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THE EFFECT OF DENSITY DISSIPATION IN THE PERTURBED SYSTEM OF EULER EQUATIONS

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ABSTRACT. In this paper, we studied the perturbation in the system of Euler equations of two dimensional case in order to gain a better understanding regarding the causes of the shock instabilities found in several high speed flow simulations. This perturbation analysis is performed on the linearized Euler equations taking the inspirations from the original work of Dyakov. The instability mechanisms are studied analytically and numerical experiments are conducted to confirm the results. The fluctuation in density is found to be one of the major contributors to the problem. Then, an artificial and a tunable dissipation parameter is added only in density equation to suppress the fluctuation. This mechanism is found to be able to stabilize the solution.

1. INTRODUCTION

Shock instability is not exclusively referring to the carbuncle phenomenon. An early discovery of shock instability was from Richmeyer(1960)-Meshkov(1969) or R-M instability [4, 28] which occurs when a shockwave passed through the perturbation in between two fluids of different density [1]. There are general variations of such oddity that exist in between two distinct fluids for instance the Rayleigh(1900)-Taylor(1950), R-T and Kelvin-Helmhotz,K-H instability [1]. All these are related to abnormal appearance at the interface of two fluids of

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distinct density subjected to acceleration field [1]. Note that instability can also be in the form of numerical solution growing erratically to a point of divergence which can happen if the CFL condition is not met when solving transport problems using explicit schemes. However in this paper, we define the instability to be of numerical results having multiple plausible solutions and the inability to converge to a unique one.

Notwithstanding, the shock instability incurred by carbuncle phenomenon has similarly brought many researchers in grief with sweat and tears. The problem is not easily solved by simply using numerical diffusion or adding more meshes. Instead, different schemes have different conclusions on the carbuncle; thus until now a general and a robust consensus is hardly achievable. At first encountered by Peery and Imlay [31], though some had concurrently computed it [25]. They treated this problem using eigenvalue smoothing of certain values and fixing spatial parameter coefficient to be zero producing barely satisfactory results [31]. Nevertheless, the operation is scarcely explanatory since in doing so, numerical dissipation is added to flux functions which tends to eliminate numerical instabilities similarly to other dissipative schemes. In the early years of its discovery, many claimed that the problem only existed in multidimensional case as perceived earlier. The latest research refuted this by showing that the problem lies fundamentally in the one-dimensional case [5, 12] [44, 46]. A higher dimension however, does complicate the instability [18].

Another hunch is that the presence of the pressure in the density flux is the cause of such occurrence hence the development of schemes to separate these variables famously known as Advection Upstream Splitting Method or preferably AUSM [24]. These methods were claimed to have solved the anomaly but was refuted by [11, 17, 29, 40]. On top of that, there is an oscillation due to back pressure difference [22–24, 29] which is not desirable. Qu [33] took the advantage of the Liou's conjenture to the Roe's flux and modify numerical dissipation for density flux instead of pressure. Further suspicion is on the odd-even decoupling that exists along the shock to the grid alignment [34] and added that dissipation in any Riemann solvers is inevitable. It also suggested to use *ad hoc* switching scheme whenever necessary in order to keep the dissipation the least. There are other earlier schemes as well [16, 38], to name a few, that can be summarized as adding numerical dissipation to the flux function will remove the carbuncle.

The paper by [17] provides a comprehensive list of schemes that had been tested for shock instability in which almost all schemes succumbed in one way or another. A three dimensional carbuncle was also studied by [19] revealing that the instability is somewhat amplified in three dimensions but still does not unravel the root of the carbuncle. A recent approach by Zaide [47] used a flux function approach which assumes a linear Rankine-Hugoniot condition that mathematically complies with the jump conditions across a shock with multiple intermediate cells yet physically arguable. Another [40] introduced a Linearized-Riemann Solver (LRS) which exploited the averaging of interface state instead of the averaging interface flux. They however, mentioned very little on the causes of carbuncle occurrence. Tu et. al [43] also suspected the intermediate state that is closer to the downstream flow is a form of seeding into instability yet provided no reason of such occurrence. An analysis on the stages of carbuncle's swelling documented that the entropy and shear waves are signaling the stages of instability [32]. These results are numerically supported by [15] with further conclusion that the shear wave has a greater impact for instability compared to the entropy wave. An acceptable cause was the shock was not aligned correctly to the grid [13] and in some schemes, such as AUSM, it achieves marginal stability upon grid alignment and on a refined grid [27]. Ohwada [30] exploited this notion and fixed the grid meshes mechanism to be aligned with the shockfront; though not completely obliterated but managed to regress the instability growth.

Furthermore, a majority of researchers agreed that the problem is purely numerical instead of physical since the shock solution of blunt body depicted by carbuncle is experimentally possible by placing a needle [11] or a 2D slab [10]at the body's stagnation point. Robinet et al. [14] however argued that carbuncle is inherently embedded in the model of continuum equations. He found that a specific particular Mach number will give instability even for an ideal gas. Elling [9] further support this notion and said a pure Euler's scheme fails to compute the viscosity effect of the incoming flow especially at the boundaries.

The numerical dissipation insertion by fixing the grid alignment or any type of artificial measure to rectify carbuncle phenomenon is only the tip of the iceberg; there are stages of carbuncle and once the first stage has emerged, one cannot halt the progress [32], [11]. We shall take a more modest approach in this paper investigating the carbuncle problem. Rather than finding a cure, we intend to shed some light of why it happens. When [45] applied perturbation on the conservative variables using the approach from Dyakov [7,8], Kontorovich [20], and several others [21,27] [32,42], they found that several sources are seeding into the instability in the one dimensional case tests in both stationary shock as well as slowly moving shock. Furthermore, when one of the sources was suppressed by using artificial dissipation, results in both cases have shown stability. In this paper however, we will extend the method in one dimensional case into two dimensional equations. This paper is arranged such that in the next section, we will work on the pertubation analysis for two dimensional Euler's equations. Then, we will test the perturbation analysis in partial two dimension grid as taken from an ingenuine two-dimensional test case first presented by [27]. The numerical methods for the experiment includes primarily the Roe flux since it has minimal numerical diffusion thus closely follows the Euler equations. Too much of numerical diffusion may contaminate the process of replicating the analytical works in a numerical setting. We also tried AUSM [23,24] and Entropy-Consistent(EC) flux [13]. These additional schemes are complementary. After that, we will attempt to remove the source of instability by using similar artificial dissipation but adding another term for y-component due to 2D effect. The last section will conclude the findings.

2. Analysis of Shock Instability

This paper also used the linearized expression in the Euler equations by using the definition of 'small disturbance' as done by [7,8] such that

$$\phi = \overline{\phi} + \phi', \tag{2.1}$$

where ϕ is the any quantity to be linearized, $\overline{\phi}$ is the mean value and ϕ' is the small disturbance.

2.1. **Stability Analysis Using Conservative Variables.** The first approach of our analysis is to perturb Euler equations defined as

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u H \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v H \end{pmatrix}_y = 0, \tag{2.2}$$

where $E = e + \frac{u^2+v^2}{2}$ is the total energy and that the equation of state for ideal gas $P = \rho e(\gamma - 1)$. The total enthalphy is defined as $H = E + \frac{P}{\rho}$.

The arbitrary disturbance in Equation (2.1)may be resolved into independent modes with related wave numbers, frequencies and eigenvectors which express into

$$\phi' = \mathbf{R}\exp[(lyi - \omega t + kx)]. \tag{2.3}$$

The above expression has an additional y-term due to the 2D effect compared to the 1D equation using the same approach in [45]. Then, Equation (2.3) above were plugged into Equation (2.2) Unstable modes are calculated by the singularity of determinant of the eigenvector matrix. In our analysis, we shall assume that the Jacobians of Equations (2.2) and (2.3) would include perturbed values about a linearized state, similar to the approach of Rayleigh [26,35,36] in understanding the vibration of a Hamiltonian system and Schrodinger's matrix perturbation theory [39] and the work inspired by them after that [6, 41]. This is where our work differs from previous work in studying shock-instability of the system of Euler equations.

2.2. **Analysis on 2D Euler Equations.** We write the two dimensional Euler equations as

$$\mathbf{U}_t + A\mathbf{U}_x + B\mathbf{U}_y = 0 \tag{2.4}$$

where $\mathbf{U} = \overline{\mathbf{U}} + \mathbf{U}' = [\overline{\rho} + \rho', \overline{\rho u} + (\rho u)', \overline{\rho v} + (\rho v)', \overline{\rho E} + (\rho E)']$ and A and B are the Jacobian matrices. Then using the definition for the ratio of the momentum linearization and density linearization by j_u and j_v, the inverse of the density linearization j_r the ratio energy linearization with density linearization j_e and the ratio of enthalphy linearization with density linearization j_h as

$$\begin{aligned}
&j_u = \frac{\overline{\rho u} + (\rho u)'}{\overline{\rho} + \rho'}, \quad j_v = \frac{\overline{\rho v} + (\rho v)'}{\overline{\rho} + \rho'}, \quad j_r = \frac{1}{\overline{\rho} + \rho'},\\
&j_e = \frac{\overline{\rho E} + (\rho E)'}{\overline{\rho} + \rho'} = \frac{p}{(\gamma - 1)(\overline{\rho} + \rho')} + \frac{1}{2}\frac{(\overline{\rho u} + (\rho u)')^2 + (\overline{\rho v} + (\rho v)')^2}{(\overline{\rho} + \rho')^2}\\
&\quad = \frac{p j_r}{\gamma - 1} + \frac{1}{2}(j_u^2 + j_v^2)\\
&j_h = \frac{\overline{H} + (H)'}{\overline{\rho} + \rho'} = \frac{\overline{\rho E} + (\rho E)'}{\overline{\rho} + \rho'} + p\frac{1}{\overline{\rho} + \rho'} = j_e + p j_r.
\end{aligned} \tag{2.5}$$

For two dimensional case, we added another definitions from Eq. (2.5) and define $j_w^2 = (j_u^2 + j_v^2)$, we have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma-1}{2}j_w^2 - j_u^2 & (3-\gamma)j_u & (1-\gamma)j_v & \gamma-1 \\ -j_u j_v & j_v & j_u & 0 \\ j_u\left(\frac{\gamma-1}{2}j_w^2 - j_h\right) & j_h - (\gamma-1)j_u^2 & (1-\gamma)j_u j_v & \gamma j_u \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -j_u j_v & j_v & j_u & 0 \\ \frac{\gamma-1}{2}j_w^2 - j_v^2 & (1-\gamma)j_u & (3-\gamma)j_v & \gamma-1 \\ j_v\left(\frac{\gamma-1}{2}j_w^2 - j_u\right) & (1-\gamma)j_u j_v & j_h - (\gamma-1)j_v & j_v\gamma \end{pmatrix}.$$

Equation (2.3) then is plugged into Equation (2.4) resulting into

$$\mathbf{U}'[Ak - \omega I + iBl] = 0$$

whose determinant is expressed by

$$\det(Ak - \omega I + iBl) = [(\omega - kj_u - ilj_v)^2 - G^2a^2](\omega - kj_u - ilj_v)^2 = 0, \tag{2.6}$$

where $G = \sqrt{k^2 - l^2}$ and roots of $\omega = [kj_u + ilj_v, kj_u + ilj_v \pm Ga, kj_u + ilj_v]$. If the wave numbers are equal, then $G = 0$, resulting into all repeated roots. This is the case where the acoustic waves are indistinguishable from the entropy and shear wave which also in good agreement with [32]. With the eigenvalues above, the corresponding eigenvectors are given by:

$$\mathbf{R} = \begin{bmatrix}\mathbf{R}_1 & \mathbf{R}_2 & \mathbf{R}_3 & \mathbf{R}_4\end{bmatrix}$$

where

$$\mathbf{R}^1 = \begin{bmatrix} 1 \\ j_u - \dfrac{a}{\sqrt{1-l^2/k^2}} \\ j_v - \dfrac{ia}{\sqrt{l^2/k^2-1}} \\ j_h - \dfrac{a}{Q}(kj_u + ilj_v) \end{bmatrix}, \quad \mathbf{R}^4 = \begin{bmatrix} 1 \\ j_u + \dfrac{a}{\sqrt{1-l^2/k^2}} \\ j_v + \dfrac{ia}{\sqrt{l^2/k^2-1}} \\ j_h + \dfrac{a}{Q}(kj_u + ilj_v) \end{bmatrix}$$

$$\mathbf{R}^2 = \begin{bmatrix} 1 \\ j_u + \dfrac{ilj_v}{k} \\ 0 \\ \dfrac{j_u^2}{2} + \dfrac{ij_uj_vl}{k} - \dfrac{j_v^2}{2} \end{bmatrix}, \qquad \mathbf{R}^3 = \begin{bmatrix} 0 \\ -\dfrac{il}{k} \\ 1 \\ \dfrac{-ilj_u}{k} + j_v \end{bmatrix}$$

The eigenvectors are linearly dependent if

$$\det(\mathbf{R}) = 2a\gamma\left(\frac{pj_r}{\gamma-1}\right)\sqrt{1-l^2/k^2} = 0. \tag{2.7}$$

The above equation holds when any of following conditions are met:

(i) The acoustic waves coincide with both entropy wave and shear wave.
(ii) The speed of sound $\to 0$.
(iii) The fluids specific heat ratio $\gamma \to 1$.
(iv) The pressure $\to 0$.
(v) The inverse of perturbed density, $j_r = \dfrac{1}{\overline{\rho}+\rho'} \to 0$.

The first four conditions have been reported in the literature [32], whereas the last condition is reported in [45]. In addition, the first condition is where we believe that it is the multi-dimensional root for shock instability which occurs when the acoustic waves coincides with the shear and entropy waves which resulted into equal wave number which is $l = k$ as reported by [32] during the "bleeding" stage of the carbuncle. Nonetheless, the final condition indicated that the continuous growth of the density fluctuation within the system may drive the inverse of density linearization to a very large number in which will eventually lead to instability. This behavior can be apparently seen in a steady-state shock, where [14] claimed that the pre-shock fluctuations are stable and the unstable fluctuations occur in the post-shock region.

Regardless, we have discovered that the density perturbation is indeed a highly potential root for shock instability even in the two dimensional Euler equations. The second possible cause of shock instability is when the acoustic waves are indistinguishable from the entropy and shear waves.

3. Numerical Experiments on Shock Instability

The numerical test conducted in this section is by using the following mechanism:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}[F] + \xi^\star, \tag{3.1}$$

where $[F]$ refers to the flux calculated using Roe averaging method and ξ is the second order artificial dissipation that will be added later. The flux using Roe scheme is chosen due to its least dissipative method that closely follows Euler equations.

At this stage, it is a bit premature to analyze the shock instability case on the supersonic flow over a two dimensional cylinder using the analytical results of the previous section. Instead, we shall work on a simplified carbuncle problem which has been proposed by [27] and [17]. This simplified carbuncle problem is in the form of steady shocks in (quasi) two dimensions. We shall refer to this as the 1.5D carbuncle problem. The following describes the simplified (1.5D) carbuncle problem.

3.1. Initial Condition.

Stationary shock was considered in which Rankine-Hugoniot jump condition based on $\Delta F = 0$ [27]. Then the pre-shock and post-shock profiles are

(3.2)
$$U_{pre} = \left[1.0 \quad 1.0 \quad 0.0 \quad 1/2+\eta\right],$$
$$U_{post} = \left[f(M_{pre}) \quad 1.0 \quad 0.0 \quad g(M_{pre})\eta + \frac{1}{2f(M_{pre})}\right],$$

where

(3.3)
$$f(M_{pre}) = \left(\frac{1}{\dfrac{2}{(\gamma+1)M_p^2re} + \dfrac{\gamma-1}{\gamma+1}}\right),$$
$$g(M_{pre}) = \frac{2\gamma M_{pre}^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1},$$
$$\eta = \frac{1}{\gamma(\gamma-1)M_{pre}^2}.$$

We used incoming Mach number to be $M_{pre} = 5.0$, with CFL condition of $\nu = 0.2$ and 50 computational square grid cells. In addition, all types of limiters are excluded from the computation.

3.2. **Boundary Conditions.** The boundary conditions for inlet and outlet for all variables were being kept constant at the exact Rankine-Hugoniot relation using the ghost cells on the left and on the right. The top and bottom walls were set to be hard walls. The inlet and outlet conditions exactly followed the configuration of stability analysis numerical setup by [27]. All of these are done to ensure a similar conditions as close as possible to the analytical part.

3.3. **Perturbation Procedures to Induce Shock Instability.** The analysis then is followed by a series of instability test done numerically. All respected equations in the beginning were tested without any perturbation. The instability is indicated by the residual error showing a sign of instability such as the limit cycle [12]. The first instability test is using the random perturbation as done by [27] as expressed below
$$\mathbf{U}_s = \overline{\mathbf{U}} + \epsilon\mathbf{U}.$$
This method mimics the fluctuations from its mean value which is in good agreement with the linearization process. The ϵ has an interval of $[10^{-3}, 10^{-6}]$ as

practised by [19]. In addition, a second test is also being administered by introducing an intermediate point at the shock location to mimic the fully $2D$ grid orientation [11] unbiasedly on all conservative variables such that

$$\mathbf{U}_s = \delta \mathbf{U}_0 + (1-\delta)\mathbf{U}_1,$$

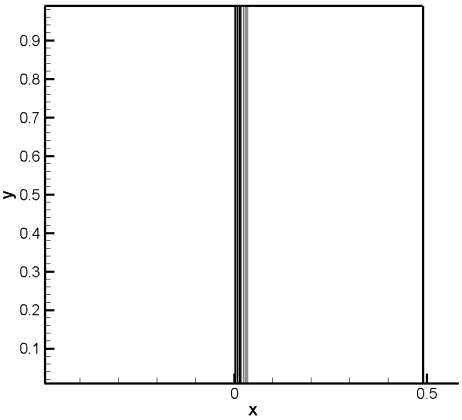
where the subscript s, 0 and 1 refer to the shock location, pre-shock and post-shock profile respectively. The range of δ is from 0.0 to 1.0.

3.4. **Observations on Shock Instability.** We did both perturbation procedures and obtained similar pattern of results. We present several figures as a sample of results for Mach 5 shock profile that leads to carbuncle from the initial conditions as expressed in Equation (3.2,3.3) with boundary conditions as explained in Section (3.2). The computed shock instability came in a sequence of three stages that is called "pimples" in the first stage, then "bleeding" for the second one and "carbuncles" at the last stage [32]. These stages are demonstrated in the following figures. The "pimples" stage depicts wiggles along the normal shock (Fig. 1b). We believe that the "pimples' stage is due to the density fluctuations based on our analysis in Section (2.2).

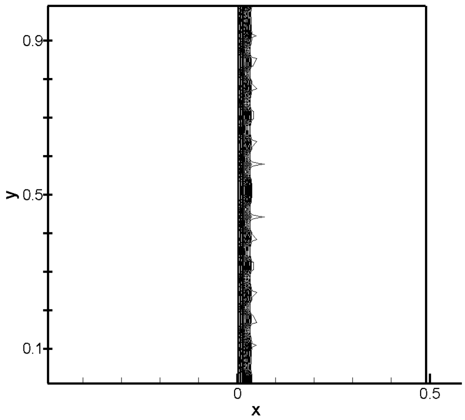
The "bleeding" stage is depicted in Figure 1c in which parallel jets emanating from the spots of "pimples" propagate downstream. According to [32], this stage occurs when the acoustic waves coincide with the shear waves, which is consistent with our analysis as shown in Equation (2.6) and (2.7) where $Q = 0$. We took a sample of these eigenvalues in the unstable numerical solutions from the Roe's flux during the "bleeding" stage to verify our claim. The values are shown in the next figure. The stable solutions as shown in Figure (2a) clearly show distinct eigenvalues. Figure (2b) has demonstrated that indeed several eigenvalues from the acoustic waves are identical (if not almost identical) with entropy/shear wave. This confirms that indeed "bleeding" stage occurs when the eigenvalues are indistinguishable.

The "carbuncles" stage in Figure (1d), occurs when the normal shock is replaced with a series of oblique shocks. We believe that this is a strongly non-linear phenomenon that the perturbation analysis cannot capture.

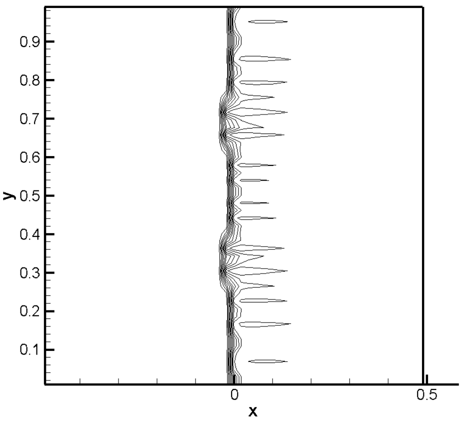
The conjecture is that if we can subdue the growth in the first ("pimples") stage from marching to the second ("bleeding") stage, perhaps stability can be achieved. Therefore, based on our numerical observations and from the analysis in the previous section, we suggest the following section to proof our conjecture.



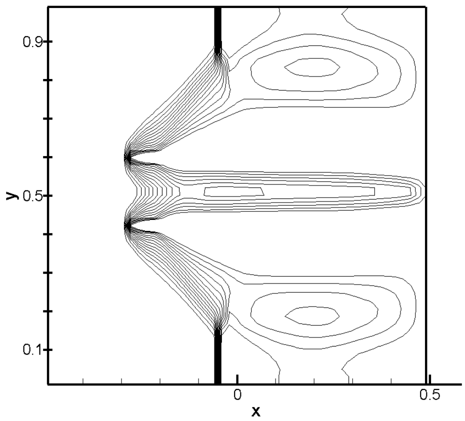
(A) Initial



(B) Pimple

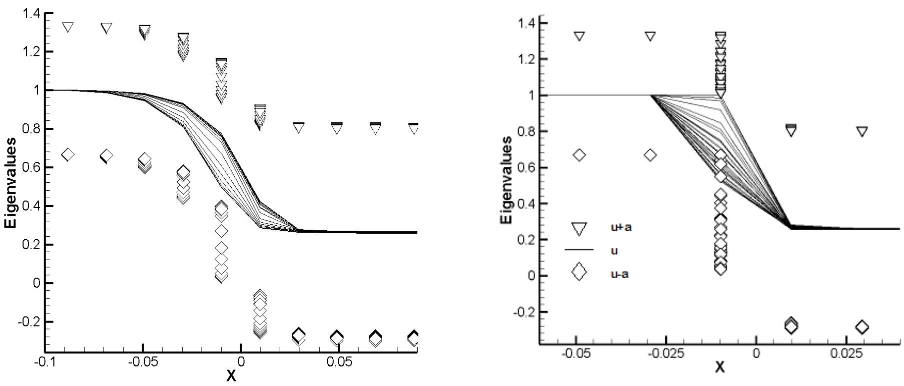


(C) Bleeding



(D) Carbuncle

FIGURE 1. Line contour corresponds to the Mach number profile whose ahead of the shock is 5. (1b) is the first stage after 400 timesteps where the contour displays some disturbance along the shock but manage to present the overall solution. After 500 timesteps, (1c), a series of parallel jets excreting from the developed pimple in stage one. In (1d), a series of wedges formed along the plane shock and moves toward the inlet at timesteps > 800.



(A) Stable (B) Unstable

FIGURE 2. Comparing the eigenvalues. The solid line representing the u-eigenvalues at different pseudo times. The 'triangles' and 'diamonds' are u+a-eigenvalue and u-a-eigenvalue respectively.

3.4.1. *Brief Remarks on Tested Schemes.* The schemes that were chosen to demonstrate the instability are the Roe's flux scheme [37], selected AUSM's flux [22–24] and the Entropy-Consistent's flux scheme [13]. Of all these three schemes, only the Roe's flux has gone through all the stages of carbuncle. The AUSM's family fluxes exhibit first stage of instability; for example the AUSM+ retained at pimple stage. The Entropy-Consistent method demonstrated the best (at least in terms of minimal range of instability) among these three schemes but it produces thicker shock profiles. Furthermore, both type of perturbations demonstrated similar stages of progression to instability. Therefore, one is free to chose either technique.

3.5. **An Attempt to Remove Density Fluctuations in 1.5D.** We attempted to fix instability by adding a diffusive factor on the right-hand-side of Equation (2.2) only to the density equation such that

$$\mathbf{U}_t + \hat{A}\mathbf{U}_x + \hat{B}\mathbf{U}_y = \xi(\rho_{xx} + \rho_{yy}),$$

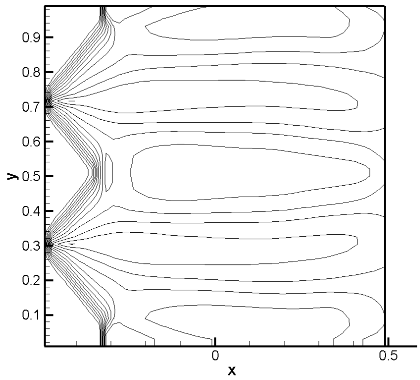
where the $\hat{\cdot}$ on the Jacobians refers to the linearized properties. Then, pure central differencing to evaluate ρ_{xx}, ρ_{yy} was used. This mechanism is applied to the Equation (3.1), where $\xi^\star$ is now added and defined by the following expression:

$$\xi(\rho_{xx}) = \xi\left(\frac{\rho_{[i+1][j]} - 2\rho_{[i][j]} + \rho_{[i-1][j]}}{\Delta x^2}\right)$$

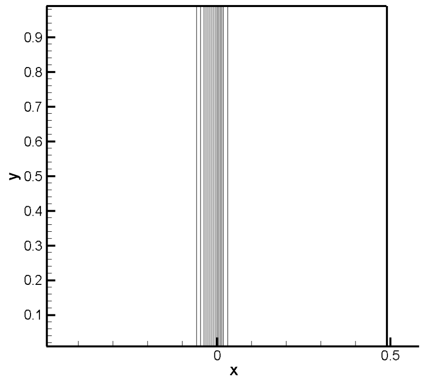
for x-direction and similarly applied in y-direction.

The purpose of this adding artificial diffusion to the density equation is to show that we can focus on curing on one variable instead of the whole system thus keeping the dissipation minimal. Furthermore, this dissipation can be looked into another perspective such as given by [2] and [3] where there is a possibility of volume dissipation accross any flow. At the moment, we are fixating on the conservative variables based on the analysis, hence the density is the target instead of narrowing down to the volume. Nonetheless, we have found that the minimum value of $\xi \geq 0.04$ is ample to resist the recurring instability for all schemes. The computed solutions in the following figures are being compared to the state before and after the inclusion of density dissipation. Note that this fix is just merely to prove our point that density is one of the potential of shock-instability, and not necessarily the best fix for the carbuncle problem. In fact, we have attempted to add similar dissipation separately to the momentum and energy equations without adding diffusion to the density but the shock instability still persists. Adding diffusion on all conservative equations would be excessive and physically unjustifiable.

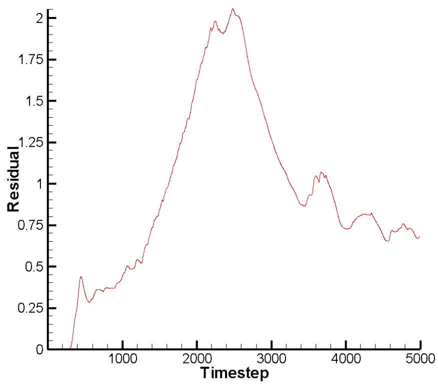
Figure (3a) above demonstrates that the shock with the carbuncle has moved uptream. This behavior is displayed by the residual in Figure (3c) where the residual is initially increasing indicating the shock movement but eventually decreasing corresponding to the fact that it is no longer within the calculated domain. The Mach's solution profile with the dissipation only in density is shown in Figure (3b) where the shock is stationary; thus, demonstrating that the artificial dissipation is preserving the conservative form. Furthermore, the carbuncle, even the pimple stage is absent. Another tested scheme is from the AUSM's family, which is AUSM+ and the solutions are displayed in Figure (4). Figure (4a) shows the pimple stage for AUSM+ and this stage is completely absent when a similar dissipation is added as presented in Figure (4b). Lastly, though not being shown, the Entropy-Consistent scheme also displayed a similar result.



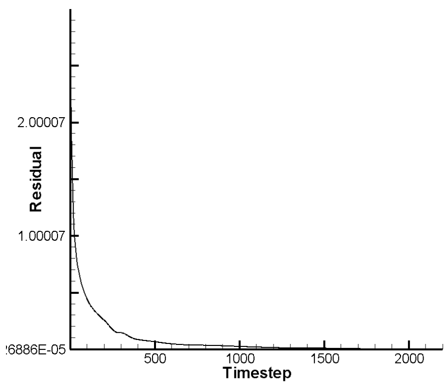
(A) Roe-before



(B) Roe-after



(C) Residual-before

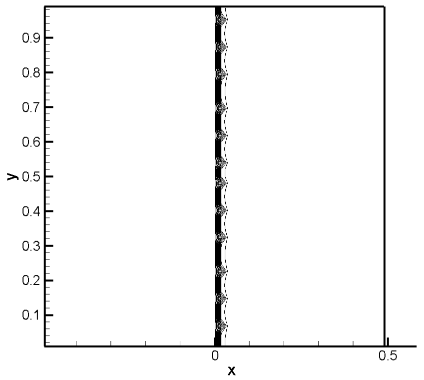


(D) Residual-after

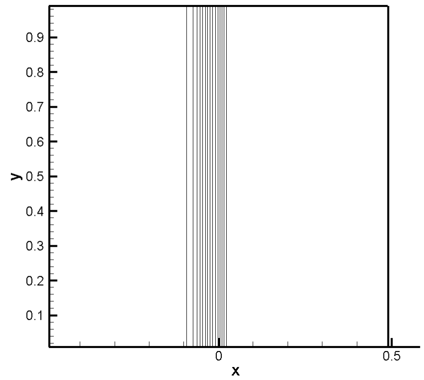
FIGURE 3. The computed profile for Mach's solution before and after adding ς on Roe's flux scheme using an error of $\epsilon = 1.0e - 6$

4. CONCLUDING REMARKS

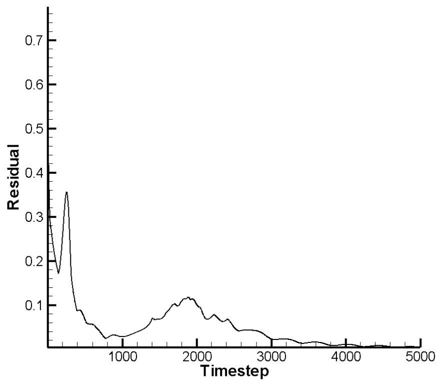
For the systems of Euler equations, we have found that density fluctuations is one possible root of shock instability. In one dimension, other than approaching vacuum state, instability is solely due to the growth in density fluctuations. In two dimensions, we have analytically discovered that density fluctuation is just one of the potential roots of shock instability. Applying artificial diffusion to the density equations reduces the fluctuations and prevents the "pimples" stage



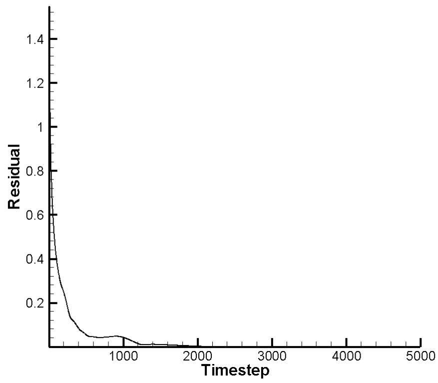
(A) AUSM+-before



(B) AUSM+-after



(C) Residual-before



(D) Residual-after

FIGURE 4. Solution's profile and residual errors comparison for AUSM+ scheme before and after the adding the diffusion with an intermediate error of $\delta = 0.5$.

to grow to the "bleeding" stage hence achieving stability in our numerical experiments. This additive is a proof that the density perturbation is a trigger to the problem and a cure can be implemented based on this discovery if we are to believe that shock instability always starts from "pimples" and progresses to "bleeding". From our analysis, instability may occur when the acoustic waves coincide with the shear and entropy waves which is depicted during the "bleeding" stage. Perhaps this is the multi dimensional root of instability being reported by [19]. There is no way we could guarantee that instability will only start from the "pimples" stage. Although it is not reported herein, there is a possibility that the process of shock instability could bypass the "pimples" stage and proceed to the "bleeding" stage during the transient phase of computations, for which we currently have no solution to overcome that.

As such, we shall not make the claim of resolving the carbuncle problem, rather pointing out to the potential sources of instability. More needs to be done in finding a way to resolve the issue when the physical characteristic waves being indistinguishable from each other. There is no restriction on the system for which the wave resonance does not happen during the transient phase of computations. We believe that the remedy is beyond just adding more diffusion but we are hopeful that there is a solution. The work is currently underway.

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