ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **10** (2021), no.1, 687–695
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.1.70

AN ALGORITHM FOR CALCULATING THE ZERO-DIVISOR GRAPH OF THE RING $Z_N[X]/(X^2)$

Rania Shaqbou'a[1] and Hani Kawariq

To The Soul of Prof. Hassan Al-Ezeh

ABSTRACT. For a commutative ring R with unity $1 \neq 0$, let $Z(R)$ be the set of zero-divisors of R. A simple graph $\Gamma(R)$ is associated to R with vertices $Z(R)^* = Z(R) - \{0\}$,the set of nonzero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. Thus $\Gamma(R)$ is the empty graph if and only if R is an integral domain. Moreover, a nonempty $\Gamma(R)$ is finite if and only if R is finite. In this article an algorithm to compute zero-divisors of the ring $Z_n[X]/(X^2)$ is developed.

1. INTRODUCTION

All rings in this paper are commutative rings with unity $1 \neq 0$. The idea of a zero-divisor graph of a commutative ring was introduced by I.Beck in 1988 [4],where he was mainly interested in colorings. Then D.D.Anderson and M.Naseer continued this investigation of colorings of a commutative ring. Their definition was slightly different from ours, they let all elements of the ring R be the vertices and had for any distinct elements x and y are adjacent if and only if $xy = 0$, [1].

[1]*corresponding author*
2020 *Mathematics Subject Classification.* 05C90, 05C85.
Key words and phrases. Simple Zero-Divisor Graph, Algorithm.
Submitted: 27.12.2020; *Accepted:* 11.01.2021; *Published:* 30.01.2021.

Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the non-zero zero-divisors, [3]. These graphs turn out to best exhibit the properties of the set of zero-divisors of a commutative ring. The zero-divisor graph helps us to study the algebraic properties of rings using graph theoretical tools. Some algebraic properties of rings can be translated into graph theory language and then the geometric properties of graphs help to explore some interesting results in the algebraic structures of rings. The zero-divisor graph of a commutative ring has been studied extensively by Anderson, Frazier, Lauve, Levy, Livingston [2, 3],Axtell, Coykendall, Stickless [6], Akbari and Mohammadian [7], and more.

This paper is organized as follows, in section 2 an algorithm to compute $\Gamma(Z_n)$ is presented. This was originally given by Krone [5]. This algorithm is developed to compute $\Gamma(Z_n[X]/(X^2))$ in section 3.

Note that $\gcd(a, b)$ and $U(R)$ are denoted for greatest common divisor between the two numbers a and b and the units of the ring R, respectively. Also, for any positive integer r, $\phi(r)$ is the Euler's ϕ-function, which is the number of positive integers that less than or equals r and relatively prime to r. The following results are necessary throughout this paper.

Lemma 1.1. *[8] Let p be a prime number and $n \geq 1$, then $\phi(p^n) = p^n(1 - \frac{1}{p})$.*

Lemma 1.2. *[8] Given two integers a and b.If $\gcd(a, b) = d$ then $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$.*

Theorem 1.1. *Let the ring $R = Z_n[X]/(X^2)$. Then $a + bx$ is zero-divisor in R if and only if a is zero-divisor in Z_n.*

Proof. Suppose that $a+bx$ is zero-divisor in $Z_n[X]/(X^2)$, then there exists $c+dx \neq 0$, such that $(a + bx)(c + dx) = 0$, hence $ac + (bc + da)x = 0$, if $c \neq 0$ then $ac = 0$, so a is zero-divisor. If $c = 0$, then $d \neq 0$ and $ad = 0$, hence a is a zero-divisor. Consequently, Suppose that a is a zero-divisor in Z_n, then there exists $c \neq 0$,such that $ac = 0$, hence $cx(a + bx) = 0$ for any $b \in Z_n$. So $a + bx$ is a zero-divisor in $Z_n[X]/(X^2)$. □

Theorem 1.2. *Let R and S be any two commutative rings with unity different than zero, then*

$$(R \oplus S)[X]/(X^2) \cong R[X]/(X^2) \oplus S[X]/(X^2).$$

Corollary 1.1. *Let $Z_n \cong Z_{p^r} \oplus Z_{q^t}$, where p and q are relatively prime integers and r and t are positive integers, then*

$$(Z_{p^r} \oplus Z_{q^t})[X]/(X^2) \cong Z_{p^r}[X]/(X^2) \oplus Z_{q^t}[X]/(X^2).$$

2. An Algorithm for calculating $\Gamma(Z_n)$

This algorithm is recursive in nature and construct the graph for a given ring from sub-graphs which themselves are zero-divisor graphs of rings of smaller orders. This was presented by Joan Krone [5].

Case 1: If $n = p^k$ for some prime p and integer $k > 1$.

(a) Find the zero-divisor of Z_{p^k}, by taking the numbers $1, 2, \ldots, p^{k-1} - 1$, then multiply those numbers by p. Hence, we get the zero-divisors of Z_{p^k}.

(b) Divide the zero-divisors into $k - 1$ sets according to how many factors of p each divisor has, they are, $S_1, S_2, \ldots, S_{k-1}$.

(c) Connected vertices in the graph from the set S_m, to the vertices in the set S_t, such that $m + t \geq k$.

Case 2: If $n = p^r q^t$ where p and q are relatively prime integers, hence $Z_n = Z_{p^r} \times Z_{q^t}$.

(a) Find the zero-divisors of Z_{p^r} and Z_{q^t}, by using (1) in case1.

(b) Put $T_1 = (x, 0) : x \in Z_{p^r} - \{0\}$ and $T_2 = (0, y) : y \in Z_{q^t} - \{0\}$.

(c) Connected every elements of T_1 to every element of T_2.

(d) For each zero-divisor d of $Z_{p^r} - \{0\}$ connected $(d, 0)$ to (e, f) where $de = 0$ and $f \in Z_{q^t}$.

(e) For each zero-divisor a of $Z_{q^t} - \{0\}$ connected $(0, a)$ to (c, b) where $ab = 0$ and $c \in Z_{p^r}$.

(f) Connected (x, y) to (w, z) where $x, w \in Z(Z_{p^r})^*$ and $y, z \in Z(Z_{q^t})^*$ and $xw = 0$ in Z_{p^r} and $yz = 0$ in Z_{q^t}.

Example 1. *Let $R = Z_{36} \cong Z_4 \times Z_9$, then $T_1 = \{(1, 0), (2, 0), (3, 0)\}$ and*

$$T_2 = \{(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8)\}.$$

Hence every element of T_1 is connected to every element of T_2. Now $Z(Z_4)^* = \{2\}$. Then $(2, 0)$ is connected to every element in the set

$$\{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (2, 8)\}.$$

Moreover $Z(Z_6)^* = \{3, 6\}$. *Then* $(0, 3)$ *is connected to every element in the set*

$$\{(0,6),(1,6),(2,6),(3,6),(1,3),(2,3),(3,3)\}$$

and $(0, 6)$ *is connected to every element in the set* $\{(0,3),(1,3),(2,3),(3,3),(1,6),(2,6),(3,6)\}$. *Also,* $(2, 3)$ *and* $(2, 6)$ *are connected.*

3. An Algorithm for calculating $\Gamma(Z_n[X]/(X^2))$

Considering the algorithm for calculating $\Gamma(Z_n[X]/(X^2))$, the following cases and sub cases are presented.

Case 1: For $Z_{p^n}[X]/(X^2)$, where p prime number and $n \geq 1$.
First,we consider the zero-divisors of $Z_{p^n}[X]/(X^2)$, they are of three types:

(a) By applying the algorithm in the previous section that calculates the zero-divisor graph of Z_{p^n}, we get the $n-1$ sets according to how many factors of p each divisor has, and they are, $S_1, S_2, \ldots, S_{n-1}$, where $S_i = \{rp^i : \gcd(r, p^{n-i}) = 1\}$.

(b) Consider the n sets, $E_0, E_1, \ldots, E_{n-1}$, where $E_i = \{bx : b \in S_i\}$. We mean by the set $S_0 = \{a : a \in U(Z_{p^n})\}$ and $E_0 = \{bx : b \in S_0\}$.

(c) Consider the $(n-1)n$ sets, $S_{i,j}$, such that $i = 1, 2, \ldots, n-1$ and $j = 0, 1, \ldots, n-1$. Where

$$\begin{aligned} S_{i,j} &= \{a + bx : a \in S_i \text{ and } b \in S_j\} \\ &= \left\{a + bx : \gcd(a, p^n) = p^i \text{ and } \gcd(b, p^n) = p^j\right\} \\ &= \left\{a + bx : \gcd(\frac{a}{p^i}, p^{n-i}) = 1 \text{ and } \gcd(\frac{b}{p^j}, p^{n-j}) = 1\right\}. \end{aligned}$$

Set

$$r = a/p^i \text{ and } s = b/p^j,$$

then

$$S_{i,j} = \{rp^i + sp^j x : \ \gcd(r, p^{n-i}) = 1 \text{ and } \gcd(s, p^{n-j}) = 1\}.$$

Second, if $z_1, z_2 \in \Gamma(Z_{p^n}[X]/(X^2))$, then the question is what are the cases for which $z_1 z_2 = 0$. We have six subcases:

Case 1.1: Every element in the set S_i is connected to every element in the set S_j if and only if $i + j \geq n$, this follows from the algorithm that calculates the zero-divisor graph of Z_{p^n}. Hence every element in S_i is connected to every element in the set S_j such that $j = n-i, n-i+1, \ldots, n-1$.

Case 1.2: Every element in the set S_i is connected to every element in the set E_j if and only if $i + j \geq n$, Hence every element in S_i is connected to every element in the set E_j such that $j = n-i, n-i+1, \ldots, n-1$.

Case 1.3: Every element in the set S_i is connected to every element in the set $S_{k,m}$ if and only if $i + k \geq n$, and $i + m \geq n$ .In other words, when the set S_i is connected to the two sets S_k and S_m. Hence, every element in S_i is connected to every element in the set $S_{k,m}$ such that $k = n-i, n-i+1, \ldots, n-1$ and $m = n-i, n-i+1, \ldots, n-1$.

Case 1.4: Every element in the set E_i is connected to every element in the set E_j for all i and j.Hence, Every element in E_i is connected to every element in the set E_j such that $j = 0, 1, \ldots, n-1$.

Case 1.5: Every element in the set E_i is connected to every element in the set $S_{k,m}$ if and only if $i + k \geq n$ .In other words,when the set S_i is connected to the set S_k. Hence, every element in E_i is connected to every element in the set $S_{k,m}$ such that $k = n-i, n-i+1, \ldots, n-1$ and $m = 0, 1, \ldots, n-1$.

Case 1.6: Consider the two sets $S_{i,j}$ and $S_{k,m}$. The essential condition in order that elements in the set $S_{i,j}$ are connected to elements in the set $S_{k,m}$ is $i + k \geq n$. Now consider for the following sub cases that $i + k \geq n$.

(a) If $i + m \geq n$ and $j + k < n$. Then no element in the set $S_{i,j}$ is connected to any element in the set $S_{k,m}$. Similarly, if $i + m < n$ and $j + k \geq n$. Then no element in the set $S_{i,j}$ is connected to any element in the set $S_{k,m}$.

(b) If $i + m < n$ and $j + k < n$ and $i + m \neq j + k$, then no element in the set $S_{i,j}$ connected to any element in the set $S_{k,m}$. To show that, let $y_1 = r_1 p^i + s_1 p^j x \in S_{i,j}$ and $y_2 = r_2 p^k + s_2 p^m x \in S_{k,m}$, if $y_1 y_2 = 0$, then $r_1 s_2 p^{i+m} + s_1 r_2 p^{j+k} = 0 \pmod{p^n}$, suppose that $i + m < j + k$, then $r_1 s_2 + s_1 r_2 p^{j+k-(i+m)} = 0 \pmod{p^{n-(i+m)}}$ $s_2 \equiv$

$-r_1^{-1}s_1r_2p^{j+k-(i+m)}$ (mod $p^{n-(i+m)}$). But $\gcd(s_2, p^{n-m}) = 1$, which is a contradiction.

(c) If $i+m \geq n$ and $j+k \geq n$, then every element in the set $S_{i,j}$ is connected to every element in the set $S_{k,m}$. In other words, every element in the set S_i is connected to the elements of the two sets S_k and S_m. and the elements of the set S_j are connected to the set S_k. Since $j+k \geq n$ and $i+k \geq n$, then $k \geq n-t$, where $t = \min\{i,j\}$ and $m \geq n-i$. Hence every element in the set $S_{i,j}$ is connected to every element in the set $S_{k,m}$, such that $k = n-t, n-t+1, \ldots, n-1$ where $t = \min\{i,j\}$ and $m = n-i, n-i+1, \ldots, n-1$.

(d) If $i+m < n$ and $j+k < n$ and $i+m = j+k$, then there exists elements in the set $S_{i,j}$ which are connected to elements in the set $S_{k,m}$.

Now, since $i+k \geq n$ and $j+k < n$, then $n-i \leq k < n-j$ and that inequality has a solution if and only if $i > j$. Also, since $i+m = j+k$ then $m = j+k-i \geq 0$, so $k \geq i-j$. Thus there exists elements in the set $S_{i,j}$ that are connected to elements in the set $S_{k,m}$ such that $k = w, w+1, \ldots, n-j-1$ where $w = max\{n-i, i-j\}$ and $m = j+k-i$.

To find these elements, let

$$S_{i,j} = \{rp^i + sp^jx : \gcd(r, p^{n-i}) = 1 \text{ and } \gcd(s, p^{n-j}) = 1\};$$
$$S_{k,m} = \{rp^k + sp^mx : \gcd(r, p^{n-k}) = 1 \text{ and } \gcd(s, p^{n-m}) = 1\}.$$

Suppose that $y_1 = r_1p^i + s_1p^jx \in S_{i,j}$ and $y_2 = r_2p^k + s_2p^mx \in S_{k,m}$, if $y_1y_2 = 0$, then

$$r_1s_2p^{i+m} + s_1r_2p^{j+k} \equiv 0 (\text{mod } p^n), \text{where } i+m = j+k$$
$$r_1s_2 + s_1r_2 \equiv 0 (\text{mod } p^{n-(i+m)})$$

Note that if $p = 2$ and $i+m = j+k = n-1$, then all elements in $S_{i,j}$ are connected all elements in $S_{k,m}$.

(1) For $Z_n[X]/(X^2)$, where $n = p^rq^t$ with p and q are relatively prime integers. Then the calculation of $\Gamma(Z_n[X]/(X^2))$ can be get by applying Theorem 1.2, case 1 of this section, and following the same manner of case 2 of the algorithm in previous section on $Z_{p^r}[X]/(X^2)$ and $Z_{q^t}[X]/(X^2)$.

Example 2. *Consider the ring $R = Z_{27}[X]/(X^2)$. then $Z(R)^*$ is divided into three types according to case 1,*

$$
\begin{aligned}
Type\ a: S_1 &= \{3, 6, 12, 15, 21, 24\} \\
S_2 &= \{9, 18\} \\
Type\ b: E_0 &= \{bx : b \in U(Z_{27})\} \\
E_1 &= \{bx : b \in S_1\} \\
E_2 &= \{bx : b \in S_2\} \\
Type\ c: S_{1,0} &= \{a + bx : a \in S_1, b \in U(Z_{27})\} \\
S_{1,1} &= \{a + bx : a \in S_1, b \in S_1\} \\
S_{1,2} &= \{a + bx : a \in S_1, b \in S_2\} \\
S_{2,0} &= \{a + bx : a \in S_2, b \in U(Z_{27})\} \\
S_{2,1} &= \{a + bx : a \in S_2, b \in S_1\} \\
S_{2,2} &= \{a + bx : a \in S_2, b \in S_2\}
\end{aligned}
$$

Now, by case 1.1, every element in the set S_1 is connected to every element in the set S_2, and every element in the set S_2 is connected to every element in the set S_2. By case 1.2, every element in the set S_1 is connected to every element in the set E_2, and every element in the set S_2 is connected to every elements in the sets E_1 and E_2. By case 1.3, every element in the set S_1 is connected to every element in the set $S_{2,2}$, and every element in the set S_2 is connected to every element in the sets $S_{1,1}, S_{1,2}, S_{2,1}$ and $S_{2,2}$. By case 1.4, every element in the sets E_0, E_1 and E_2 is connected to every element in the sets E_0, E_1 and E_2. By case 1.5, every element in the set E_1 is connected to every element in the sets $S_{2,0}, S_{2,1}$ and $S_{2,2}$, and every element in the set E_2 is connected to every element in the sets $S_{1,0}, S_{1,1}, S_{1,2}, S_{2,0}, S_{2,1}$ and $S_{2,2}$. By subcase c of case 1.6, every element in the sets $S_{1,1}, S_{1,2}$ is connected to every element in the set $S_{2,2}$, every element in the set $S_{2,1}$ is connected to every element in the sets $S_{2,1}$ and $S_{2,2}$, and every element in the set $S_{2,2}$ is connected to every element in the set $S_{2,2}$. By subcase d case 1.6, there exist elements in the set $S_{2,0}$ which are connected to elements in the same set. Also, there exist elements in the set $S_{1,0}$ which are connected to elements in the set $S_{2,1}$.

Example 3. *Let $R = Z_6[X]/(X^2)$. Then by Corollary 1.1,*

$$Z_6[X]/(X^2) \cong Z_2[X]/(X^2) \oplus Z_3[X]/(X^2).$$

By using case 1 of the algorithm, the zero divisor

$$Z\left(Z_2[X]/(X^2)\right)^* \quad \textit{Type b } E_0 = \{X\},$$
$$Z\left(Z_3[X]/(X^2)\right)^* \quad \textit{Type b } E_0 = \{X, 2X\}.$$

According to case 2 in section 2 :

$$T_1 = \{(1,0),(X,0),(1+X,0)\},$$
$$T_2 = \{(0,1),(0,2),(0,X),(0,2X),(0,1+X),$$
$$(0,1+2X),(0,2+X),(0,2+2X)\}.$$

Now every element in T_1 is connected to every element in T_2 and $(X,0)$ is connected to every element in

$$\{(X,1),(X,2),(X,X),(X,1+X),(X,2X),$$
$$(X,1+2X),(X,2+X),(X,2+2X)\}.$$

Also, every element in $\{(0,X),(0,2X)\}$ is connected to every element in

$$\{(1,X),(X,X),(1+X,X),(1,2X),(X,2X),(1+X,2X)\}.$$

Moreover, $(0,X)$ is connected to $(0,2X)$ and (X,X) is connected to $(X,2X)$.

References

[1] D. D. Anderson, M. Naseer: *Beck's coloring of commutative ring*, J. Algebra, **159** (1993), 500-514.

[2] D. F. Anderson, A. Frazier, A. Lauve, P. S. Livingston: *the zero-divisor graph of a commutative ring*, II, Ideal Theoretic Methods in commutative Algebra, **220** (2001), 61-72.

[3] D. F. Anderson, P. S. Livingston: *the zero-divisor graph of a commutative ring*, J. Algebra, **217** (1999), 434-447.

[4] I. Beck: *coloring of commutative rings*, J. Algebra, **116** (1988), 208-226.

[5] J. Krone: *Algorithms for constructing zero-divisor graphs of commutative rings*, preprint.

[6] M. Axtell, J. Coykendall, J. Stickless: *Zero-divisor of polynomials and power series over commutative rings*, Communications in Algebra, **33** (2003), 2043-2050.

[7] S. Akbari, A. Mohammadian: *On the zero-divisor of commutative rings*, J. Algebra, **274** (2004), 847-855.

[8] U. Dudley: *Elementry Number Theory, W.H. Freeman and Company*, San Frncisco, 1969.

Department of Mathematics
The Hashemite University
Zarqa, Jordan
Email address: rania.shaqboua@hu.edu.jo

Department of Basic Sciences
Al-Ahliyya Amman University
Amman, Jordan
Email address: h.qwareeq@ammanu.edu.jo