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STABILITY OF DISCRETE FRACTIONAL JOSEPHSON JUNCTION MODEL

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ABSTRACT. The Josephson Junction (JJ) is one of the vital element in the conversion of quantum phenomena into technology that are used in daily life. There is occurrence of the phase transition in many metals and alloys from normal state with resistance to super conducting state when temperature is very low. In this paper, we consider the discrete fractional version of the Resistively and Capacitively shunted JJ model. The stability of the solution is carried out in the sense of Hyers-Ulam (HU). The stability conditions are numerically calculated and tabulated.

1. INTRODUCTION

In 1962, Brian Josephson was the first person to predict that at equal voltage, tunneling super-currents exists between coupled superconductors that are separated by thin insulated barrier. The applications of the josephson junctions are digital and analog electronics, amplifiers, mixers and so on [3]. There have been increasing interest for the researchers in dynamical systems with nonlinearties that exhibit the complex behavior. One such model is the Resistive and Capacitively Shunted Junction model (RCSJ) describing the Josephson junction (JJ) biased by ideal current source.

The arbitrary order calculus originated in 17^{th} century from Leibnitz's note has recently attracted mathematicians, physicists and engineers due to its wide

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application in different fields like viscoelasticity, mechanics, chemical reaction, biological models and neural networks. Greater level of accuracy has been achieved in modeling of physical systems by fractional calculus. Initially the Hyers-Ulam (HU) stability was proposed for the functional equations, but later HU stability was extended for differential equations of both integer and arbitrary order [8, 11].

The discrete version of the fractional calculus has gained significant growth due to the works of Atici and Eloe [1, 2]. The analysis of qualitative properties like oscillation and non oscillation, asymptotic behavior and stability of solutions of discrete fractional equations has been carried out recently in [4–7].

Denote $\mathbb{Q} : [\kappa+1, \kappa+\mathcal{T}] \cap \mathbb{N}_{\kappa+1}$, where $\mathcal{T} \in \mathbb{N}$ and $\mathbb{N}_\kappa = \{\kappa, \kappa+1, \ldots\}, \kappa \in \mathbb{R}$. Inspired by the works mentioned above we study the uniqueness and stability of discrete fractional RCSJ model of the form

$$
\begin{cases} \nabla^{\sigma}_{\kappa*}\left[\nabla^{\gamma}_{\kappa*} + \dfrac{1}{RC}\right] y(\ell) + \dfrac{I_c 2e}{\hbar C}\left[\sin(y(\ell)) - I + I_F(\ell)\right] = 0, \\ \qquad \text{if } \ell \in [\kappa+2, \kappa+\mathcal{T}] \cap \mathbb{N}_{\kappa+2}, \\ y(\kappa) = A, \text{ if } \nabla^{\gamma}(y(\kappa)) = B \end{cases} , \tag{1.1}
$$

where ∇^{σ}_{*} is a caputo fractional nabla difference operator, $0 < \sigma, \gamma < 1$ are the fractional orders such that $1 < \sigma + \gamma < 2$, y denotes the phase difference and other parameters include elementary charge e, Resistance R, Capacitance C, Critical current in the junction I_c. Here, $I = \frac{J}{I_c}$ and $I_F(\ell) = \frac{J_F}{I_c}$ where J and J_F represents the total current through the circuit and the fluctuating current.

The paper is organized with preliminary mathematical concepts in section 2. The stability of the solution of (1.1) are presented in section 3 followed with an application in section 4.

2. Preliminaries

Some essential concepts used in this work are provided in this section.

Definition 2.1. *[9] The ϑ^{th} rising factorial function for any $\vartheta, \delta \in \mathbb{R}$ is*

$$\delta^{\overline{\vartheta}} = \frac{\Gamma(\delta+\vartheta)}{\Gamma(\delta)}, \ \delta \in \mathbb{R} \backslash \{\ldots, -2, -1, 0\}.$$

Definition 2.2. *[9] Nabla sum of fractional order $\vartheta \in \mathbb{R}^+$ of $u : \mathbb{N}_\kappa \to \mathbb{R}$ is*

$$\nabla^{-\vartheta} u(\delta) = \frac{1}{\Gamma(\vartheta)} \sum_{r=\kappa+1}^{\delta} (\delta - r + 1)^{\overline{\vartheta-1}} u(r), \ \delta \in \mathbb{N}_\kappa$$

Definition 2.3. *[9] Let $\vartheta \in \mathbb{R}^+$ and choose $M \in \mathbb{N}_1$ such that $M - 1 < \vartheta < M$. Caputo fractional nabla difference of $u : \mathbb{N}_\kappa \to \mathbb{R}$ is given by*

$$\nabla^{\vartheta}_{\kappa *} u(\delta) = \nabla^{-(M-\vartheta)} \left[\nabla^M u(\delta) \right], \ \delta \in \mathbb{N}_{\kappa+M}.$$

Theorem 2.1. *[9] Let $\vartheta > 0$ and $q > -1$. Then,*

1. $\nabla^{-\vartheta}_{\kappa} (\delta - \kappa)^{\overline{q}} = \frac{\Gamma(q+1)}{\Gamma(q+\vartheta+1)} (\delta - \kappa)^{\overline{\vartheta+q}}, \ \delta \in \mathbb{N}_\kappa.$
2. $\nabla^{\vartheta}_{\kappa} (\delta - \kappa)^{\overline{q}} = \frac{\Gamma(q+1)}{\Gamma(q-\vartheta+1)} (\delta - \kappa)^{\overline{q-\vartheta}}, \ \delta \in \mathbb{N}_{\kappa+M}.$

The function $y : \mathbb{N}_\kappa \to \mathbb{R}$ is a solution of (1.1) if and only if it has following representation

$$\begin{aligned} y(\ell) =& A + \frac{1}{\Gamma(\gamma)} \sum_{s=\kappa+1}^{\ell} (\ell - s + 1)^{\overline{\gamma-1}} \left(B + \frac{1}{RC}(A - y(s)) \right) \\ &+ \frac{2I_c e}{\hbar C} \frac{1}{\Gamma(\gamma+\sigma)} \sum_{s=\kappa+1}^{\ell} (\ell - s + 1)^{\overline{\gamma+\sigma-1}} (I - I_F(s) - \sin(y(s))), \end{aligned} \tag{2.1}$$

for $\ell \in \mathbb{Q}$.

3. Hyers-Ulam Stability

Let $\mathfrak{B}$ be the linear space of all functions of $y(\ell)$ with norm $\|y\| = \sup_{\ell \in \mathbb{Q}} |y(\ell)|$. Clearly, $\mathfrak{B}$ is a banach space. This section presents the Hyers Ulam stability results of (1.1).

Definition 3.1. *[5] Initial value nabla fractional discrete equation* (1.1) *is Hyers-Ulam stable, if there exist a positive constant $\mathbb{H}$ such that for every $\varepsilon > 0$, $\phi \in \mathbb{R}$ satisfies*

$$\left| \nabla^{\sigma}_{\kappa *} \left[\nabla^{\gamma}_{\kappa *} + \frac{1}{RC} \right] \phi(\ell) + \frac{I_c 2e}{\hbar C} \left[\sin(\phi(\ell)) - I + I_F(\ell) \right] \right| \leq \varepsilon, \quad \ell \in \mathbb{Q}, \tag{3.1}$$

with $\phi(\kappa) = A, \nabla^{\gamma}(\phi(\kappa)) = B$ then the solution $y(\ell)$ of (1.1) *exists such that $|\phi(\ell) - y(\ell)| \leq \mathbb{H}\varepsilon$.*

Remark 3.1. *A function $\phi(\ell) \in \mathbb{R}$ solves* (3.1) *if and only if there exists $\xi : \mathbb{Q}\times\mathbb{R} \to \mathbb{R}$ that satisfy*

A1: $|\xi(\ell, \phi(\ell))| \le \varepsilon, \qquad \ell \in \mathbb{Q}.$
A2: $\nabla^{\sigma}_{\kappa*}\left[\nabla^{\gamma}_{\kappa*} + \frac{1}{RC}\right]\phi(\ell) + \frac{I_c 2e}{\hbar C}\left[\sin(\phi(\ell)) - I + I_F(\ell)\right] = \xi(\ell, \phi(\ell)).$

Theorem 3.1. *Let $\phi \in \mathbb{R}$ satisfy* (3.1) *for some $\varepsilon > 0$ and let $y \in \mathbb{R}$ be the solution of*

$$\begin{cases} \nabla^{\sigma}_{\kappa*}\left[\nabla^{\gamma}_{\kappa*} + \frac{1}{RC}\right] y(\ell) + \frac{I_c 2e}{\hbar C}\left[\sin(y(\ell)) - I + I_F(\ell)\right] = 0, \\ \quad \text{if } \ell \in [\kappa+2, \kappa+\mathcal{T}] \cap \mathbb{N}_{\kappa+2}, \\ y(\kappa) = \phi(\kappa), \quad \text{if } \nabla^{\gamma}(y(\kappa)) = \nabla^{\gamma}(\phi(\kappa)) \end{cases}, \tag{3.2}$$

then (1.1) *is Hyers Ulam Stable with $\mathbb{H} = \frac{(\kappa+\mathcal{T})^{\overline{\sigma+\gamma}}}{\Gamma(\sigma+\gamma+1)(1-\Omega)}$ where $\Omega < 1$.*

Proof. If $\phi(\ell)$ solves (3.1), then

$$\Bigg|\phi(\ell) - A - \frac{1}{\Gamma(\gamma)}\sum_{s=\kappa+1}^{\ell}(\ell-s+1)^{\overline{\gamma-1}}\left(B + \frac{1}{RC}(A - \phi(s))\right) - \frac{2I_c e}{\hbar C}\frac{1}{\Gamma(\gamma+\sigma)} \sum_{s=\kappa+1}^{\ell}(\ell-s+1)^{\overline{\gamma+\sigma-1}}(I - I_F(s) - \sin(\phi(s)))\Bigg| \le \varepsilon\frac{\Gamma(\kappa+\sigma+\gamma+\mathcal{T})}{\Gamma(\kappa+\mathcal{T})\Gamma(\sigma+\gamma+1)}$$

for $\ell \in \mathbb{Q}$.

Using the solution $y(\ell)$ of (3.2) obtained from (2.1), we have

$$\begin{aligned} |\phi(\ell) - y(\ell)| =& |\phi(\ell) - \phi(\kappa) - \frac{1}{\Gamma(\gamma)}\sum_{s=\kappa+1}^{\ell}(\ell-s+1)^{\overline{\gamma-1}}\left(\nabla^{\gamma}(\phi(\kappa)) + \frac{\phi(\kappa)}{RC}\right) \\ &+ \frac{1}{\Gamma(\gamma)}\sum_{s=\kappa+1}^{\ell}(\ell-s+1)^{\overline{\gamma-1}}\left(\frac{1}{RC}(y(s))\right) \\ &- \frac{2I_c e}{\hbar C}\frac{1}{\Gamma(\gamma+\sigma)}\sum_{s=\kappa+1}^{\ell}(\ell-s+1)^{\overline{\gamma+\sigma-1}}(I - I_F(s) - \sin(y(s)))| \\ =& |\phi(\ell) - \phi(\kappa) - \frac{1}{\Gamma(\gamma)}\sum_{s=\kappa+1}^{\ell}(\ell-s+1)^{\overline{\gamma-1}}\left(\nabla^{\gamma}(\phi(\kappa)) + \frac{\phi(\kappa)}{RC}\right) \\ &+ \frac{1}{\Gamma(\gamma)}\sum_{s=\kappa+1}^{\ell}(\ell-s+1)^{\overline{\gamma-1}}\left(\frac{1}{RC}(\phi(s))\right) \end{aligned}$$

$$- \frac{2I_c e}{\hbar C} \frac{1}{\Gamma(\gamma+\sigma)} \sum_{s=\kappa+1}^{\ell} (\ell - s + 1)^{\overline{\gamma+\sigma-1}} (I - I_F(s) - \sin(\phi(s)))$$
$$+ \frac{1}{RC} \frac{1}{\Gamma(\gamma)} \sum_{s=\kappa+1}^{\ell} (\ell - s + 1)^{\overline{\gamma-1}} (y(s) - \phi(s))$$
$$+ \frac{2I_c e}{\hbar C} \frac{1}{\Gamma(\gamma+\sigma)} \sum_{s=\kappa+1}^{\ell} (\ell - s + 1)^{\overline{\gamma+\sigma-1}} (\sin(y(s)) - \sin(\phi(s)))|$$
$$\|\phi(\ell) - y(\ell)\| \leq \varepsilon \mathbb{H}.$$

Thus (1.1) Hyers-Ulam Stable with constant $\mathbb{H} = \frac{(\kappa+\mathcal{T})^{\overline{\sigma+\gamma}}}{\Gamma(\sigma+\gamma+1)(1-\Omega)}$, where $\Omega = \frac{1}{\Gamma(\kappa+\mathcal{T})} \left(\frac{\Gamma(\kappa+\mathcal{T}+\gamma)}{RC\Gamma(\gamma+1)} + \frac{2I_c e}{\hbar C} \frac{\Gamma(\kappa+\sigma+\gamma+\mathcal{T})}{\Gamma(\gamma+\sigma+1)} \right) < 1$. This completes the proof. □

4. An Application

The thickness of the barriers between the superconductors is important for the electron pairs to get through [10]. The electron pairs tunnel if the thickness of insulating barrier is less than $10nm$. The RCSJ model equation can be related to the simple driven pendulum and tilted washboard. The application of RCSJ model includes detecting electromagnetic waves, logical gates construction with flux quanta, SQUID and so on.

Example 1. *This example considers the discrete fractional RSCJ model described by*

$$\begin{aligned} &\nabla_{\kappa*}^{0.8} \left[\nabla_{\kappa*}^{0.7} + \omega \right] \phi(\ell) + \lambda \sin(\phi(\ell)) = I, \\ &\phi(0) = 0, \quad \nabla^{0.7}(\phi(0)) = 1 \end{aligned} \tag{4.1}$$

where $\ell \in [1, 12] \cap \mathbb{N}_1$, $\lambda = \frac{2I_c e}{\hbar C}$, *$I$ are constants and* $\omega = \frac{1}{RC}$ *represents the damping coefficient. Let the parameters take the values* $\lambda = 0.01, I = 0.75, \omega = 0.015$. *It is clear that* $\Omega = 0.4155 < 1$. *Let* $\varepsilon = 0.67$ *and* $\phi(\ell) = \frac{\ell^{\overline{2}}}{6}, \ell \in [1, 8] \cap \mathbb{N}_1$. *The inequality* (3.1) *implies*

$$|\nabla_{\kappa*}^{0.8} \left[\nabla_{\kappa*}^{0.7} + 0.015 \right] \phi(\ell) + 0.01 \sin(\phi(\ell)) - 0.75| \leq 0.6596 < \varepsilon.$$

From Theorem (3.1), *we guarantee that the solution of* (4.1) *is Hyers-Ulam stable with constant* $\mathbb{H}$.

TABLE 1. Illustration of γ and Ω

γ	$\sigma = 0.15$	$\sigma = 0.30$	$\sigma = 0.45$	$\sigma = 0.60$	$\sigma = 0.75$	$\sigma = 0.9$
	Ω					
0.1	*0.0405*	*0.0503*	*0.0638*	*0.0823*	*0.1071*	*0.1401*
0.2	*0.0532*	*0.0654*	*0.0820*	*0.1045*	*0.1346*	*0.1744*
0.3	*0.0691*	*0.0841*	*0.1045*	*0.1319*	*0.1681*	*0.2157*
0.4	*0.0889*	*0.1073*	*0.1322*	*0.1652*	*0.2087*	*0.2655*
0.5	*0.1134*	*0.1359*	*0.1660*	*0.2057*	*0.2577*	*0.3251*
0.6	*0.1434*	*0.1708*	*0.2070*	*0.2546*	*0.3165*	*0.3962*
0.7	*0.1801*	*0.2132*	*0.2567*	*0.3134*	*0.3867*	*0.4807*
0.8	*0.2247*	*0.2645*	*0.3165*	*0.3838*	*0.4704*	*0.5809*
0.9	*0.2787*	*0.3262*	*0.3881*	*0.4678*	*0.5697*	*0.6991*

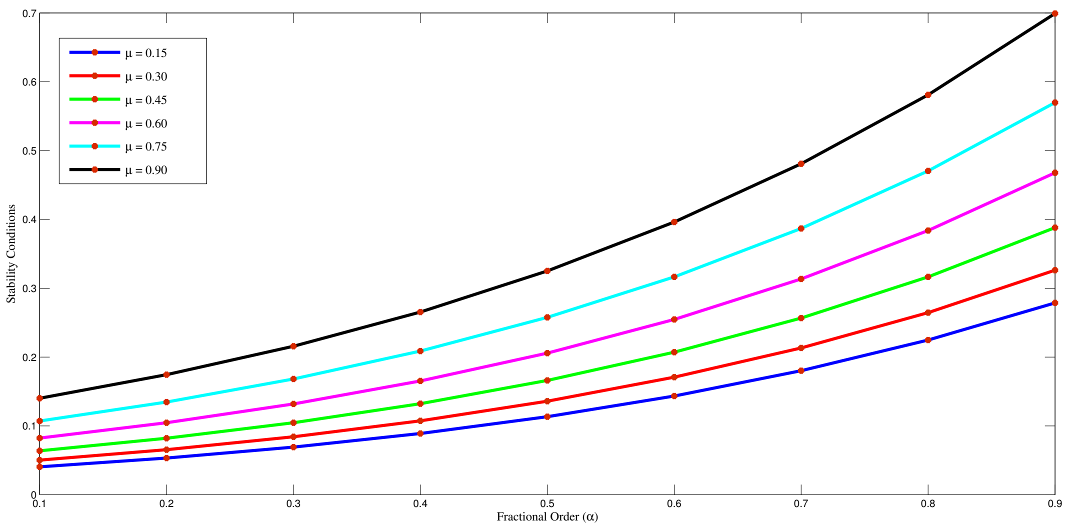

FIGURE 1. γ versus Ω

The stability conditions obtained from Theorem (3.1) *are tabulated in Table* 1 *considering different order and are plotted in Figure 1.*

5. CONCLUSION

Discrete fractional version of the resistive and capacitively shunted junction model is considered in this work. The stability in the sense of Hyers and Ulam for the solution of RCSJ model is established. Numerical values are provided for the parameters in the model and simulation is performed for the stability conditions with values tabulated.

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