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APPROXIMATE SOLUTION OF FOURTH ORDER DIFFERENTIAL EQUATION

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ABSTRACT. The present papers adopts five point ILMM (Implicit Linear Multistep Method) by interpolation and collocation on the basic of power series and its derivatives respectively for approximate solution of fourth order ordinary differential equations. The objective of the method is to obtain the zero stability, consistency, convergence and order of the method by Taylor's series approximation. Comparison of the exact solutions with an approximate solutions along with ODE-45 for three numerical tests are obtained .

1. INTRODUCTION

This study has a lot of applications in Sciences and engineering especially in control theory, hence the study of the methods of solution is of great interest to researchers. Solution of higher order ODEs with given initial conditions by the method of reduction to a system of first order differential equations and its integration needs a lot of human effort and computational time [1,3,4]. A general fourth order ODE of initial value problems is given by

$$y^{iv} = f(x, y, y', y'', y'''), \tag{1.1}$$

with, $y(a) = \gamma_0, y'(a) = \gamma_1, y''(a) = \gamma_2, y'''(a) = \gamma_3;\ a \leq x \leq b.$

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The conventional method of solving equation (1.1) is first reduce it to a system of first order differential equation. Direct method of solution of equation (1.1) using implicit linear multistep method has found to be more efficient in terms of speed and accuracy than the method of reduction to a system of first order ordinary differential equation [5, 6]. Implicit linear multistep method is selected because it has better stability properties than the explicit methods. Direct method of solving higher order ordinary differential equations by continuous collocation multistep methods have been extensively discussed in [7]. Several continuous LMM [8, 9] have developed for the direct solution of equation (1.1). The methods developed by some of these authors were implemented in predictor -corrector mode while those of the others were combined with additional methods obtained from continuous k-step LMMs to solve fourth orders ODES directly. Although the predictor corrector methods yield good results, the major drawback of the method is apart from the inherent computational burden, the predictors which were developed have reducing order of accuracy .The other methods such as block method, differential transformation method, numerical integration of real and analytic function [2,12–28] are efficient to develop the present method. In this paper we have proposed a five point fully Implicit Linear Multistep Method (ILLM) by interpolation and collocation on the basis of power series for solution of fourth order differential equations to avoid the above described difficulties. Our results are compared with results of ODE-45 and contribute better results. This paper is organized as follows: Section-1 is an Introduction, Section-2 contains derivation of the method, The method is analyzed in Section-3. Stability and Convergence condition are developed in Section-4. Numerical verifications and conclusions are reported in Section-5 and Section-6 respectively.

2. Derivation of the method

According to [10], k-step, Linear Multistep Method(LLM) is given by

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^n \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{2.1}$$

where α_j, β_j are unique and $\alpha_0 + \beta_0 \neq 0, \alpha_k = 1, n$ is the order of the differential equation. Using interpolation and collocation equation (2.1) can be transformed to a continuous LMM of the form

$$Y(x) = \sum_{j=0}^{k} \alpha_j y_{n+j} + h^n \sum_{j=0}^{k} \beta_j f_{n+j},$$

where α_j, β_j are continuous differentiable functions of x . The error estimation in continuous collation method is better than the discrete methods used in reducing higher order ODEs to first order differential equations. The approximate solution $y(x)$ of equation (1.1) is derived by approximating the polynomial of degree m given by

$$y(x) \approx Y(x) = \sum_{j=0}^{m} a_j (x - x_n)^j, \; x_n \leq x \leq x_{n+k}, a_0 \neq 0 \; . \tag{2.2}$$

Differentiation of equation (2.2) up to order four gives

$$Y^{(iv)}(x) = \sum_{j=4}^{m} j(j-1)(j-2)(j-3) a_j (x - x_n)^{(j-4)}. \tag{2.3}$$

Now interpolating $x = x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+k-1}$ in equation (2.2), we have the following results

(2.4)

$$\begin{aligned} Y(x_n) &= y_n = a_0 \\ Y(x_{n+1}) &= y_{n+1} = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + ... + a_m h^m \\ &\vdots \\ Y(x_{n+k-1}) &= y_{n+k-1} = a_0 + (k-1) a_1 h + (k-1)^2 a_2 h^2 + \cdots + (k-1)^m a_m h^m. \end{aligned}$$

Collocating equation (2.3) at $x = x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+k}$ we have

$$f_{n+k} = Y^{(iv)}(x_{n+k}) = \sum_{j=4}^{m} j(j-1)(j-2)(j-3) a_j (x_{n+k} - x_n)^{(j-4)}. \tag{2.5}$$

In particular we consider equation (2.4) and (2.5) for $k = 5$, $m = 10$ and form the matrix for our proposed method to evaluate the required continuous coefficients. By matrix inversion method the coefficients are obtained as functions of y_n and f_n. Putting them in equation (2.2) and substitution at $x = x_{n+5}$ gives the discrete solution as

$$\begin{aligned} y_{n+5} &= y_n - 5y_{n+1} + 10y_{n+2} - 10y_{n+3} + 5y_{n+4} \\ &\quad + \frac{h^4}{720}[f_n - 125 f_{n+1} - 350 f_{n+2} + 350 f_{n+3} + 125 f_{n+4} - f_{n+5}]. \end{aligned} \tag{2.6}$$

3. Analysis of the Method

Equation (2.6) represents a particular case of standard LMM as

$$\sum_{j=0}^{5} \alpha_j y_{n+j} = h^4 \sum_{j=0}^{5} \beta_j f_{n+j}. \tag{3.1}$$

In order to find the error constant and order of this method we write equation (3.1) in difference operator form

$$L(y(x), h) = \sum_{j=0}^{5} (\alpha_j y(x_n + jh) - h^4 \beta_j f_(x_n + jh)), \tag{3.2}$$

where $y(x)$ is assumed to be a continuously differentiable function of higher order. Using Taylor's series expansion, the difference operator (3.2) can be expressed as

$$L(y(x), h) = \gamma_0 y(x) + \gamma_1 h y'(x) + \gamma_2 h^2 y''(x) + \gamma_3 h^3 y'''(x) + ... + \gamma_{p+2} h^{p+2} y^{p+2}(x).$$

Using [10] the order of the method is p if $\gamma_0 = \gamma_1 = \gamma_2 = \cdots = \gamma_p = \gamma_{p+1} = \gamma_{p+2} = 0$ and $\gamma_{p+3} \neq 0$. In the present method $p = 8$ and the error constant is $\gamma_{11} = \frac{1}{3024}$.

Zero stability and convergence

The first characteristic polynomial from equation (2.6) is considered for analysing the zero stability, i.e., $r^5 - 5r^4 + 10r^3 - 10r^2 + 5r - 1 = 0$ which gives $r = 1$ of multiplicity not exceeding the value of k . Hence the method is zero stable and since the order $p > 1$ the method is consistent. As per [11] the convergence of the method is also established.

4. Numerical Examples

Application of the method in fourth order IVPs has been carried on by calculating y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4} by Taylor's series expansion and their first, second, third and fourth derivatives up to $p = 8$. Higher order derivatives are obtained by using partial differentiation. Two examples are considered for two IVPs to implement the ILMM and the results have produced marginal errors as compared with the exact solution.

Example 1. $y^{(iv)} = 16y + 128\cosh 2x, y(0) = 1, y'(0) = 24, y''(0) = 20, y'''(0) = -160, 0 \leq x \leq 1, h = 0.1.$ *The exact solution is* $y = -\frac{3}{2}e^{2x} + \frac{5}{2}e^{-2x} + 2x(e^{2x} - e^{-2x}) + 16\sin 2x.$

TABLE 1. Computational data of Example-1

x	Exact Solution	Approx. Solution	Error in ODE45	Absolute Error
0.1	3.473966439192117	3.473966439192127	$2.070193e^{-06}$	$1.0214051 \times 10^{-14}$
0.2	5.997358406207853	5.997358406207840	$2.561710e^{-06}$	$1.3322676 \times 10^{-14}$
0.3	8.437114762547758	8.437114762547504	$2.808199e^{-06}$	2.072751×10^{-14}
0.4	10.683678043446914	10.683678043447408	$2.863389e^{-06}$	4.938272×10^{-13}
0.5	12.656214004453984	12.656214004452369	$2.794668e^{-06}$	1.614708×10^{-12}
0.6	14.308142774140519	14.308142774139625	$2.684461e^{-06}$	8.935074×10^{-13}
0.7	15.632932343466662	15.632932343464278	$2.631204e^{-06}$	2.38387×10^{-12}
0.8	16.670187757298784	16.670187757299818	$2.750214e^{-06}$	1.033839×10^{-12}
0.9	17.512165555130117	17.512165555138381	$3.174765e^{-06}$	8.263612×10^{-12}
1.0	18.310954520294541	18.310954520353818	$8.510556e^{-07}$	5.927702×10^{-11}

TABLE 2. Computational data of Example-2

x	Exact Solution	Approx. Solution	Error in ODE45	Absolute Error
0.1	11.710717749892432	11.710717749635059	$1.425637e^{-05}$	2.5737×10^{-10}
0.2	11.501533206561351	11.501533204623515	$5.149911e^{-05}$	1.93783×10^{-9}
0.3	11.467221263557018	11.467221291768432	$1.023663e^{-06}$	2.8211413×10^{-8}
0.4	11.566935513414810	11.566935347735537	$6.1683250e^{-06}$	1.656792×10^{-7}
0.5	11.774738297187065	11.774738912461428	$1.064189e^{-07}$	6.152743×10^{-7}
0.6	12.073845248837996	12.073843502291169	$9.623654e^{-07}$	1.746546×10^{-6}
0.7	12.453123902379366	12.453128073933891	$6.064467e^{-07}$	4.171554×10^{-6}
0.8	12.904960459474843	12.904951606536839	$1.000908e^{-07}$	8.85293×10^{-6}
0.9	13.423957295241237	13.423974557104920	$5.682060e^{-07}$	1.72618×10^{-5}
1.0	14.006135152856297	14.006260343085421	$1.357815e^{-03}$	1.251902×10^{-4}

Example 2. $y^{(iv)} = 26y' - 25y + 50(x+1)^2, y(0) = 12.16, y'(0) = -6, y''(0) = 34, y'''(0) = -130, h = 0.1, 0 \leq x \leq 1$. *The exact solution is* $y = e^{-5x} + 5e^{-x} + 2x^2 + 4x + \frac{154}{25}$.

5. CONCLUSIONS

Application of proposed method to selected numerical problems with non overlapping intervals obtains higher degree of accuracy when compared with the results of exact solutions as well as solutions obtained by ODE45. Since this method is convergent, consistent and stable, computation in larger intervals are also possible .As our numerical tests provide better results than ODE 45 (Runge-kutta method), it implies the present method contributes a better agreement towards the approximate solution than reduction method.

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