ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **10** (2021), no.1, 385–389
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.1.38 Spec. Iss. on AMCSE

TOPOLOGICAL GROUPS: VIRTUE OF PRE-OPEN SETS

P. Gnanachandra[1] and A. Muneesh Kumar

ABSTRACT. In this paper, we introduce notions of p-topological group and p-irresolute topological group which are generalizations of the notion topological group. We discuss the properties of p-topological group with illustrated examples. Also, we prove that translation and inversion in p-topological group are p-homeomorphism.

1. INTRODUCTION

Topological group is a mathematical structure on a set which is defined by underlying two distinguished structures on that set namely group and a topology. A topological group in modern notion is defined as, a group binded with a topology such that the binary operations are continuous. Based on this, some generalizations of topological groups such as paratopological groups, semitopological groups and quasitopological groups are defined. In a finite group, all the above mentioned generalizations coincide [7]. The concepts of S-topological group and s-topological group were discussed in [1] and the theory of almost topological group was initiated in [5]. In this paper, we discuss some more

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 40B05, 33E99.
Key words and phrases. Topological group, p-topological group, p-irresolute topological group, pre -connectedness.
Submitted: 27.12.2020; *Accepted:* 06.01.2021; *Published:* 20.01.2021.

generalizations which defined based on pre-open sets and present a new generalization of topological group called p-topological group.

2. Preliminaries

Throughout this paper, the pair (A, Υ) denotes a group A together with a topology Υ. For any $g \in \mathsf{A}$, g^{-1} denotes the inverse of g in A. Let $\mathsf{S}, \mathsf{T} \subseteq \mathsf{A}$, then $\mathsf{ST} = \{s.t : s \in \mathsf{S}, t \in \mathsf{T}\}$. The notions pre-open, pre-closed and pre-continuous map follows [6]. For a set C on a topological space X, the notions pre-interior and pre-closure are denoted by $pint(\mathsf{C})$ and $pcl(\mathsf{C})$ follows [6]. For a set R, the power set of R is denoted by $\mathcal{P}(\mathsf{R})$ and for a topology Υ on A, the collection of open sets, closed sets, pre-open sets are denoted by $\mathsf{O}(\mathsf{A}), \mathsf{C}(\mathsf{A})$ and Υ_p.

3. p-topological group

We introduce the concept of p-topological group and investigate its basic properties with illustrated examples in this section.

Definition 3.1. *A pair (A, Υ) is said to be* p***-topological group****, for $m, n \in \mathsf{A}$:*

- *for each open neighbourhood K of mn, $\exists$ pre-open neighbourhoods M of m and N of $n \ni \mathsf{MN} \subseteq \mathsf{K}$*
- *for each open neighbourhood S of g^{-1} $\exists$ pre-open neighbourhood T of $g \ni$ $\mathsf{T}^{-1} \subseteq \mathsf{S}$.*

In other words, multiplication and inversion mappings are pre-continuous.

Example 3.2. *Consider the group $\mathsf{A} = (\mathbb{Z}_3, \oplus)$ with topology $\Upsilon = \{\emptyset, \{1,2\}, \mathsf{A}\}$. For the topology Υ, $\Upsilon_p = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{0,1\}, \{0,2\}, \mathsf{A}\}$. Then (A, Υ) is a p-topological group. Finite group with indiscrete topology is the only connected topological group. But here, the above mentioned p-topological group is connected.*

Proposition 3.3. *Let A be a p-topological group and $\mathsf{S} \in \mathsf{O}(\mathsf{A})$. Then for any $g \in \mathsf{A}$, $g\mathsf{S}$ and $\mathsf{S}g$ are pre-open.*

Proof. Let $h \in g\mathsf{S}$, then $h = gs$ for some $s \in \mathsf{S}$. Now, $s = g^{-1}h$ and by the pre-continuity of multiplication $\exists$ pre-open sets M and N of g^{-1} and $h \ni \mathsf{MN} \subseteq \mathsf{S}$ which implies $h \in \mathsf{N} \subseteq g\mathsf{S}$. Hence $g\mathsf{S}$ is pre-open. Similarly we can prove that $\mathsf{S}g$ is pre-open. $\square$

Corollary 3.4. *Let* $\mathsf{T} \in \mathsf{C}(\mathsf{A})$ *in a* p*-topological group* A*. Then for any* $a \in \mathsf{A}$*,* $a\mathsf{T}$ *and* $\mathsf{T}a$ *are pre-closed.*

Proposition 3.5. *Let* A *be a* p*-topological group. Then* $\mathsf{D} \in \Upsilon_p$ *if and only if* $\mathsf{D}^{-1} \in \Upsilon_p$*.*

Proof. Let $\mathsf{D} \in \Upsilon_p$, then $\exists\ \mathsf{M} \in \mathsf{O}(\mathsf{A}) \ni \mathsf{D} \subseteq \mathsf{M} \subseteq cl(\mathsf{D})$. Now, $\mathsf{D}^{-1} \subseteq \mathsf{E}^{-1} \subseteq (cl(\mathsf{D}))^{-1}$. Since inversion is pre-continuous, we have D^{-1} is pre-open and $(cl(\mathsf{D}))^{-1}$ is pre-closure of D^{-1}. By using that, for a set C, $pcl(\mathsf{C}) \subseteq cl(\mathsf{C})$ [4], we have $\mathsf{D}^{-1} \subseteq \mathsf{M}^{-1} \subseteq int(cl(\mathsf{M}^{-1})) \subseteq (cl(\mathsf{M}))^{-1} \subseteq cl(\mathsf{M}^{-1})$. Hence $\exists$ $int(cl(\mathsf{M}^{-1})) \in \mathsf{O}(\mathsf{A}) \ni \mathsf{D}^{-1} \subseteq int(cl(\mathsf{M}^{-1})) \subseteq cl(\mathsf{D}^{-1})$ and so D^{-1} is pre-open. Proof of the converse is similar. ☐

Theorem 3.6. *Let* S *and* R *be* p*-topological groups with* R *is submaximal and* g *be a pre-irresolute homomorphism at identity* e_{S}*. Then* g *is pre-irresolute.*

Proof. Let $n \in \mathsf{S}$ and $\mathsf{M} \in \Upsilon_p$ in R containing $g(n) = m$. Since R is submaximal, each pre-open set is open [3] and so M is open. By Proposition 3.3, $m^{-1}\mathsf{M}$ is pre-open in T containing e_{T}. Since η is pre-irresolute at identity e_{S}, $\exists\ \mathsf{N} \in \Upsilon_p$ in S containing $e_{\mathsf{S}} \ni g(\mathsf{N}) \subset m^{-1}\mathsf{M}$. Given that g is homomorphism and so $g(n\mathsf{N}) = g(n)g(\mathsf{N}) \subseteq \mathsf{M}$. Hence g is pre-irresolute. ☐

One may remind that, A bijective mapping $\mu : \mathsf{S} \mapsto \mathsf{T}$ is p**-homeomorphism** [2] if μ is pre-continuous and $\mu(\mathsf{D})$ is pre-open for every $\mathsf{D} \in \mathsf{O}(\mathsf{S})$.

Theorem 3.7. *Let* S *be a* p*-topological group and* $k \in \mathsf{S}$*. Then for all* $s \in \mathsf{S}$*,*

(i) *The mappings* $\lambda_k(s) = ks$ *and* $\rho_k(s) = sk$ *are* p*-homeomorphism.*
(ii) *Inversion mapping is* p*-homeomorphism.*

Proof left to the reader.

Theorem 3.8. *Let* R, M *be a* p*-topological group and its subgroup,*

(i) *If* $\exists\ \mathsf{D} \in \mathsf{O}(\mathsf{R})$ *and* $\mathsf{D} \subseteq \mathsf{M}$*, then* $\mathsf{M} \in \Upsilon_p$*.*
(ii) *If* $\mathsf{M} \in \mathsf{O}(\mathsf{R})$*, then it is pre-closed.*
(iii) *If* $\mathsf{M} \in \mathsf{O}(\mathsf{R})$*, then* M *itself a* p*-topological group.*

Proof left to the reader.

4. p-IRRESOLUTE TOPOLOGICAL GROUP AND PRE-CONNECTEDNESS

We discuss the independency of p-topological group from other generalization concepts of topological group. We also explore pre-connectedness properties of p-irresolute topological group through this section.

Example 4.1. *Consider the group* $\mathsf{S} = (\mathbb{Z}_n, \oplus)$ *with topology* $\Upsilon = \{\emptyset, \mathbb{Z}_n, \{0\}\}$. *Then* (S, Υ) *is* S*-topological and almost topological groups but not* p*-topological and* s*-topological group.*

Theorem 4.2. Let (S, Υ) be a pair satisfies $\exists$ at least one singleton is open in S then Υ is discrete if and only if (S, Υ) is a s-topological group.

By considering (S, Υ) in Example 4.1, the above result need not be true in S-topological and almost topological groups.

Example 4.3. *Consider the group* $\mathsf{T} = (\mathbb{Z}_n, \oplus)$ *with the topology* $\Upsilon = \{\emptyset, \{0\},$ $\{0,1\}, \{0,2,1\}, \ldots, \mathbb{Z}_n\}$. *Then* (T, Υ) *is* S*-topological and an almost topological groups. But* (T, Υ) *is not* p*- topological and* s*-topological groups.*

Example 4.4. *Consider* (S, Υ) *in Example 3.2, which is a* p*-topological group. Here* (S, Υ) *is an almost topological group but not* s*-topological group.*

Definition 4.5. *The pair* (T, Υ) *is said to be* ***p-irresolute topological group*** *if binary operations are* pre*-irresolute.*

A topological space Y is **pre-connected** [8] if $\mathsf{Y} \neq \mathsf{E} \cup \mathsf{F}$, where E, F are two disjoint non-empty pre-open sets.

Theorem 4.6. *If* A *is a pre-connected,* p*-irresolute topological group and* H*, a discrete invariant subgroup of* A*, then* $\mathsf{H} \subseteq Z(\mathsf{A})$*, where* $Z(\mathsf{A})$ *denotes center of* A*.*

Proof. Suppose $\mathsf{H} = \{e\}$, then the result is trivial. Suppose H is non-trivial. Let $g \neq e \in \mathsf{H}$. Since, H is discrete, we can find $\mathsf{D} \in \mathsf{O}(\mathsf{A})$ of g in $\mathsf{A} \ni \mathsf{D} \cap \mathsf{H} = \{g\}$. Now, A is p-irresolute topological group, $\exists$ $\mathsf{E} \in \Upsilon_p$ of e and $\mathsf{E}.g \in \Upsilon_p$ of g in $\mathsf{A} \ni (\mathsf{E}.g).\mathsf{E}^{-1} \subset \mathsf{D}$. Let $b \in \mathsf{E}$. Since H is an normal subgroup of A, we have $b.\mathsf{H} = \mathsf{H}.b$ which implies that $b.g \in \mathsf{H}.b$ and so $b.g.b^{-1} \in \mathsf{H}$. It is also clear that $b.g.b^{-1} \in \mathsf{E}g\mathsf{E}^{-1} \subset \mathsf{D}$. Therefore, $b.g.b^{-1} \in \mathsf{D} \cap \mathsf{H} = \{g\}$ which implies $b.g.b^{-1} = g$. Thus, $b.g = g.b$ for each $b \in \mathsf{E}$. Since, A is pre-connected, E^n with $n \in \mathbb{N}$ covers A. Thus, every element $a \in \mathsf{A}$ can be written in the form

$a = b_1.b_2...b_n$ where $b_1, b_2, ..., b_n \in \mathsf{E}$ and $n \in \mathbb{N}$. Since g commutes with every element of E, we have $a.g = b_1.b_2....b_n.g = b_1.b_2....g.b_n = \cdots = b_1.g.b_2....b_n = g.b_1.b_2....b_n = g.a$. Hence, $g \in \mathsf{H} \in Z(\mathsf{A})$ and so $\mathsf{H} \subseteq Z(\mathsf{A})$. □

REFERENCES

[1] M.S. BOSAN: *Moiz Ud Din Khan and Ljubisa D.R. Kocinac, On s-topological groups*, Mathematica Moravica, **18**(2) (2014), 35-44.

[2] C. DEVAMANOHARAN, S. PIOUS MISSIER, S. JAFARI: *On p-homeomorphisms in topological spaces*, Italian Journal of Pure and Applied Mathematics, **30** (2013), 195-214.

[3] J. DONTCHEV: *Survey on Preopen sets*, arXiv: math. GN / 9810177.

[4] G. NAVALAGI: *Pre-neighbourhoods*, The Mathematics Education, **32**(4) (1998), 201-206.

[5] M. RAM: *On almost topological groups*, Mathematica Moravica, **23**(1) (2019), 97-106.

[6] A.S. MASHHOUR, M.E. ABD EL-MONSEF, S.N. EL-DEEB: *On precontinuous and precontinuous mappings*, Proceedings of Mathematical Physics Society, Egypt, **53** (1982), 47-53.

[7] A. MUNEESH KUMAR, P. GNANACHANDRA: *Exploratory Results on finite Topological groups*, JP journal of geometry and topology, **24**(1-2) (2020), 1-15.

[8] V. POPA: *Properties of H-almost continuous functions*, Bulletin Mathematique de la Societe des Sciences Mathematiques de la Republique Socialiste de Roumanie Nouvelle Serie, **31**(79)(2) (1987), 163-168.

CENTRE FOR RESEARCH AND POST GRADUATE STUDIES IN MATHEMATICS,
AYYA NADAR JANAKI AMMAL COLLEGE, SIVAKASI, TAMIL NADU, INDIA.
Email address: pgchandra07@gmail.com

CENTRE FOR RESEARCH AND POST GRADUATE STUDIES IN MATHEMATICS,
AYYA NADAR JANAKI AMMAL COLLEGE, SIVAKASI, TAMIL NADU, INDIA.
Email address: muneeshkumarar@gmail.com