ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **10** (2021), no.1, 665–671
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.1.67

THE BI-COMPACT-G$_\delta$-OPEN TOPOLOGY: SEPARATION AXIOMS

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ABSTRACT. Let $C(X)$ denotes the set of all continuous functions from the space X to the space $\mathbb{R}$. This paper studies separation axioms of the space $C(X)$ equipped with the bi-compact-G$_\delta$-open topology.

1. INTRODUCTION

Let $C(X)$ denotes the set of all continuous functions from the space X to the space $\mathbb{R}$, where X is a completely regular space and $\mathbb{R}$ denotes the space of all real numbers with its usual topology. While defining "set-open" topology, we take sets from the space X and open sets from the space $\mathbb{R}$. Let $\mathcal{E}$ denotes a family subsets of space X, $\phi \in \mathcal{E}$. Let $[A, U]^+ = \{g \in C(X) : g(A) \subseteq U\}$, where U is open in $\mathbb{R}$, A is a set in X. The collection $\mathcal{B} = \{[A, U]^+ : A \in \mathcal{E}, U \text{ is open in } \mathbb{R}\}$ forms subbase for some set-open topology. There are various set-open topologies on $C(X)$ (see [4], [10], [7], [11] and [12], [5]). If $\mathcal{E}$ denotes the family of all compact-G_δ subsets of X, then collection $\mathcal{B}$ generates the compact-G_δ-open topology.

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2020 *Mathematics Subject Classification.* 54C35, 54D10, 54D15.

Key words and phrases. The point-open topology, Open-point topology, Bi-point-open topology, Separation axioms, G_δ-dense, Zero set.

Submitted: 30.12.2020; *Accepted:* 14.01.2021; *Published:* 30.01.2021.

Anubha Jindal, R. A. McCoy and S. Kundu found another way of defining a topology on $C(X)$. They defined a "open-set" topology on $C(X)$ known as open-point topology. let $[V, r]^- = \{f \in C(X) : f^{-1}(r) \cap V \neq \emptyset, r \in \mathbb{R}, V$ is open in $X\}$. While defining open-set topology we take open sets from space X and sets from space $\mathbb{R}$. The collection $\{[V, r]^- : r \in \mathbb{R}, V$ is open in $X\}$ forms subbase for open-point topology (see [2]).

Later, the join of set-open and open-set topologies is also studied, known as bi-point-open and bi-compact-open topologies (see [1], [2]). The collection that contains subbase of both point-open and open-point topologies, is subbase of the bi-point-open topology. Similarly, The collection that contains subbase of both compact-open and open-point topologies, is subbase of the bi-compact-open topology. A.R. Prasannan and Barkha defined a new topology called bi-compact-G_δ-open topology in [3]. The bi-compact-G_δ-open topology is the join of open-point and compact G_δ-open topologies. This topology has subbasic open sets of both kinds: $[V, r]^- = \{f \in C(X) : f^{-1}(r) \cap V \neq \emptyset\}$ and $[K, U]^+ = \{g \in C(X) : g(K) \subseteq U\}$, where V is open in X, $r \in \mathbb{R}$, K is a compact-zero set in X and U is open in $\mathbb{R}$. The space $C(X)$ equipped with bi-compact-G_δ-open topology is denoted by $C_{kz,h}(X)$.

In [2], authors investigated separation axioms of bi-compact-open topology. In this paper, we analogously study separation axioms of bi-compact-G_δ-open topology.

A subset A of a space X is said to be a zero set if there exists a real-valued continuous function f on X such that $A = \{x \in X : f(x) = 0\}$. We say, a set A is compact G_δ in X, if it is both compact and G_δ. A compact set is a zero-set if and only if it is a G_δ-set. So a set is compact G_δ-set if and only if it is compact zero-set. For more references related to compact G_δ-set, see [9]. Let $\mathcal{KZ}(X)$ denotes the set of all compact zero sets in the space X.

Throughout the rest of the paper, all spaces are assume to be completely regular Hausdorff which has atleast one nonempty compact-zero set, later we may put some other conditions on X. For other basic notations, see [8]. $\mathbb{R}$ is space of all real number numbers with its usual topology. 0_X denotes constant zero function defined on the space X.

2. Separation axioms

Theorem 2.1. *[2] For every space X, $C_h(X)$ is a T_1-space.*

Corollary 2.1. *For every space X, $C_{kz,h}(X)$ is a T_1-space.*

Proof. Since $C_h(X) \leq C_{kz,h}(X)$ and $C_h(X)$ is T_1-space, $C_{kz,h}(X)$ is a T_1-space. □

Theorem 2.2. *If set of all isolated points in X is dense in X. Then $C_{kz,h}(X)$ is Hausdorff.*

Proof. Since $C_h(X) \leq C_{kz,h}(X)$, $C_{kz,h}(X)$ is Hausdorff (use Theorem 3.2 in [2]). □

Definition 2.1. *[2] A subset S of a space X is called G_δ-dense in space X if every G_δ-set in X intersects S.*

Let x be a point in X and for each open set U containing x, there exists an open set V in X such that $x \in V \subset \overline{V} \subset U$, where $\overline{V}$ is a compact G_δ-set. let $lkz(X)$ denotes the set of all these type of points in the space X.

For example, Since $\mathbb{R}$ is locally compact and each compact set is compact-G_δ set in $\mathbb{R}$, so $lkz(\mathbb{R}) = \mathbb{R}$.

Definition 2.2. *[7] A space X is of point pseudocountable type if each point in X is contained in a compact G_δ-subset of X, that is, $X = \{A : A \in \mathcal{KZ}(X)\}$.*

Theorem 2.3. *The space $C_{kz,h}(X)$ has a base consisting of the sets of the form $[A_1, W_1]^+ \cap ... \cap [A_n, W_n]^+ \cap [U_1, y_1]^- \cap ... \cap [U_m, y_m]^-$, where $m, n \in \mathbb{N}$, A_i is a compact zero subset of the space X, U_j is open in X, W_i is open in $\mathbb{R}$, $y_j \in \mathbb{R}$, whenever $1 \leq i \leq n$, $1 \leq j \leq m$ and $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \neq j$.*

Proof. It directly follows from Proposition 2.1 in [2] □

Theorem 2.4. *Let X is of point pseudocountable type and $C_{kz,h}(X)$ is regular space, then $lkz(X)$ is G_δ-dense in X.*

Proof. If possible, suppose $lkz(X)$ is not G_δ-dense in X. Then there exists a G_δ-set A in X such that $lkz(X) \cap A = \emptyset$. Also, since A is nonempty G_δ-set in X, there exists $f \in C(X)$ such that $f \geq 0$, $f^{-1}(0) \neq \emptyset$ and $f^{-1}(0) \subseteq A$ (see 3.11.(b) in [6]). Also, since $f^{-1}(0) \cap A \neq \emptyset$, $F = C_{kz,h}(X) \setminus [X, 0]^-$ is closed in $C_{kz,h}(X)$

which does not contain f. Let $N = [A_1, U_1]^+ \cap ... \cap [A_n, U_n]^+ \cap [V_1, y_1]^- \cap ... [V_m, y_m]^-$ be a subbasic open set and W be any open set in $C_{kz,h}(X)$ such that $f \in N$ and $F \subset W$, where $\overline{V_i} \cap \overline{V_j} = \emptyset$ for $i \neq j$. We will show that $N \cap W \neq \emptyset$ and thus $C_{kz,h}(X)$ is not regular.

Since $f \in N$ and $f \geq 0$, there exists $x_j \in V_j$ such that $f(x_j) = y_j \geq 0$ for all $j = 1, \ldots, m$. Let $I = \{i \in \{1, \ldots, m\} : y_i = 0\}$ and $J = \{1, \ldots, m\} \setminus I$. Also, let $a_i = \min f(A_i)$ for each $i = 1, \ldots, n$. If $a_i = 0$ for some $i \in \{1, \ldots, n\}$, then there $0 \in U_i$. Since U_i is open for each $i = 1, \ldots, n$, there exists $\epsilon_i > 0$ such that $[0, \epsilon_i) \subset U_i$. Take $S = \{\frac{\epsilon_i}{2} : a_i = 0\} \cup \{\frac{a_i}{2} : a_i \neq 0\} \cup \{\frac{y_j}{2} : j \in J\}$ and $\delta = \min\{s : s \in S\}$. Define a function $g : X \to \mathbb{R}$ by $g(x) = \max\{f(x), \delta\}$. Note that, $g \in C(X)$ and $g > 0$, $g \in [A_1, U_1]^+ \cap \cdots \cap [A_n, U_n]^+$ and $g(x_j) = y_j$ for each $j \in J$. Hence $g \in F \subset W$.

Let $N' = [C_1, W_1]^+ \cap ... \cap [C_l, W_l]^+ \cap [U'_1, r_1]^- \cap ... \cap [U'_q, r_q]^-$ be a basic neighbourhood of g in $C_{kz,h}(X)$ with $N' \subset W$, and let $K = \{1, \ldots, q\}$. Since $g \in N'$, there exists $z_k \in U'_k$ such that $g(z_k) = r_k$. Also, since $X = \bigcup\{A : A \in \mathcal{KZ}(X)\}$, there exist compact zero sets A_{x_j} and A_{z_k} such that $x_j \in A_{x_j}$ and $z_k \in A_{z_k}$ for $1 \leq k \leq q$, $j \in J$. Take $A = \{A_{x_j} : j \in J\} \cup \{A_{z_k} : k \in K\} \cup A_1 \cup \cdots \cup A_n \cup C_1 \cup \cdots \cup C_l$. Since finite unions of compact-zero sets is again a compact zero set, so is A. For each $i \in I$, since $x_i \notin lkz(X)$ and X is regular space, $V_i \not\subseteq A$. For each $i \in I$, take $x'_i \in V_i \setminus A$. Define a function $h : A \cup \{x'_i : i \in I\} \to \mathbb{R}$ such that $h(x'_i) = 0$ for each $i \in I$ and $h(x) = g(x)$ for all x in A. Since $A \cup \{x'_i : i \in I\}$ is compact, h has continuous extension on X. Then $h \in N \cap N' \subset N \cap W$. □

Theorem 2.5. *If $lkz(X)$ is a G_δ-subset in X, then $C_h(X)$ has base consisting of open sets of the form $[V_1, r_1]^- \cap ... \cap [V_n, r_n]^-$, where $\overline{V_i} \cap \overline{V_j} = \phi$ for $i \neq j$, $1 \leq i, j \leq n$ and each open set $\overline{V_i}$ is a compact-G_δ set.*

Proof. Let $B = [U_1, r_1]^- \cap ... \cap [U_n, r_n]^-$ be a basic open set in $C_h(X)$, where U_i is open in $\mathbb{R}$, $r_i \in \mathbb{R}$ and $\overline{U_i} \cap \overline{U_j} = \phi$, $i \neq j$, $i \in \{1, \ldots, n\}$. Let $f \in B$. Since each singleton set $\{r_i\}$ in $\mathbb{R}$ is a zero set and inverse image of a zero set under a continuous function is again a zero set, so is $f^{-1}(r_i)$, for each $i \in \{1, \ldots, n\}$. Since $lkz(X)$ is a G_δ-subset in X, for each $i \in \{1, \ldots, n\}$, we can find x_i in $lkz(X)$ such that $x_i \in f^{-1}(r_i) \cap U_i$. Also, $x_i \in lkz(X)$, so there exists an open set V_i such that $x_i \in V_i \subset \overline{V_i} \subset U_i$ and $\overline{V_i}$ is a compact G_δ-set, for each $i \in \{1, \ldots, n\}$. Since $\overline{U_i} \cap \overline{U_j} = \phi$ for $i \neq j$, $\overline{V_i} \cap \overline{V_j} = \phi$. So, $[V_1, r_1]^- \cap ... \cap [V_n, r_n]^- \subset B$. □

Theorem 2.6. *Let X is of point pseudocountable type and $lkz(X)$ is G_δ-dense in X, then $C_{kz,h}(X)$ is completely regular.*

Proof. $[V, y]^-$ be a subbasic open set in $C_{kz,h}(X)$ such that $\overline{V}$ is compact G_δ-set. let $f \in [V, y]^-$. It follows that, $f^{-1}(y) \cap V \neq \phi$. Let $x \in f^{-1}(y) \cap V$. Since X is completely regular space, there exists a continuous function $h : X \mapsto [0, 1]$ such that $h(x) = 0$ and $h(v) = 1$ for $v \in V^c$. Now, define a function $\phi : C_{kz,h}(X) \mapsto [0, 1]$ such that for each $g \in C_{kz}(X)$, $\phi(g) = \inf h(g^{-1}(y) \cap V)$ for $g^{-1}(y) \cap V \neq \phi$, and $\phi(g) = 1$, for $g^{-1}(y) \cap V = \phi$. Note that, $\phi(f) = 0$ and if $g \notin [V, y]^-$, then $\phi(g) = 1$. Now, we will show that $\phi : C_{kz,h}(X) \mapsto [0, 1]$ is continuous. Let $g \in C_{kz,h}(X)$. Let $\phi(g) = c$ and $\epsilon > 0$. First, we will prove for $c > 0$ and $g^{-1}(y) \cap V \neq \phi$. We can assume that $c - \epsilon > 0$. Note that, $[0, c - \frac{\epsilon}{2}]$ is a zero set in space $\mathbb{R}$ and inverse image of a zero set under a continuous function is again a zero set, so $h^{-1}([0, c - \frac{\epsilon}{2}]$ is a zero set. Finite intersection of zero set is again a zero set and $\overline{V}$ is compact G_δ-set, so $\overline{V} \cap h^{-1}([0, c - \frac{\epsilon}{2}]$ is a zero set. Since $\overline{V} \cap h^{-1}([0, c - \frac{\epsilon}{2}]$ is a closed subset of compact set $\overline{V}$, $\overline{V} \cap h^{-1}([0, c - \frac{\epsilon}{2}]$ is also compact. Thus, $\overline{V} \cap h^{-1}([0, c - \frac{\epsilon}{2}]$ is a compact zero-set. A compact zero-set is a compact G_δ-set. So the set $B = [V \cap h^{-1}(c - \frac{\epsilon}{2}, c + \frac{\epsilon}{2}), y]^- \cap [\overline{V} \cap h^{-1}([0, c - \frac{\epsilon}{2}]), \{y\}^c]^+$ is an open set $C_{kz,h}(X)$. We will prove that $\phi(B) \subset (c - \epsilon, c + \epsilon)$ and $g \in B$. We have $\phi(g) = c > 0$, so there is $z \in g^{-1}(y) \cap V$ such that $h(z) \in [c, c + \epsilon)$. For each $t \in V^c$, $h(t) = 1$, so $h(x) \geq c$ for all $x \in g^{-1}(y) \cap \overline{V}$. It follows that, $g^{-1}(y) \cap \overline{V} \cap h^{-1}[0, c - \epsilon] = \phi$ and hence $g \in B$. Let $\psi \in B$, then there is $x \in V$ such that $\psi(x) = y$, $h(x) \in (c - \frac{\epsilon}{2}, c + \frac{\epsilon}{2})$ and $\psi(u) \neq y$, for all $u \in \overline{V} \cap h^{-1}[0, c - \frac{\epsilon}{2}]$. Hence, $\inf h(\psi^{-1}(y) \cap V) \subset [c - \frac{\epsilon}{2}, c + \frac{\epsilon}{2}]$. Therefore, in case of $c > 0$, $\phi(B) \subset (c - \epsilon, c + \epsilon)$. For $c = 0$, use $B = [V \cap h^{-1}(\frac{-\epsilon}{2}, \frac{\epsilon}{2}), y]^-$ and follow the same procedure as for $c > 0$.

Now, suppose if $g^{-1}(y) \cap V = \phi$, then in this case, $\phi(g) = 1$. Consider a neighbourhood $(1 - \epsilon, 1]$ of $\phi(g)$, where $0 < \epsilon < 1$. Let $B = [\overline{V} \cap h^{-1}([0, 1 - \frac{\epsilon}{2}]), \{y\}^c]^+$, we will show that $\phi(B) \subset (1 - \epsilon, 1]$ and $g \in B$. If $a \in \overline{V} \setminus V$, then $h(a) = 1$ and $g^{-1}(y) \cap V = \phi$. It follows that, $\overline{V} \cap g^{-1}(y) \cap h^{-1}[0, 1 - \frac{\epsilon}{2}] = \phi$, hence $g \in B$. So, if $f_1 \in B$, then $f_1(x) \neq y$ for all $x \in \overline{V} \cap h^{-}[0, 1 - \frac{\epsilon}{2}]$. Therefore, for each such f_1, $h(f_1(y) \cap V) \subset [1 - \frac{\epsilon}{2}, 1]$. It follows that, $\phi(B) \subset (1 - \epsilon, 1]$. Thus, ϕ is continuous.

Now, consider a subbasic open set $[K, U]^+$ in $C_{kz,h}(X)$. Take $f \in [K, U]^+$. Since $X = \bigcup\{A : A \in \mathcal{KZ}(X)\}$, $C_{kz,h}(X)$ is Hausdorff (see Theorem 2.2 in [7].

Also, $C_{kz}(X)$ is a locally convex space (see Theorem 2.1 in [7]). It is known that a locally convex Hausdorff space is completely regular space, so is $C_{kz}(X)$. It follows that, there exists a continuous function $\Psi : C_{kz,h}(X) \to [0,1]$ such that $\Psi(f) = 0$ and $\Psi(g) = 1$ for $g \notin [A,U]^+$. Since ϕ and Ψ are continuous and maximum of a finite number of continuous functions is continuous, it is easy to see that $C_{kz,h}(X)$ is completely regular space. □

Theorem 2.7. *Let X is of point pseudocountable type, then the following statements are equivalent.*

(a) $C_{kz,h}(X)$ *is completely regular.*
(b) $C_{kz,h}(X)$ *is regular.*
(c) $lkz(X)$ *is G_δ-dense in X.*

Proof. Proof follows from Theorem 2.4 and Theorem 2.6. □

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