}(h_j)\right]; \quad i = \{1, 2, \ldots, n\} \\
v^*(a_k) &= 1 - (v \text{ strength } of \ \ h_j) = 1 - (I_{v(h_j)}) \\
&= 1 - \min(\Theta_{v(h_j)}, v_{V'}(h_j)) \\
&= 1 - \min[\min\left(\frac{1}{I_{v(h_i,h_j)}}\right), v_{V'}(h_j)]; \\
&= 1 - \min\left[\min\left(\frac{1}{\frac{v_{E'}(h_i,h_j)}{\max(v_{V'}(h_i)), v_{V'}(h_j)}}\right), v_{V'}(h_j)\right]; \quad i = \{1, 2, \ldots, n\}
\end{aligned}$$

Step 4: If the arc (h_i, h_j) has both strong and weak nodes which are colored then (h_i, h_j) takes the color distinct to arc incident with strong node by taking the same color of arc incident to weak node.

Step 5: If the arc (h_i, h_j) has both strong and weak nodes with same color then arc (h_i, h_j) takes distinct color to those incident arcs.

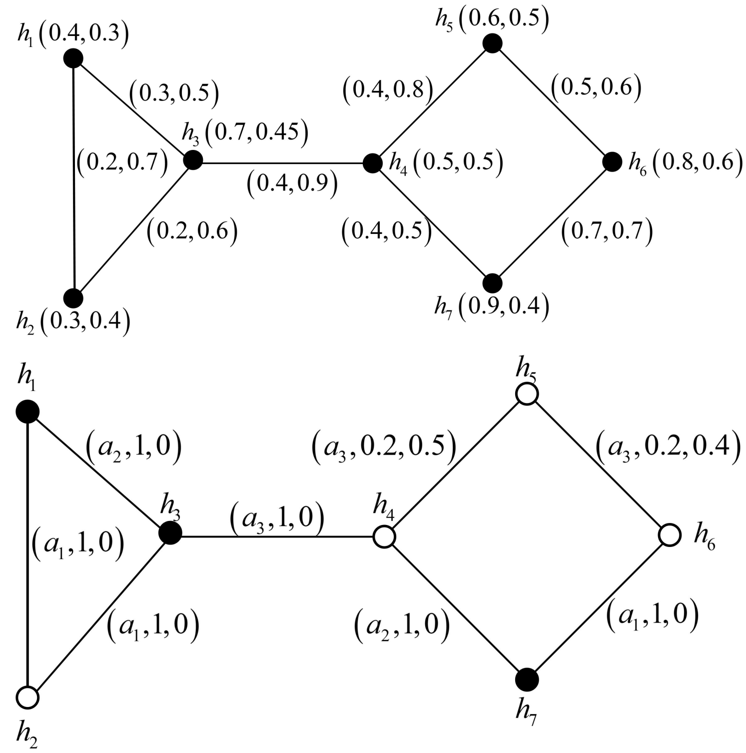

FIGURE 2. Pythagorean fuzzy graph and its arc coloring with $\chi^P(G) = 3$

Step 6: If the node h_j is weak and all the incident arcs are not colored then (h_i, h_j) takes the color $(a_1, 1, 0)$ and then the incident arcs are colored as of step3.
Step 7: If the node h_j is weak and all the incident arcs are colored expect (h_i, h_j) then (h_i, h_j) takes color of any one of the incident arc with different membership and non membership value $v^*(a_k)$ which is calculated as $\mu^*(a_k) = \max(1 - (I_{\mu(h_j)}))$ and $v^*(a_k) = \min(1 - (I_{v(h_j)}))$.

6. REMARKS

(i) Pythagorean fuzzy chromatic number $(\chi^P(G))$ is always less than or equal to chrisp chromatic number $(\chi(G))$.
(ii) $\chi^P(G)) - \chi(G) =$ number of weak edges in Pythgorean fuzzy graph.

7. Conclusion

We have introduced a new concept of coloring the Pythagorean fuzzy graph with Pythagorean fuzzy colors and explained the detailed procedure for Pythagorean fuzzy node and arc coloring with suitable example.

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