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# A NUMERICAL SCHEME FOR THE SOLUTION OF $q$-FRACTIONAL DIFFERENTIAL EQUATION USING $q$-LAGUERRE OPERATIONAL MATRIX

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ABSTRACT. In this paper, we develop a numerical scheme for the solution of $q$-fractional order differential equation using $q$-Laguerre operational matrix($q$-LOM). Here, we consider the $q$-fractional derivative in the Caputo sense. An operational matrix of $q$-Laguerre polynomials for $q$-fractional order derivatives are determined and utilized along with the spectral tau method for converting the $q$-fractional differential equations($q$-FDEs) into a system of algebraic equations. This method is applied to solve linear $q$-fractional differential equations.

## 1. INTRODUCTION

The investigation of $q$-calculus began during the 1740s. The $q$-calculus is the most part settled on inferring $q$-analogous to the traditional analytical results without utilizing limits. The subject arrangement with the properties of the $q$-special functions, which are the development of the ordinary special functions dependent on a parameter, or the base $q$. The significant and principal instrument of $q$-calculus is $q$-derivative. The pioneer who deals with quantum calculus is Jackson. For basic definitions and properties of $q$-calculus we refer [8–10].

Fractional order derivatives generalizes integer-order differentiation and integration. In 1695 Leibniz's noted the Fractional derivative in his list to L'Hospital and now we have many definitions of fractional derivatives. we can refer the

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fundamental definitions on fractional differential equations (FDEs) [1,2]. Fractional calculus have many applications and used in many fields and few are LTE networks [19], two loop controlled micro gird [18], evaluation of closed-loop-PID [20], optimization based multilevel thresholding for medical images [21], closed-loop MLI based DP-FC for fourteen-bus system [15]. The $q$-fractional calculus is $q$-expansion of the classical fractional calculus. Al-Salam [6], and Agarwal [7] introduced and developed different types of $q$-fractional integral operators and $q$-fractional derivatives. Since most of the FDEs does not have exact analytic solutions, then it is required to develop approximate and numerical techniques. Many authors and researches have been discussed various numerical and approximate methods to solve the FDEs, for example, variational iteration method, homotopy perturbation method, Adomain's decomposition method, homotopy analysis method, operational matrix method by collocation [5] and finite difference method. Where as in $q$-calculus, there are not many known methods.

In recent years various operational matrices for the polynomials have been developed to obtain the the numerical solution. The polynomials have been frequently used in the solution of integral, differential and approximation theory. We can refer some hyper geometric polynomials in [12–14, 16, 17]. In the operational matrix method, we can reduce an FDE to algebraic equations with the help of operational matrices and orthogonal polynomials, and get the approximate solution. We get the operational matrices by approximating the integral of orthogonal polynomials. For example, Saadatmandi and Dehghan [22] generalized the Legendre operational matrix, Abdelkawy and Taha [3] developed the Laguerre operational matrix and Bhrawy and Alofi [4] introduced a new shifted Chebyshev operational matrix of fractional integration in the R–Liouville sense, to the FDEs for linear and non-linear cases and also discussed spectral techniques based on operational matrices of fractional derivatives and integrals for solving FDEs.

This paper deals with numerical solutions of $q$-fractional differential equations ($q$-FDEs) using the $q$-Laguerre polynomials($q$-LOM. Our main aim is to generalize the $q$-LOM to $q$-fractional calculus. The advantage of this method is that the $q$ operational matrix of orthogonal functions for solving $q$-FDEs is a computer-oriented. The rest of this paper is presented as follows. In section 2, the basic definitions of $q$-fractional integrals and derivatives are given. In section

3, we present $q$-Laguerre polynomials and obtain an operational matrix for the $q$-fractional derivative. In section 4, the main result of this paper is represented and numerical examples are given. Finally, conclusions have been drawn in the last.

## 2. Preliminaries

**Definition 2.1.** *[6] Let $\mu > 0$, The R-Liouville definition of $q$-fractional integral of $h(z)$ is defined as*

$$J_q^{\mu} h(z) = \frac{1}{\Gamma_q(\mu)} \int_0^z (z-qt)^{\mu-1} h(t) d_q(t)$$
$$J_q^0 h(z) = h(z).$$

**Definition 2.2.** *[6] Let $\mu > 0$, The Caputo definition $q$-fractional integral of $h(z)$ is defined as*

$$D_q^{\mu} h(z) = J^{(m-\mu)} D^m h(z) = \frac{1}{\Gamma_q(m-\mu)} \int_0^z (z-qt)^{m-\mu-1} \frac{d_q^m}{dz_q^m} h(t) d_q t,$$

*$(m-1) < \mu < m, z > 0$, where $D^{\mu}$ is the differential operator of order $\mu$ and satisfies the following*

$$D^{\mu} C = 0, \quad \text{(C is a constant)},$$

$$D_q^{\mu} z^{\alpha} = \begin{cases} 0, & \text{for } \alpha \in \eta_0 \text{ and } \alpha < \lceil \mu \rceil \\ \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha+1-\mu)} z^{\alpha-\mu}, & \text{for } \alpha \in \eta_0 \text{ and } \alpha \geq \lceil \mu \rceil \text{ or } \alpha \notin \eta \text{ and } \alpha > \lfloor \mu \rfloor. \end{cases} \tag{2.1}$$

Here $\lceil \mu \rceil$ denotes the ceiling function and $\lfloor \mu \rfloor$ denotes the floor functions respectively. Moreover $\eta = \{1, 2, \ldots\}$ and $\eta_0 = \{0, 1, 2, \ldots\}$.

The Caputo $q$-fractional differentiation is a linear operator

$$D_q^{\mu}(\lambda h(z) + \delta g(z)) = \lambda D_q^{\mu} h(z) + \delta D_q^{\mu} g(z), \tag{2.2}$$

where $\lambda$ and $\delta$ are constants.

$q$-Laguerre Polynomials: [11] The k'th degree of $q$-Laguerre polynomials in the interval $\Lambda \equiv (0, \infty_q)$ are defined as

$$L_{k,q}(z) = \frac{1}{[-1]^k \, k_q!} \sum_{v=0}^{k} \frac{[-1]^k \ (k_q!)^2}{((k-v)_q!)^2 \ v_q!} z^{k-v}, \quad k = 0, 1, \ldots. \tag{2.3}$$

The orthogonality condition is

$$\int_0^\infty E_q(-qz)L_n(z)L_m(z)d_qz = q^n\delta_{mn}.$$

## 3. GENERALISED $q$-LOM OF $q$-FRACTIONAL CALCULUS

Let us consider $p(z) \in L_w^2(\Lambda)$, then $p(z)$ may be expressed in terms of $q$-Laguerre polynomial as

$$p(z) = \sum_{j=0}^{\infty} a_j L_j(z), \quad a_j = \int_0^\infty p(z)L_j(z)w(z)dz, \quad j = 0, 1, 2, \ldots. \tag{3.1}$$

First, consider the $(n+1)$ terms of $q$-Laguerre polynomials. At that point

$$p_n(z) = \sum_{j=0}^{n} a_j L_j(z) = C^T\phi(z).$$

Here $C$ is the $q$-Laguerre coefficient vector and $\phi(z)$ is the $q$-Laguerre vector and are given by

$$C^T = [c_0, c_1, \ldots, c_n],\ \phi(z) = [L_0, L_1, \ldots, L_n]^T.$$

Now, we express The $q$-fractional derivative of a vector $\phi(z)$ as

$$\frac{d_q\phi(z)}{dz_q} = D_q^1\phi(z), \tag{3.2}$$

where $D_q^1$ is the$(n+1)\times(n+1)$ is given by

$$D_q^1 = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 0 & 0 & \ldots & 0 & 0 \\ -1, & -\frac{1}{q} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ -1, & -\frac{1}{q} & -\frac{1}{q^2} & \ldots & -\frac{1}{q^{n-1}} & 0 \end{pmatrix}$$

From (3.2), it is clear that

$$\frac{d_q^n\phi(z)}{dz_q^n} = (D_q^{(1)})^n\phi(z),$$

where $(D_q^1)$ stands for matrix powers and $n \in N$. Hence

$$D_q^{(n)} = (D_q^{(1)})^n, \quad n = 1, 2, 3, \ldots. \tag{3.3}$$

**Lemma 3.1.** *Let $L_k(z)$ be a $q$-Laguerre polynomial, then*

$$D_q^{\mu} L_k(z) = 0, \quad k = 0, 1, \ldots, \alpha < \lceil \mu \rceil - 1, \;\; \mu > 0.$$

*Proof. By using* (2.1) *and* (2.2) *in* (2.3)*, the lemma can be easily proved.*

**Theorem 3.1.** *Suppose $\phi(z)$ be $q$-Laguerre vector defined and also $\mu > 0$, then*

$$D_q^{\mu}\phi(z) = D_q^{(\mu)}\phi(z), \tag{3.4}$$

*where $D_q^{\mu}$ is the $(n+1)$ dimension operational matrix of $q$-fractional derivatives of order $\mu$ and is defined, as follows*

$$\begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \zeta_{\mu,q}(\lceil \mu \rceil, 0) & \zeta_{\mu,q}(\lceil \mu \rceil, 1) & \ldots & \zeta_{\mu,q}(\lceil \mu \rceil, n) \\ \vdots & \vdots & \vdots & \vdots \\ \zeta_{\mu,q}(k, 0) & \zeta_{\mu,q}(k, 1]) & \ldots & \zeta_{\mu,q}(k, n) \\ \vdots & \vdots & \vdots & \vdots \\ \zeta_{\mu,q}(n, 0) & \zeta_{\mu,q}(n, 1) & \ldots & \zeta_{\mu,q}(n, n) \end{pmatrix},$$

*where*

$$\zeta_{\mu,q}(k, j) = \frac{1}{q^j [-1]^{k+j} k_q! j_q!} \sum_{v=\lceil \mu \rceil}^{k} \sum_{l=0}^{j} \frac{[-1]^{v+l} \; (k_q!)^2 \; (j_q!)^2 \; \Gamma_q(k - v - \mu + l + 1)}{((j-l)_q!)^2 \; (k-v)_q! \; v_q! \; \Gamma_q(k - v - \mu + 1) \; l_q!},$$

*and the starting $\lceil \mu \rceil$ rows of $D_q^{\mu}$ are all zeros.*

*Proof.* From (2.1), (2.2) and (3.1), we have

$$\begin{aligned} D_q^{\mu} L_k(z) &= \frac{1}{[-1]^k k_q!} \sum_{v=0}^{k} \frac{[-1]^v \; (k_q!)^2}{((k-v)_q!)^2 \; v_q!} \; D_q^{\mu} z^{k-v} \\ &= \frac{1}{[-1]^k k_q!} \sum_{v=\lceil \mu \rceil}^{k} \frac{[-1]^v \; (k_q!)^2}{(k-v)_q! \; \Gamma_q(k - v - \mu + 1)! \; v_q!} \; z^{k-v-\mu}, \end{aligned}$$

$k = 0, 1, \ldots$. Now, applying $z^{k-v-\mu}$ by $n+1$ terms of $q$-Laguerre series, we have

$$z^{k-v-\mu} = \sum_{j=0}^{n} b_j L_j(z),$$

where $b_j$ is given from (3.1) with $p(z) = z^{(k-v-\mu)}$ and

$$b_j = \frac{1}{q^j[-1]^j \ j_q!}\sum_{l=0}^{j}[-1]^l \frac{\Gamma_q(k-v-\mu+l+1) \ (j_q!)^2}{((j-l)_q!)^2 \ l_q!},$$

$$D_q^{\mu} L_k(z) = \sum_{j=0}^{n} \zeta_{\mu,q}(k,j) L_j(z), \quad k = \lceil \mu \rceil, \dots n, \tag{3.5}$$

$$\zeta_{\mu,q}(k,j) = \frac{1}{q^j[-1]^{k+j} j_q! k_q!}\sum_{v=\lceil \mu \rceil}^{k}\sum_{l=0}^{j} \frac{[-1]^{v+l} \ (k_q!)^2 \ (j_q!)^2 \ \Gamma_q(k-v-\mu+l+1)}{((j-l)_q!)^2 \ (k-v)_q! \ v_q! \ \Gamma_q(k-v-\mu+1) \ l_q!}.$$

From (3.5), it can be composed in a vector form

$$D_q^{\mu} L_k(z) = [\zeta_{\mu,q}(k,0), \zeta_{\mu,q}(k,1), \zeta_{\mu,q}(k,2), \dots, \zeta_{\mu,q}(k,n)]\phi(z). \tag{3.6}$$

According to the Lemma 3.1, we can write

$$D_q^{\mu} L_k(z) = [0, 0, \dots, 0]\phi(z), \quad k = 0, 1, 2, \dots, \quad \lceil \mu \rceil - 1. \tag{3.7}$$

From (3.6) and (3.7), we will get expected result. □

## 4. Applications of $q$-LOM for $q$-FDEs

On the basis of $q$-LOM, we are going to execute the technique to the linear multi-order $q$- FDEs with constant coefficients with the tau method.

### 4.1. Linear Multi-term $q$- FDEs.

Let us consider the following linear Caputo $q$- FDEs

$$D_q^{\mu} p(z) = \sum_{j=1}^{v} \gamma_v D^{\alpha_j} p(z) + \gamma_q(v+1) p(z) + g(z), \lambda \in (0, \infty_q) \tag{4.1}$$

with the initial conditions

$$p'(0) = d_k, \ \ k = 0, 1, \dots, m-1. \tag{4.2}$$

To solve the linear Caputo $q$-FDE (4.1) with conditions (4.2), we imprecise $p(z)$ and $g(z)$ by $q$-Laguerre polynomials as

$$p(z) = \sum_{k=0}^{n} c_k L_k(z) = C^T \phi(z), \tag{4.3}$$

(4.4) $$g(z) = \sum_{k=0}^{n} g_k L_k(z) = G^T \phi(z).$$

Here $G = [g_0, g_1, g_2, .., g_n]^n$ is known vector, but $C = [c_0, c_1, c_2, \ldots, c_n]^n$ is an unknown vector.

From (3.4), (4.3) and Theorem 3.1, we get

(4.5) $$D_q^{\mu} p(z) = C^T D_q^{\mu} \phi(z),$$

(4.6) $$D_q^{\alpha_j} p(z) = C^T D_q^{\alpha_j} \phi(z), \quad j = 1, 2, \ldots, k.$$

Take on (4.3)- (4.6), the residual $R_n(x)$ for (4.1) can be composed as,

$$R_n(z) = (C^T D_q^{\mu} - C^T \sum_{j=1}^{v} \gamma_j D_q^{\alpha_j} - \gamma_(v+1) C^T - G^T)\phi(z).$$

As mentioned in a regular tau method [5], we can produce $n - m + 1$ linear equations by applying,

(4.7) $$< R_n(z), L_j(z) > = \int_0^{\infty} W(z) R_n(z) L_j(z) = 0, \quad j = 0, 1, 2, \ldots, n - m,$$

and furthermore substituting (3.3) and (4.3) in (4.2), we get

(4.8) $$p^k(0) = C^T D_q^k \phi(z) = d_k, \quad k = 0, 1, 2, \ldots, m - 1.$$

From (4.7) and (4.8) develop $(n - m + 1)$ and $m$ set of linear equations respectively. These linear equations can be solved for unknown coefficient vector $C$ and furthermore $p(z)$ given in (4.2) can be calculated, which gives the required solution.

### 4.2. **Numerical results.**

**Example 1.** *Applications to the Bagely-Torvik equation.*
*Consider the Bagely-Torvik equation*

$$D_q^2 p(z) + D_q^{\frac{3}{2}} p(z) + p(z) = 1 + z, \; p(0) = 1, \; p^{\cdot}(0) = 1$$

*The exact solution of the given problem is* $p(z) = 1 + z$. *By implementing the method described in the previous section 4.1 with n=2, we imprecise the solution as*

$$p(z) = c_0 L_0(z) + c_1 L_1(z) + c_2 L_2(z) = c^T \phi(z),$$

*where*

$$D_q^1 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & -\frac{1}{q} & 0 \end{pmatrix}, \quad D_q^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{q^2} & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix},$$

$$D_q^{\frac{3}{2}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -\frac{1}{q^3} + \frac{\Gamma_q(\frac{1}{2})}{\Gamma_q(-\frac{1}{2})} & \frac{1}{q^7} - \frac{2_q!\Gamma_q(\frac{1}{2})}{q^5\Gamma_q(-\frac{1}{2})} + \frac{\Gamma_q(\frac{3}{2})}{q^2\Gamma_q(-\frac{1}{2})}. \end{pmatrix}$$

*Therefore, using* (4.7)*, we obtain*

$$c_0 + (\frac{1}{q^2} + 1)c_2 - 2 = 0. \tag{4.9}$$

*Also by using* (4.8)*, we have*

$$-c_1 - (1 + \frac{1}{q})c_2 - 1 = 0 \tag{4.10}$$

*and*

$$c_0 + c_1 + c_2 = 1. \tag{4.11}$$

*By solving above* (4.9)–(4.11)*, we get*

$$c_0 = 2, \ \ c_1 = -1, \ c_2 = 0.$$

*Thus, we can write*

$$p(z) = \begin{pmatrix} 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 - z \\ \frac{1}{q\,2_q}(z^2 - 2_q^2 z + q\,2_q) \end{pmatrix} = 1 + z,$$

*which is the exact solution.*

## 5. Conclusion

This paper deals with solutions of $q$-fractional differential equations via operational matrix method with the help $q$-Laguerre polynomials. First, we obtain the $q$-LOM of $q$-fractional derivatives. The benefit of the present operational matrix technique has less computational and multifaceted nature since each operational matrix of differentiation incorporates generally zeros areas and in like manner decrease the time and gives courses of action high accuracy.

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