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NEARLY EXPANDABILITY IN BITOPOLOGICAL SPACES

Jamal Oudetallah

ABSTRACT. The concept of nearly Pairwise expandable spaces is a well known weaker form of expandable spaces, Katetove and L. Krajewski in [5] and in [6] respectively, followed by its further pursuit by many others. In the present paper, the same concept has been investigated in terms of a certain type of cover,called regular cover.and The purpose of this paper is study the properties of a new generalizations of pairwise expandable spaces called nearly pairwise expandable space.

1. INTRODUCTION

An expandability studied by Katetove and L. Krajewski in [5] and in [6] respectively, which is one of the most famous concepts of general topology, and many of its various forms have been investigated. Among the various discoloration generalizations found in the literature, the compactness, paracompact spaces can be indicated, and locally finite. In [3], Jamal A. Oudetallah introduced and studied pairwise expandability spaces as for m be an infinite cardinal, then a bitopological space (X, τ_1, τ_2) is called $\tau_\iota - m-$expandable space with respect to τ_j if for every $\tau_{\iota -}$ locally finite $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ with $|\triangle| \leq m$, there exist τ_j - locally finite collection $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open

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subsets of X such that $F_\alpha \subset G_\alpha$for all $\alpha \in \Delta$ and for $i \neq j, i, j = 1, 2$. A bitopological space (X, τ_1, τ_2) is called τ_i-expandable with respect to τ_j.If it is $\tau_i - m-$expandable for every cardinal m and $i \neq j, i, j = 1, 2$. A bitopological space (X, τ_1, τ_2) is called a pairwise expandable ($P-$expandable) proved that it is $P - T_2-$space and it is τ_1-expandable with respect to τ_2 and τ_2-expandable with respect to τ_1.In what follows, by a space X we shall mean a bitopological space (X, τ_1, τ_2).In [4] the authors of this paper introduced the concepts of pairwise paracompact spaces. For a subset A of X, $Int(A)$and $Cl(A)$ will stand respectively for interior and closure of A in X.We say that a set A in $X = (X, \tau_1, \tau_2)$ be a τ_i-regularly open set with respect to τ_j if $Int^{\tau_j}(Cl^{\tau_j}(A)) = A$ for $i \neq j, i, j = 1, 2$. Moreover A is called pairwise regularly open set in a space X if its τ_1 regularly open set with respect to τ_2 and τ_2 regularly open set with respect to τ_1.Clearly every pairwise regularly open set is pairwise open set. and a subset B in a space X is pairwise regularly closed set if its a complement of pairwise regularly open set.

Definition 1.1. *Let $X = (X, \tau_1, \tau_2)$ be a bitopological space, and let A be a τ_i - subset of X. then A is called $\tau_i - S-$open set with respect to τ_j if for every $x \in A$ there is a τ_j- regularly open set U of X such that $x \in U \subseteq A$ for $i \neq j, i, j = 1, 2$. Moreover A is called pairwise $-S-$ open set in a space X if its τ_1 $-S-$ open set with respect to τ_2 and τ_2 $-S-$ open set with respect to τ_1.*

Clearly, every pairwise $-S-$ open set is pairwise open set, and a subset B in a space X is pairwise $-S-$closed set if its a complement of pairwise $-S-$open set.

Definition 1.2. *Let $X = (X, \tau_1, \tau_2)$ be a bitopological space, and let τ_1^* be a collection of all $\tau_2 - S-$open sets and τ_2^* be a collection of all $\tau_1 - \delta-$open sets then $\mathrm{X} = (\mathrm{X}, \tau_1^*, \tau_2^*)$ is a bitopological space on X called semi$-$regular bitopological space. $\mathrm{X} = (\mathrm{X}, \tau_1^*, \tau_2^*)$ is weaker than $X = (X, \tau_1, \tau_2)$ and the collection of all pairwise regularly open sets of X form a base for $\mathrm{X} = (\mathrm{X}, \tau_1^*, \tau_2^*)$.*

Definition 1.3. *Let $X = (X, \tau_1, \tau_2)$ be a bitopological space, and let A be a τ_i-subset of X. A point $x \in X$ is said to be $\tau_i - S-$ adeherent point of A with respect to τ_j if every τ_j- regularly open set U containing x intersect A for $i \neq j, i, j = 1, 2$. Point x is called pairwise $-S-$ adeherent of A if its $\tau_1 - S-$ adeherent point of A with respect to τ_2 and its $\tau_2 - S-$ adeherent point of A with respect to τ_1. The set* *of all pairwise $-S-$ adeherent point of A is called pairwise $-S-$ closure of A and denoted by $p-S-Cl(A)$.*

As a result, a subset A a bitopological space $X=(X,\tau_1,\tau_2)$ is pairwise $-S-$ closed if and only if $A=p-S-Cl(A)$.

Definition 1.4. *A cover $\tilde{U}$ of the bitopological space (X,τ_1,τ_2) is called pairwise regular ($P-$ regular) open cover if $\tilde{U}$ contain at least one non-empty τ_1- regular open set with respect to τ_2 and at least one non-empty τ_2- regular open set with respect to τ_1.*

Definition 1.5. *[3] A collection subset $\tilde{F}=\{F_\alpha : \alpha \in \Delta\}$ of a bitopological space (X,τ_1,τ_2) is said to be pairwise locally finite if for each $x\in X$ there exist an τ_1-open set U containing x such that U intersects only finitely many members of $\tilde{F}$, or there exist τ_2-open V containing x such that V intersects only finitely many members of $\tilde{F}$.*

Definition 1.6. *[2]. A pairwise open cover ($P-$open cover) $\tilde{V}$ of a bitopological space (X,τ_1,τ_2) is called parallel refinement of a $P-$open cover $\tilde{U}$ of X if each τ_i-open set of $\tilde{V}$ is contained in some τ_i-open set of $\tilde{U}$ $(i=1,2)$.*

Definition 1.7. *[1].A bitopological space X is called pairwise - $m-$paracompact ($P-m-$paracompact) where m be an infinite cardinal if every $P-$open cover $\tilde{U}$ of X, such that $\left|\tilde{U}\right| \leq m$,has a pairwise locally finite open parallel refinement.If $m=\omega_0$ then a space X is called $P-$countably paracompact. If the space X is $P-m-$paracompact for every infinite cardinal m, then X is called $P-$paracompact.*

2. Characterizations of nearly expandability in bitopological spaces

In this section we study the relationship of nearly expandability with other bitopological spaces, obtain some characterization of pairwise nearly paracompactness and a pairwise expandability.

Definition 2.1. *A bitopological space X is called pairwise nearly - $m-$paracompact ($P-N-m-$paracompact) where m be an infinite cardinal if every $P-$regular open cover $\tilde{U}$ of X, such that $\left|\tilde{U}\right| \leq m$, has a pairwise locally finite open parallel refinement. If $m=\omega_0$, then a space X is called $P-$mildly paracompact. If the space X is $P-N-m-$paracompact for every infinite cardinal m, then X is called $P-N-$paracompact.*

Theorem 2.1. *A bitopological apace $X=(X,\tau_1,\tau_2)$ is $P-N-$paracompact if and only if every p-S-open cover of X has a pairwise locally finite open refinement.*

Proof. Follows from the fact that pairwise regularly open sets from a base for pairwise S-open sets and every pairwise regularly open set is pairwise S-open set □

Remark 2.1.

(i) Every $P-$Paracompact space is $P-N-$Paracompact.

(ii) In $\mathrm{X}=(\mathrm{X},\tau_1^*,\tau_2^*)$ Paracompact is equivalent to P-N-Paracompact.

Proof. Both (i) and (ii) follow immediately from definitions. □

Definition 2.2. *let $X=(X,\tau_1,\tau_2)$ be a bitopological space, A collection $\tilde{F}=\{F_\alpha : \alpha\in\Delta\}$ of subsets of X is called τ_i-S-locally finite w.r.t τ_j if for each $x\in X$, there exists a τ_j-S- open set u of X such that $x\in u$ and u intersect only finitely many members of $\tilde{F}$ for $i\neq j, i,j=1,2$. if $\tilde{F}$ is τ_1-S-locally finite w.r.t τ_2 and τ_2-S-locally finite w.r.t. τ_1 then $\tilde{F}$ is called pairwise $S-$locally finite and denoted by $P-S-$locally finite.*

Obviously,every $p-S-$locally finite collection in $X=(X,\tau_1,\tau_2)$ is $p-$locally finite, but the converse need to be true by the following example:

Example 1. *let X be an infinite set, let $q\in X$.define $C=\{A\subseteq X : q\in A\}$ then $X=(X,\tau_1,\tau_2)$ where $\tau_1=\tau_2=\{\phi\}\cup C$be a bitopological space so that the collection $\tilde{F}=\{\{x\} : x\in X-\{q\}\}$ of subset of X is p-locally finite but not $P-S-$locally finite.*

Now, we discuss some properties of $P-S-$locally finite collections.

Theorem 2.2. *A collection $\tilde{F}$=$\{F_\alpha : \alpha\in\Lambda\}$ of subset of abitopological space $X=(X,\tau_1,\tau_2)$is $P-S-$locally finite in $\mathrm{X}=(\mathrm{X},\tau_1^*,\tau_2^*)$.*

Proof. The proof is obvious in view of the fact that a subset of a space $\mathrm{X}=(\mathrm{X},\tau_1^*,\tau_2^*)$ is $p-$ open if and only if it is $p-S-$open. □

Theorem 2.3. *let $i\neq j, i,j=1,2$ and $\tilde{F}$=$\{F_\alpha : \alpha\in\Lambda\}$ be a collection of τ_i-subset of a bitopological space $X=(X,\tau_1,\tau_2)$. Then:*

(i) *$\tilde{F}\prime$ is $p-S-$locally finite if and only if $\{S-CL(F_\alpha) : \alpha\in\Lambda\}$ is $p-S-$locally finite.*

(ii) *if $\tilde{F}$ is $p-S-$locally finite, then* $\underset{\alpha\in\Lambda}{\cup}(S-CL(F_\alpha)) = S-CL(\underset{\alpha\in\Lambda}{\cup}F_\alpha)$.

In particular, the union of $p-S-$locally finite collection of closed sets is closed.

Proof.

(i) pick $x \in X$ then there exist $u_x$be a $\tau_i - S$-open set such that $u_x \cap F_\alpha = \phi$ except for finitely many α. but then $u_x \cap CL(F_\alpha) = \phi$except for finitely many α. Thus, $\{S-CL(F_\alpha) : \alpha \in \Lambda\}$ is $p-S-$ locally finite.

Conversely, pick $x \in X$ then there exist u_x be a $S-\tau_i-$open set such that $u_x \cap CL(F_\alpha) = \phi$except for finitely many α since $F_\alpha \subset CL(F_\alpha)$ and $u_x \cap CL(F_\alpha) = \phi$so $u_x \cap F_\alpha = \phi$; except for finitely many α, therefore $\tilde{F} = \{F_\alpha : \alpha \in \Lambda\}$ is $p-S-$ locally finite.

(ii) Easily $\bigcup\limits_{\alpha\in\Lambda} S-CL(F_\alpha) \subset S-CL(\bigcup\limits_{\alpha\in\Lambda} F_\alpha)$. On other hand, suppose $x \in S-CL(\bigcup\limits_{\alpha\in\Lambda} F_\alpha)$.

Now some $\tau_i - -S-$open sets of x meets only finitely many of the sets $S-F_\alpha$, say $S-F_{\alpha_1}$,$S-F_{\alpha_2}, \ldots, S-F_{\alpha_n}$. Since every $\tau_i - S-$open sets of x meets $S-\cup F_\alpha$, every $\tau_i - S-$open set of x must then meet $S-F_\alpha$, say $S-F_{\alpha_1}$,$S-F_{\alpha_2}\ldots$,$S-F_{\alpha_n}$. Hence $x \in \delta - CL(S-F_{\alpha_1}$,$S-F_{\alpha_2} \ldots, S-F_{\alpha_n}$)$= S-CL(F_{\alpha_1}) \cup S-CL(F_{\alpha_2}) \cup \cdots \cup S-CL(F_{\alpha_n}) = \bigcup\limits_{i=1}^{n} S-CL(F_i)$, establishing the theorem part (ii). □

Theorem 2.4. *Let $\tilde{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a $P-S-$ locally finite collection of subsets of a space X. let Γ be the collection of all finite subsets of Λ. then the collection $\tilde{F}^* = \{\underset{\alpha\in\gamma}{\cap} F_\alpha : \gamma \in \Gamma\}$is P-S-locally finite.*

The proof follows from the definition of P-S-locally finite collection.

Definition 2.3. *Let $i \neq j, i, j = 1, 2$. and m be an infinite cardinal number a bitopological space $X = (X, \tau_1, \tau_2)$ is said to be $\tau_i - m - nearly$expandable with respect to $\tau_j$if for each $\tau_i - S-$locally finite collection $\tilde{F} = \{F_\alpha : \alpha \in \Lambda\}$ with $|\Lambda| \leq m$of subsets of X, there exist a τ_j-locally finite collection $G = \{G_\alpha : \alpha \in \Lambda\}$with $|\Lambda| \leq m$ of open subsets of X such that $F_\alpha \subset G_\alpha$, for each $\alpha \in \Lambda$.*

A bitopological space $X = (X, \tau_1, \tau_2)$is said to be pairwise m-nearly expandable space if it is τ_1-nearly expandable space w.r.t $\tau_2$and conversely.

If X is pairwise m-nearly expandable space for every cardinal number m then X is P-nearly expandable ($P-N-$ expandable).

Remark 2.2. *A according to Theorem 2.1 we may replace the τ_i collection $\tilde{F}$of subsets of X by the τ_i-collection of $S-$closedsubsets of X, in the Definition 2.3 above, also in this definition , we may replace a τ_j-locally finite collection of open subsets of X by $\tau_j - S-$locally finite collection of open subsets of X.*

Theorem 2.5. *Let $i \neq j, i, j = 1, 2$, a bitopological space $X = (X, \tau_1, \tau_2)$ is $P - N-$expandable space space if and only if for each $\tau_i - S-$locally finite collection $\tilde{F}$of subsets of$X$ there exist τ_i-locally finite open cover $\tilde{U}$of X such that each member of $\tilde{U}$ intersected at most finitely many members of $\tilde{F}$.*

Proof. Let X be a $P - N-$expandable space. Let $\tilde{F} = \{F_\alpha : \alpha \in \Lambda\}$ be $\tau_i - S-$locally finite collection of subsets of X.

By Theorem 2.1 $S - CL(F) = \{S - CL(F_\alpha) : \alpha \in \Lambda\}$ is a $\tau_i - S-$locally finite collection of closed subsets of X. By assumption, there exist a τ_i-locally finite collection of open subsets of X say $\tilde{G} = \{G_\alpha : \alpha \in \Lambda\}$ such that $CL(F_\alpha) \subseteq G_\alpha$, for each $\alpha \in \Lambda$. Let Γ be the collection of all finite subsets of Λ define $U_\gamma = \bigcap\limits_{\alpha\in\gamma} G_\alpha - \bigcup\limits_{\alpha\in\gamma} CL(F_\alpha)$. Let $\tilde{U} = \{\cup_\gamma : \gamma \in \Gamma\}$. Then we claim that $\tilde{U}$is $\tau_j - S-$locally finite open cover of X.so for $x \in X$ there exist τ_j-open set $v(x)$ in X such that $x \in v(x)$ and intersects at most finitely many members of $\tilde{F}$ say $F_{\alpha_1}, F_{\alpha_2}, \ldots, F_{\alpha_n}$ and $x \in \bigcap\limits_{i=1}^{n} F_{\alpha_i}$. But for each F_{α_i}, there exist G_{α_i}, such that $F_{\alpha_i} \subseteq G_{\alpha_i}$. there fore $\bigcap\limits_{i=1}^{n} F_{\alpha_i} \subseteq \bigcap\limits_{i=1}^{n} G_{\alpha_i}$. This implies $x \in \bigcap\limits_{i=1}^{n} G_{\alpha_i}\dot{S}$o that there exist $\Upsilon \in \Gamma$ such that $x \in U_\gamma$ and finally, $\tilde{U}$ is a τ_j- opencover of X. Clearly for each $\gamma \in \Gamma$, U_γ intersects at most finitely many members of $\tilde{F}$.

Conversely, suppose $\tilde{F} = \{F_\alpha : \alpha \in \Lambda\}$ is a $\tau_i - S-$locally finite collection of subsets of X and $\tilde{U} = \{U_\beta : \beta \in B\}$ is a τ_j-locally finite open cover of X such that for all $\beta \in B$, U_β intersects at most finitely many members of $\tilde{F}$. Now for each $\alpha \in \Lambda$, let $B_\alpha = \{B \in B : U_\beta \cap F_\alpha \neq \phi\}$ and let $G_\alpha = \bigcup\limits_{\beta\in B_\alpha} U_\beta$, then $\tilde{G} = \{G_\alpha : \alpha \in \Lambda\}$is a $\tau_i - S-$locally finite collection of open subsets of X, such that $F_\alpha \subseteq G_\alpha$, for each $\alpha \in \Lambda$. Hence X is $\tau_i - N-$expandable space with respect to τ_j for $i \neq j, i, j = 1, 2$. Therefore X is $P - N-$expandable space. □

Corollary 2.1. *Every $P-$expandable space is $P - N-$expandable. and every $P - m_0-$expandable space is $P - N-$expandable.*

The proof follows from the fact that every $P - S-$locally finite collection is $P-$locally finite.

Now, let us discuss sufficient conditions for pairwise nearly expandability to be equivalent to pairwise expandability.

Theorem 2.6. *Let $X = (X, \tau_1, \tau_2)$ be a pairwise-semi-regular space then X is $P-$expandable if X is $P - N-$expandable.*

Proof. If X is $P-$semi-regular expandable space, then it is $P-$ regular expandable space so by Corollary 2.1, X is $P - N-$expandable.

Conversely, suppose X is $P - N-$ expandable, let $i \neq j, i, j = 1, 2$ and let $\tilde{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a τ_i-locally finite collection of subsets of X. Since X is $p-$semi regular space, then $\tilde{F}$ is equivalent to a $\tau_i - S-$locally finite collection of subsets of X by Theorem 2.5. So by assumption, $\tilde{F}$ can be expanded to an τ_j-open locally finite collection, therefore X is τ_i-expandable with respect to τ_j for all $i \neq j, i, j = 1, 2$ and hence X is $P-$expandable space. □

Corollary 2.2. *A bitopological space (X, τ_1, τ_2) is $P - N - m_0-$expandable if and only if (X, τ_1^*, τ_2^*) is $P - m_0-$expandable.*

The proof follows from Theorem 2.6 pairwise near expandabiliy looks like pairwise expandability as it has a lot of characterization in terms of covering.

Theorem 2.7. *Every $P - N-$Paracompact space is $P - N-$expandable.*

Proof. Let $i \neq j, i, j = 1, 2$ and m be an infinite cardinal number. Let $\tilde{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a $\tau_i - S-$locally finite collection of closed subsets of the $P - m - N-$paracompact space X with $|\Lambda| \leq m$. let Γ be the collection of all finite subsets of X and define $\upsilon_\Upsilon = X - \cup\{F_\alpha : \alpha \in \Lambda\}, \Upsilon \in \Gamma$, clearly υ_Υ is $\tau_i - openset$ and υ_Υ intersects only finitely many members of $\tilde{F}$ and $\tilde{v} = \{\upsilon_\Upsilon : \Upsilon \in \Gamma\}$ is τ_i-covers of X. For this, let $x \in X$, since $\tilde{F}$ is $\tau_i - S-$locally finite of subsets of X, then there is a $\tau_j - S-$openset $U(x)$ such that $U(x)$intersects at most finitely many members of $\tilde{F}$ say F_{α_1},$F_{\alpha_2}, \ldots, F_{\alpha_n}$. Let $\Upsilon = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, then V_{Υ_n} is $\tau_j - S-$open set contain x and therefore $\tilde{V}$ is $\tau_i - S-$open set cover of X with $|\Gamma| \leq m$. Since X is $_j-$locally finite open refinement $\tilde{W} = \{w_s : s \in B\}$, now put $U_\alpha = St(F_\alpha, \tilde{W}) = \cup\{W_\alpha \in \tilde{W} : W_\alpha \cap F_\alpha \neq \varnothing\}, \alpha \in \wedge$, hence $F_\alpha \subseteq U_\alpha$and U_α is $\tau_j - S-$open set for each $\alpha \in \wedge$. Now it suffices to show that $\tilde{U} = \{U_\alpha : \alpha \in \wedge\}$ is $\tau_j - S-$locally finite collection of subsets of X. Let $x \in X$ then there exists a $\tau_j - S-$open set O which contains x and intersects only finitely many members of $\tilde{W}$. Thus $O \cap U_\alpha \neq \varnothing$ if and only if $O \cap W_s \neq \varnothing$ and $W_s \cap F_\alpha \neq \varnothing$, for some

$s \in B$. But $\tilde{W}$is a $\tau_j - S-$open refinement of V_s. So W_s contained in some V_s which intersects only finitely many of F_α $'^s$. thus U is τ_j-locally finite. So X is $\tau_i - m-$nearly expandable space for all $i \neq j, i, j = 1, 2$ and for every cardinal number $m.$ thus X is $P - N-$expandable space. □

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND INFORMATION TECHNOLOGY
IRBID NATIONAL UNIVERSITY
Email address: jamalayasrah12@gmail.com