ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **10** (2021), no.1, 279–285
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.1.28

ON GENERALIZATIONS OF 2-ABSORBING PRIMARY IDEALS IN SEMIGROUPS

Pairote Yiarayong

ABSTRACT. Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function where $\mathcal{I}(S)$ is a set of all ideals of a semigroup. We extend the concept of primary and 2-absorbing ideals in semigroups to the context of ϕ-2-absorbing primary ideals. We say that a proper ideal A of a semigroup S is a ϕ-2-absorbing primary ideal if $a, b, c \in S$ with $abc \in A - \phi(A)$ implies that $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$. The aim of this paper is to investigate the concept of ϕ-2-absorbing primary ideals in semigroups. Finally, we obtain sufficient conditions of a 2-absorbing primary ideal in order to be rephrased a ϕ-2-absorbing primary ideal in a semigroup.

1. ϕ-2-ABSORBING PRIMARY IDEALS

In this section, we introduce the concept of ϕ-2-absorbing primary ideals in semigroups and give its characterizations corresponding to ϕ-2-absorbing primary ideals in semigroups.

Let A be a subset of a semigroup S. Then, the **radical** (see [1]) of A is defined as $\sqrt{A} = \{a \in S : a^n \in A$ for some positive integer $n\}$.

Definition 1.1. *Let S be a semigroup and let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function where $\mathcal{I}(S)$ be the set of all ideals of S. A proper ideal A of S is called a* ***ϕ-2-absorbing primary ideal*** *if for each $a, b, c \in S$ with $abc \in A - \phi(A)$, then $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$.*

2020 *Mathematics Subject Classification.* 13A15, 13F05.
Key words and phrases. semigroup, ideal, 2-absorbing primary ideal, ϕ-2-absorbing primary ideal.

We now present the following example satisfying above definition.

Example 1. *Let $S = \{a, b, c, d, e\}$ be a semigroup with following multiplication given by*

$\circ$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

It is easy to see that $\{a, b, d\}$ is a ϕ-2-absorbing primary ideal of a semigroup S.

Remark 1.1. *It is easy to see that every 2-absorbing primary ideal of a semigroup S is a ϕ-2-absorbing primary ideal of S.*

The following example shows that the converse of Remark 1.1 is not true.

Example 2. *Let $S = \mathbf{Z}^+$. Consider the proper ideal $P = 30\mathbf{Z}^+$ of the semigroup S. Define $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ by $\phi(A) = A$ for every $A \in \mathcal{I}(S)$. It is easy to see that P is a ϕ-2-absorbing primary ideal of S. Notice that $2 \cdot 3 \cdot 5 \in P$, but $2 \cdot 3 \notin P, 3 \cdot 5 \notin \sqrt{P}$ and $2 \cdot 5 \notin \sqrt{P}$. Therefore P is not a 2-absorbing primary ideal of S.*

Let S be a semigroup and let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function. Since $A - \phi(A) = A - (A \cap \phi(A))$ for all $A \in \mathcal{I}(S)$, without loss of generality, we will assume that $\phi(A) \subseteq A$. Throughout this paper, as it is noted earlier, if ϕ is a function, then we always assume that $\phi(A) \subseteq A$.

Theorem 1.1. *Let A be a non empty subset of a commutative semigroup S. Then the following properties hold.*

(1) *If A is an ideal of S, then $\sqrt{A}$ is an ideal of S containing A.*
(2) $\sqrt{A} = \sqrt{\sqrt{A}}$.
(3) *For each ϕ-2-absorbing primary ideal A of S if $\sqrt{\phi(A)} \subseteq \phi\left(\sqrt{A}\right)$, then $\sqrt{A}$ is a ϕ-2-absorbing primary ideal of S.*
(4) *For each element s of $S - \sqrt{A}$ if A is a ϕ-2-absorbing primary ideal of S such that $\sqrt{\phi(A)} \subseteq \phi\left(\sqrt{A}\right)$, then $(\sqrt{A} : s)$ is a ϕ-2-absorbing primary ideal of S with $\left(\phi(\sqrt{A}) : s\right) \subseteq \phi\left(\sqrt{A} : s\right)$.*

Proof.

1. Assume that A is an ideal of S. It is easy to see that, $A \subseteq \sqrt{A}$. Let a and s be any elements of S such that $a \in \sqrt{A}$. Then we have, $a^n \in A$ for some positive integer n, which implies that $(sa)^n = s^n a^n \in s^n A \subseteq A$. Therefore, $sa \in \sqrt{A}$ and hence $\sqrt{A}$ is an ideal of S containing A.

2. Assume that A is a subset of S. Obviously, $\sqrt{A} \subseteq \sqrt{\sqrt{A}}$. On the other hand, let $x \in \sqrt{\sqrt{A}}$. Then we have, $x^n \in \sqrt{A}$ for some positive integer n, which means that $x^{nm} \in A$ for some positive integer m. Therefore, $x \in \sqrt{A}$ and hence $\sqrt{A} = \sqrt{\sqrt{A}}$.

3. Assume that A is an ideal of S. Then by part 1, $\sqrt{A}$ is an ideal of S. Let a, b and c be any elements of S such that $abc \in \sqrt{A} - \phi(\sqrt{A})$. Thus we have, $abc \notin \phi(\sqrt{A})$ and $(abc)^n \in A$ for some positive integer n. Since $\sqrt{\phi(A)} \subseteq \phi\left(\sqrt{A}\right)$, we have $(abc)^n \notin \phi(A)$, which implies that $(abc)^n \in A - \phi(A)$. In fact, since A is a ϕ-2-absorbing primary ideal of S, we have $(ab)^n \in A$ or $(bc)^n \in \sqrt{A}$ or $(ac)^n \in \sqrt{A}$. Therefore $ab \in \sqrt{A}$ or $bc \in \sqrt{\sqrt{A}}$ or $ac \in \sqrt{\sqrt{A}}$ and hence $\sqrt{A}$ is a ϕ-2-absorbing primary ideal of S.

4. Let a, b, c and s be any elements of S such that $abc \in (\sqrt{A} : s) - \phi(\sqrt{A} : s)$. Since $(\phi(\sqrt{A}) : s) \subseteq \phi(\sqrt{A} : s)$, we have $ab(cs) \in \sqrt{A} - \phi(\sqrt{A})$. Then by parts 2 and 3, $ab \in \sqrt{A}$ or $bcs \in \sqrt{A}$ or $acs \in \sqrt{A}$, which implies that $ab \in (\sqrt{A} : s)$ or $bc \in \sqrt{(\sqrt{A} : s)}$ or $ac \in \sqrt{(\sqrt{A} : s)}$. Consequently, $(\sqrt{A} : s)$ is a ϕ-2-absorbing primary ideal of S. □

In the light of the definition of ϕ-2-absorbing primary ideal in commutative semigroups, we can obtain the following properties.

Theorem 1.2. *Let S be a commutative semigroup and let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function. If A is a ϕ-2-absorbing primary ideal of S such that $\sqrt{A}$ is a primary ideal of S, then $(A : s)$ is a ϕ-2-absorbing primary ideal of S for every $s \in S - \sqrt{A}$ with $(\phi(A) : s) \subseteq \phi(A : s)$.*

Proof. Let a, b, c and s be any elements of S such that $abc \in (A : s) - \phi(A : s)$. Since $(\phi(A) : s) \subseteq \phi(A : s)$, we have $a(bc)s \in A - \phi(A)$. In fact, since A is a ϕ-2-absorbing primary ideal of S, we have $abc \in A$ or $bcs \in \sqrt{A}$ or $as \in \sqrt{A}$.

If $bcs \in \sqrt{A}$ or $as \in \sqrt{A}$, then $bc \in \sqrt{(A : s)}$ or $a \in \sqrt{(A : s)}$, since $\sqrt{A}$ is a primary ideal of S and $s \in S - \sqrt{A}$. Next, if $abc \in A$, then $abc \in A - \phi(A)$. Therefore, $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$. In any case, we have $ab \in (A : s)$ or

$bc \in \sqrt{(A:s)}$ or $ac \in \sqrt{(A:s)}$. Consequently, $(A:s)$ is a ϕ-2-absorbing primary ideal of S. □

In the following result, we give an equivalent definition of ϕ-2-absorbing primary ideals in a commutative semigroup.

Theorem 1.3. *Let $\phi : \mathcal{I}(S) \rightarrow \mathcal{I}(S) \cup \{\emptyset\}$ be a function. The following conditions are equivalent:*

(1) *A is a ϕ-2-absorbing primary ideal of S.*
(2) *For each elements a and b of S if $ab \in S - A$, then $(A : ab) \subseteq (\phi(A) : ab) \cup \sqrt{(\sqrt{A} : a^n)} \cup \sqrt{(\sqrt{A} : b^n)}$ for some positive integer n.*

Proof. First assume that (1) holds. Let a, b and c be any elements of S such that $c \in (A : ab)$. Then we have, $abc \in A$. If $abc \notin \phi(A)$, then $abc \in A - \phi(A)$. Since A is a ϕ-2-absorbing primary ideal of S, we have $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$. By assumption, $c \in \sqrt{(\sqrt{A} : a^n)}$ or $c \in \sqrt{(\sqrt{A} : b^n)}$ for some positive integer n that is, $c \in \sqrt{(\sqrt{A} : a^n)} \cup \sqrt{(\sqrt{A} : b^n)} \subseteq (\phi(A) : ab) \cup \sqrt{(\sqrt{A} : a^n)} \cup \sqrt{(\sqrt{A} : b^n)}$. If $abc \in \phi(A)$, then $c \in (\phi(A) : ab) \subseteq (\phi(A) : ab) \cup \sqrt{(\sqrt{A} : a^n)} \cup \sqrt{(\sqrt{A} : b^n)}$. Consequently, $(A : ab) \subseteq (\phi(A) : ab) \cup \sqrt{(\sqrt{A} : a^n)} \cup \sqrt{(\sqrt{A} : b^n)}$.

Conversely, assume that (2) holds. Let a, b and c be any elements of S such that $abc \in A - \phi(A)$. Then we have, $c \in (A : ab)$ and $c \notin (\phi(A) : ab)$. If $ab \in A$, then there is nothing to prove. If $ab \notin A$, then $(A : ab) \subseteq (\phi(A) : ab) \cup \sqrt{(\sqrt{A} : a^n)} \cup \sqrt{(\sqrt{A} : b^n)}$ for some positive integer n. Since $c \in (A : ab)$ and $c \notin (\phi(A) : ab)$, we have $c \in \sqrt{(\sqrt{A} : a^n)} \cup \sqrt{(\sqrt{A} : b^n)}$. Therefore, $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$ and hence A is a ϕ-2-absorbing primary ideal of S. □

The next theorem gives the relationships between 2-absorbing primary ideals and ϕ-2-absorbing primary ideals of a semigroup S.

Theorem 1.4. *Let $\phi : \mathcal{I}(S) \rightarrow \mathcal{I}(S) \cup \{\emptyset\}$ be a function and let $\phi(A)$ be a 2-absorbing primary ideal of a semigroup S. Then A is a ϕ-2-absorbing primary ideal of S if and only if A is a 2-absorbing primary ideal of S.*

Proof. First assume that A is a 2-absorbing primary ideal of S. Obviously, A is a ϕ-2-absorbing primary ideal of S.

Conversely, assume that A is a ϕ-2-absorbing primary ideal of S. Let a, b and c be any elements of S such that $abc \in A$. If $abc \notin \phi(A)$, then $abc \in A - \phi(A)$. By assumption, $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$. Now if $abc \in \phi(A)$, then $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$. In any case, we have A is a ϕ-2-absorbing primary ideal of S. □

In the following we shall introduce the notion of ϕ-2-absorbing primary triple zero of a ϕ-2-absorbing primary ideal A in a semigroup S.

Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function and let A be a ϕ-2-absorbing primary ideal of a semigroup S a triple $(a, b, c), a, b, c \in S$ is a **ϕ-2-absorbing primary triple zero** if

(1) $abc \in \phi(A)$
(2) $ab \notin A$ and $bc \notin \sqrt{A}$ and $ac \notin \sqrt{A}$.

Remark 1.2. *Note that a proper ideal A of a semigroup S is a ϕ-2-absorbing primary ideal of S that is not a 2-absorbing primary ideal of S if and only if A has a ϕ-2-absorbing primary triple-zero (a, b, c) for some $a, b, c \in S$.*

Now we investigate the ϕ-2-absorbing primary triple zero of a ϕ-2-absorbing primary ideal A in a semigroup S.

Theorem 1.5. *Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function and let A be a ϕ-2-absorbing primary ideal of a semigroup S. For each elements a, b and c of S if (a, b, c) is a ϕ-2-absorbing primary triple zero of A, then the following statements hold:*

(1) $abA \subseteq \phi(A)$;
(2) $aAc \subseteq \phi(A)$;
(3) $Abc \subseteq \phi(A)$;
(4) $A^2c \subseteq \phi(A)$;
(5) $aA^2 \subseteq \phi(A)$.

Proof.

1. Suppose that $abA \nsubseteq \phi(A)$. Then there exists an element d of A such that $abd \notin \phi(A)$. Thus we have, $\{abc\} \cup \{abd\} \nsubseteq \phi(A)$, which implies that $\{ab\}(\{c\} \cup \{d\}) \subseteq A - \phi(A)$. Since A is a ϕ-2-absorbing primary ideal of S, we have $ab \in A$ or $b(\{c\} \cup \{d\}) \subseteq \sqrt{A}$ or $a(\{c\} \cup \{d\}) \subseteq \sqrt{A}$. Therefore, $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$, which is a contradiction. Consequently, $abA \subseteq \phi(A)$.

2. Suppose that $aAc \nsubseteq \phi(A)$. Then there exists an element r of A such that $arc \notin \phi(A)$. Since $r \in A$, we have $a(\{b\} \cup \{r\})c \subseteq A$, which implies that $a(\{b\} \cup \{r\})c \subseteq A - \phi(A)$. In fact, since A is a ϕ-2-absorbing primary ideal of S, we have $a(\{b\} \cup \{r\}) \subseteq A$ or $(\{b\} \cup \{r\})c \subseteq \sqrt{A}$ or $ac \in \sqrt{A}$. Thus, $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$, which is a contradiction. Consequently, $aAc \subseteq \phi(A)$.

3. The proof is similar to part 2.

4. Assume that $A^2c \nsubseteq \phi(A)$. Then there exist elements r, s of A such that $rsc \notin \phi(A)$. Then by parts 2 and 3, $\{abc\} \cup \{rbc\} \cup \{asc\} \cup \{rsc\} \nsubseteq \phi(A)$, which implies that $(\{a\} \cup \{r\})(\{b\} \cup \{s\})c \subseteq A - \phi(A)$. In fact, since A is a ϕ-2-absorbing primary ideal of S, we have $(\{a\} \cup \{r\})(\{b\} \cup \{s\}) \subseteq A$ or $(\{b\} \cup \{s\})c \subseteq \sqrt{A}$ or $(\{a\} \cup \{r\})c \subseteq \sqrt{A}$. Therefore, $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$, which is a contradiction. Consequently, $A^2c \subseteq \phi(A)$.

5. Suppose that $aA^2 \nsubseteq \phi(A)$. Then there exist elements r, s of A such that $ars \notin \phi(A)$. Therefore by parts 1 and 2 we conclude that $\{abc\} \cup \{abs\} \cup \{arc\} \cup \{ars\} \nsubseteq \phi(A)$, which implies that $a(\{b\} \cup \{r\})(\{c\} \cup \{s\}) \subseteq A - \phi(A)$. In fact, since A is a ϕ-2-absorbing primary ideal of S, we have $a(\{b\} \cup \{r\}) \subseteq A$ or $(\{b\} \cup \{r\})(\{c\} \cup \{s\}) \subseteq \sqrt{A}$ or $a(\{c\} \cup \{s\}) \subseteq \sqrt{A}$. Thus, $ab \in A$ or $bc \in \sqrt{A}$ or $ac \in \sqrt{A}$, which is a contradiction. Consequently, $aA^2 \subseteq \phi(A)$. □

As a simple consequence of Theorem 1.5, we give the following result.

Corollary 1.1. *Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function and let A be a ϕ-2-absorbing primary ideal of a commutative semigroup S. For every $a, b, c \in S$ if (a, b, c) is a ϕ-2-absorbing primary triple zero of A, then the following statements hold:*

(1) *$abA \subseteq \phi(A)$ and $acA \subseteq \phi(A)$ and $bcA \subseteq \phi(A)$;*
(2) *$aA^2 \subseteq \phi(A)$ and $bA^2 \subseteq \phi(A)$ and $cA^2 \subseteq \phi(A)$.*

Now we arrive at one of our main theorem.

Theorem 1.6. *Let $\phi : \mathcal{I}(S) \to \mathcal{I}(S) \cup \{\emptyset\}$ be a function and let A be a ϕ-2-absorbing primary ideal of a commutative semigroup S. Suppose that B is an ideal of S and $a, b \in S$ such that $abB \subseteq A$. If (a, b, c) is not a ϕ-2-absorbing primary triple zero of $A, \sqrt{A}$ for every $c \in B$, then $ab \in \sqrt{A}$ or $bB \subseteq \sqrt{A}$ or $aB \subseteq \sqrt{A}$.*

Proof. Suppose that $ab \notin \sqrt{A}$ and $bB \nsubseteq \sqrt{A}$ and $aB \nsubseteq \sqrt{A}$. Then there are exist elements $d_1, d_2 \in B$ such that $bd_1 \notin \sqrt{A}$ and $ad_2 \notin \sqrt{A}$. If $abd_1 \notin \phi(A)$, then $abd_1 \in A - \phi(A)$. By assumption, $ad_1 \in \sqrt{A}$ or $bd_1 \in \sqrt{A}$. Next, let

$abd_1 \in \phi(A)$. By hypothesis, $ad_1 \in \sqrt{A}$ or $bd_1 \in \sqrt{A}$. Now if $abd_2 \notin \phi(A)$, then $abd_2 \in A - \phi(A)$. By the given hypothesis, $ad_2 \in \sqrt{A}$ or $bd_2 \in \sqrt{A}$. So let $abd_2 \in \phi(A)$. By given hypothesis, $ad_2 \in \sqrt{A}$ or $bd_2 \in \sqrt{A}$. In any case, we have $bd_1, bd_2 \in \sqrt{A}$. Since $abB \subseteq A$, we have $ab(\{d_1\} \cup \{d_2\}) \subseteq \sqrt{A}$. If $ab(\{d_1\} \cup \{d_2\}) \nsubseteq \phi(\sqrt{A})$, then $ab(\{d_1\} \cup \{d_2\}) \subseteq \sqrt{A} - \phi(\sqrt{A})$. Now by our hypothesis, $a(\{d_1\} \cup \{d_2\}) \subseteq \sqrt{A}$ or $b(\{d_1\} \cup \{d_2\}) \subseteq \sqrt{A}$, which implies that $bd_1, ad_2 \in \sqrt{A}$, which is a contradiction. Assume that $ab(\{d_1\} \cup \{d_2\}) \subseteq \phi(\sqrt{A})$. From our hypothesis, $a(\{d_1\} \cup \{d_2\}) \subseteq \sqrt{A}$ or $b(\{d_1\} \cup \{d_2\}) \subseteq \sqrt{A}$. Clearly, $bd_1 \in \sqrt{A}$ or $ad_2 \in \sqrt{A}$, which again is a contradiction. Hence $ab \in \sqrt{A}$ or $bB \subseteq \sqrt{A}$ or $aB \subseteq \sqrt{A}$. □

REFERENCES

[1] J. LI: *On the radicals of linear algebraic monoids*, Semigroup Forum, **96** (2018), 1–20.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND TECHNOLOGY
PIBULSONGKRAM RAJABHAT UNIVERSITY
PHITSANULOK 65000, THAILAND
Email address: pairote0027@hotmail.com