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# REST OF A VERTEX IN A GRAPH

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ABSTRACT. We define the rest of a vertex $v$ in a graph as the number of geodesics passing through $v$ minus the degree of $v$. The total rest of a graph is the sum of rests of all the vertices in that graph. We made some observations, compute rest of vertices in some standard graphs and obtain some interesting results.

## 1. INTRODUCTION

For standard terminology and notion in graph theory, we follow the text-book of Harary [4].

Let $G = (V, E)$ be a graph (finite, undirected, simple). We say that a graph $G$ is vertex transitive if the automorphism group of $G$ acts transitively on $V(G)$. A regular graph $G$ with $v$ vertices and degree $k$ is said to be strongly regular, denoted by $G = \text{srg}(v, k, \lambda, \mu)$, if there exist integers $\lambda$ and $\mu$ such that any two adjacent vertices have $\lambda$ common neighbors and any two non-adjacent vertices have $\mu$ common neighbors [2].

The concept of stress of a vertex in a graph was defined by Alfonso Shimbel [10] in 1953. The concept has many applications in the study of biological

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networks, social networks etc. Some related works can be found in [5–7, 9, 11]. Further, the concepts of stress number of a graph and stress regular graphs have been studied by K. Bhargava, N.N. Dattatreya, and R. Rajendra in their paper [1]. By the motivation of stress of a vertex and stress number of a graph, Rajendra et al. [8] defined stress indices for graphs.

## 2. Definitions

**Definition 2.1** (Alfonso Shimbel [10]). *Let $G$ be a graph and $v$ be a vertex in $G$. The stress of $v$, denoted by $str_G(v)$ or simply $str(v)$, is defined as the number of geodesics in $G$ passing through $v$.*

We denote the maximum stress among all the vertices of $G$ by $\Theta_G$ and minimum stress among all the vertices of $G$ by $\theta_G$.

**Definition 2.2.** *Let $G$ be a graph and $v$ be a vertex in $G$. The rest of $v$, denoted by $r_G(v)$ or simply $r(v)$, is defined as*

$$r(v) = str(v) - \deg(v). \tag{2.1}$$

The rest of a vertex $v$ is the number of geodesics passing through $v$ minus the number of edges incident on $v$.

We denote the maximum rest among all the vertices of $G$ by $R_G$ and minimum rest among all the vertices of $G$ by $\rho_G$.

**Definition 2.3** (S. Arumugam). *A graph is said to be $k$-stress regular if all of its vertices have stress $k$.*

**Definition 2.4.** *A graph is said to be $k$-rest regular if all of its vertices have rest $k$.*

**Definition 2.5.** *Let $G = (V, E)$ be a graph. The total stress of $G$, denoted by $N_{str}(G)$, is defined as,*

$$N_{str}(G) = \sum_{v \in V} str(v).$$

**Definition 2.6.** *Let $G = (V, E)$ be a graph. The total rest of $G$, denoted by $\mathcal{R}(G)$, is defined as,*

$$\mathcal{R}(G) = \sum_{v \in V} r(v). \tag{2.2}$$

## 3. Some Observations

(1) Since for any vertex $v$ in a graph $G$, str$(v) \geq 0$, by the Definition 2.2, $r(v) \geq -\deg(v)$. It follows that the rest of a pendant vertex in a graph is $-1$ because stress is zero and the degree is $1$ for that vertex.

(2) If $N$ is the number of geodesics of length at least 2 in a graph $G$, then by the Definitions 2.1 and 2.2, for any vertex $v$ in $G$, we have

$$-\deg(v) \leq r(v) \leq N - \deg(v)$$

and $-\Delta \leq r(v) \leq N - \delta$.

(3) If there is no geodesic of length $\geq 2$ passing through a vertex $v$ in a graph $G$, then str$(v) = 0$ and $r(v) = -\deg(v)$. Hence for any vertex $v$ in a complete graph $K_n$, we have $r(v) = 1 - n$.

(4) For a pendant vertex $v$ in a graph $G$, $r(v) = -1$.

(5) By the Definition 2.2, it follows that,
   (a) A regular graph is rest regular if and only if it is stress regular.
   (b) A stress regular graph is rest regular if and only if it is regular.

   A regular graph may not be rest regular because a regular graph need not be stress regular. A stress regular graph may not be rest regular because a stress regular graph need not be regular.

(6) A graph $G$ is $0$-rest regular if and only if str$(v) = \deg(v)$, $\forall v \in V(G)$.

(7) If $\eta$ is an automorphism of a graph $G$ and $v$ is any vertex in $G$, then $r(v) = r(\eta(v))$. Hence it follows that any vertex transitive graph is rest regular. However the converse is not true. The path $P_3$ is rest regular, but it is not vertex transitive since it is not regular (it is not stress regular also).

## 4. Some Results

**Theorem 4.1.** *Let $G$ be any graph and let $v$ be any vertex in $G$. Then $r(v) = -\deg(v)$ if and only if the neighbors of $v$ induce a complete subgraph.*

*Proof.* We have,

$$\begin{aligned} r(v) = -\deg(v) &\iff \text{str}(v) = 0 \qquad \text{(By Definition 2.2)} \\ &\iff \text{the neighbors of } v \text{ induce a complete subgraph.} \end{aligned}$$

□

The following corollary is immediate from the Theorem 4.1.

**Corollary 4.1.** *A connected graph is* $(1 - n)$*-rest regular if and only if it is a complete graph.*

**Proposition 4.1.** *For any graph* $G$ *of diameter* $d$ *with* $e$ *edges, the total rest of* $G$*, is given by*

$$\mathcal{R}(G) = N_{str}(G) - 2e = -2f_1 + \sum_{i=2}^{d-1}(i-1)f_i, \tag{4.1}$$

*where* $f_i$ *is the number of geodesics of length* $i$ *in* $G$*.*

*Proof.* By the Definition 2.6, we have

$$\begin{aligned}\mathcal{R}(G) = \sum_{v \in V} r(v) &= \sum_{v \in V} \text{str}(v) - \deg(v) \\ &= \sum_{v \in V} \text{str}(v) - \sum_{v \in V} \deg(v) \\ &= N_{\text{str}}(G) - 2e.\end{aligned}$$

It is easy to see that

$$N_{\text{str}}(G) = \sum_{i=2}^{d-1}(i-1)f_i. \tag{4.2}$$

Using (4.2) and $e = f_1$ in (2.2), we get

$$\mathcal{R}(G) = -2f_1 + \sum_{i=2}^{d-1}(i-1)f_i.$$

□

**Proposition 4.2.**

(i) *In a complete bipartite* $K_{mn}$*, if* $A$ *and* $B$ *are the partite sets of* $K_{mn}$ *with* $|A| = m$ *and* $|B| = n$*, then*

$$r(v) = \begin{cases} \dfrac{n(n-1)}{2} - n, & \text{if } v \in A, \\ \dfrac{m(m-1)}{2} - m, & \text{if } v \in B, \end{cases}; \tag{4.3}$$

*and*

$$R(K_{m,n}) = \frac{mn}{2}(m+n-2) - 2mn. \tag{4.4}$$

(ii) *In a cycle $C_n$ on $n$ vertices, for any vertex $v$,*

$$r(v) = \begin{cases} \dfrac{(n-1)(n-3)}{8} - 2, & \text{if } n \text{ is odd}, \\ \dfrac{n(n-2)}{8} - 2, & \text{if } n \text{ is even}, \end{cases}; \tag{4.5}$$

*and*

$$R(C_n) = \begin{cases} \dfrac{n(n-1)(n-3)}{8} - 2n, & \text{if } n \text{ is odd}, \\ \dfrac{n^2(n-2)}{8} - 2n, & \text{if } n \text{ is even}. \end{cases} \tag{4.6}$$

(iii) *Let $Wd(n, m)$ denote the windmill graph [3] constructed for $n \geq 2$ and $m \geq 2$ by joining $m$ copies of the complete graph $K_n$ at a shared universal vertex $v$. Then*

$$r(v) = \frac{m(m-1)(n-1)^2}{2} - m(n-1), \tag{4.7}$$

$$r(w) = -(n-1), \tag{4.8}$$

*and*

$$R(Wd(n, m)) = \frac{m(m-1)(n-1)^2}{2} - mn(n-1). \tag{4.9}$$

*Proof.*

(i) In a complete bipartite $K_{mn}$, if $A$ and $B$ are the partite sets of $K_{mn}$ with $|A| = m$ and $|B| = n$, then

$$\text{str}(v) = \begin{cases} \dfrac{n(n-1)}{2}, & \text{if } v \in A, \\ \dfrac{m(m-1)}{2}, & \text{if } v \in B, \end{cases}; \tag{4.10}$$

and

$$N_{\text{str}}(K_{m,n}) = \frac{mn}{2}(m+n-2). \tag{4.11}$$

Also,

$$\deg(v) = \begin{cases} n, & \text{if } v \in A, \\ m, & \text{if } v \in B, \end{cases}; \tag{4.12}$$

and number of edges in $K_{mn}$ is $mn$. Using (4.10) and (4.12) in (2.1), we get (4.3) and using (4.11) and $e = mn$ in (4.1), we get (4.4).

(ii) For any vertex $v$ in a cycle $C_n$,

$$\text{str}(v) = \begin{cases} \dfrac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd}, \\ \dfrac{n(n-2)}{8}, & \text{if } n \text{ is even}, \end{cases}; \tag{4.13}$$

and

$$N_{\text{str}}(C_n) = \begin{cases} \dfrac{n(n-1)(n-3)}{8}, & \text{if } n \text{ is odd}, \\ \dfrac{n^2(n-2)}{8}, & \text{if } n \text{ is even}, \end{cases}. \tag{4.14}$$

Since every vertex has degree $2$ and the number of edges $e = n$ in $C_n$, using (4.13) and (4.14) in (2.1) and (4.1), respectively, we get (4.5) and (4.6).

(iii) In the windmill $Wd(n,m)$, for the shared universal vertex $v$,

$$\text{str}(v) = m(m-1)(n-1)^2/2,\ \deg(v) = m(n-1), \tag{4.15}$$

and for any vertex $w \neq v$,

$$\text{str}(w) = 0,\ \deg(v) = (n-1).$$

Using (4.15) and (4.15) in (2.1), we get (4.7) and (4.8), respectively.

Finally, we have

$$N_{\text{str}}(Wd(n,m)) = m(m-1)(n-1)^2/2, \tag{4.16}$$

Since the number of edges in $Wd(n,m)$ is $mn(n-1)/2$, using (4.16) in (4.1), we get, (4.9).

□

**Proposition 4.3.** *Let $v$ be an internal vertex of a tree $T$ and let $C_1, \ldots, C_m$ be the components of $T - v$ (so that $\deg(v) = m$). Then*

$$r(v) = \sum_{i<j} |C_i||C_j| - m. \tag{4.17}$$

*Proof.* It is easy to see that

$$\text{str}(v) = \sum_{i<j} |C_i||C_j|. \tag{4.18}$$

Using (4.18) and $\deg(v) = m$, in (2.1), we get (4.17). □

**Theorem 4.2.** *Let $G = (V, E)$ be a connected graph with at least 3 vertices. In $G$, $r(v) = -\deg(v)$ for all vertices $v$ except for one if and only if $G$ is a graph with a unique cut-vertex such that all its blocks are complete subgraphs of $G$.*

*Proof.* In [1], it is proved that the graph $G$ has all vertices of zero stress except for one if and only if $G$ is a graph with a unique cut-vertex such that all its blocks are complete subgraphs of $G$. Hence the proof follows by Definition 2.2. □

By Theorem 4.2, the following Corollary is immediate:

**Corollary 4.2.** *Let $G$ be a connected graph on $n + 1$ vertices. Then $G = K_{1n}$ if and only if $G$ has exactly one vertex of rest $n(n - 3)/2$ with the rest of remaining vertices equal to their degrees.*

**Theorem 4.3.**

(i) *For any vertex $v$ in a graph $G$ of diameter 2, $r(v)$ equals the number of unordered pairs of non-adjacent vertices in $N(v)$ minus $\deg(v)$.*

(ii) *Any strongly regular graph $G =$ srg$(v, k, \lambda, \mu)$ is rest regular.*

*Proof.*

(i) For any vertex $v$ in a graph $G$ of diameter 2, str$(v)$ equals the number of unordered pairs of non-adjacent vertices in $N(v)$ (See [1]). Hence by Definition 2.2 the result follows.

(ii) Any strongly regular graph $G =$ srg$(v, k, \lambda, \mu)$ is stress regular (See [1]). Since a strongly regular graph is regular, by Definition 2.2 the result follows.

□

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