

A NOTE ON COMPUTATIONAL METHOD FOR SHEHU TRANSFORM BY ADOMIAN DECOMPOSITION METHOD

Duangkamol Poltem and Sineenart Srimongkol¹

ABSTRACT. In this paper, we apply the well known method which is Adomian decomposition method to formulate the Shehu transform. The Shehu transform is a Laplace-type integral transform for solving the differential equations. Some examples are shown to illustrate the simplicity and reliability of the proposed method.

1. INTRODUCTION

Adomian decomposition method is a useful tool to solve not only ordinary differential equations but also partial differential equations including both linear and non-linear equations [2]. Some applications in physics, chemistry, and biology are applied Adomian decomposition method (ADM). ADM is applied to solve Klein-Gordon equation which is used to study the quantum movement, and the distribution of physic plasma [6]. ADM was used in nonlocal Timoshenko beam theory to study a nano-switch [5]. Fokker-Planck equation which is using in solid-state physics, quantum optics, and theoretical biology is solved by ADM [7]. However, some complicated computations still have to investigated.

¹*corresponding author*

2020 *Mathematics Subject Classification.* 34A25.

Key words and phrases. Shehu transform, Adomian decomposition method.

Submitted: 17.01.2021; *Accepted:* 01.02.2021; *Published:* 06.02.2021.

Shehu transform is a Laplace-type integral transform for solving the differential equations in the time domain [4]. Although, Shehu transform for some functions are computed and shown in [4], the complicated of improper integral may be occurred for some complicated functions. Alternative method to find the Shehu transform is crucial. Therefore, the Adomian decomposition method of the first order linear differential equation is applied to formulate the Shehu transform.

2. SHEHU TRANSFORM

In this section, Shehu transform definition is presented.

Theorem 2.1. [4] *The Shehu transform of the function $v(t)$ of exponential order is defined over the set of functions*

$$A = \left\{ v(t) : \exists N, \eta_1, \eta_2 > 0, |v(t)| < N \exp\left(\frac{|t|}{\eta_i}\right), t \in (-1)^i \times [0, \infty) \right\},$$

by the following integral

$$\mathbb{S}[v(t)] = V(s, u) = \int_0^\infty \exp\left(\frac{-st}{u}\right) v(t) dt,$$

where $\mathbb{S}[v(t)] = V(s, u)$ is the Shehu transformation of the time function $v(t)$ and the variables $s > 0$ and $u > 0$ are the Shehu transform variables. $\mathbb{S}^{-1}[V(s, u)]$ is called inverse Shehu transform of $v(t)$ defined by

$$v(t) = \mathbb{S}^{-1}[V(s, u)] = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{u} \exp\left(\frac{st}{u}\right) V(s, u) ds.$$

3. ADOMAIN DECOMPOSITION METHOD

Adomain decompositon method is a method for solving both linear and non-linear differential equations. In this work, the first order ordinary differential equation is taken into account. Consider the following equation with an initial condition,

$$(3.1) \quad \frac{dy}{dt} + p(t)y = v(t), \quad y(0) = 0.$$

The analytical solution of (3.1) is

$$\mu(t)y(t) = \int \mu(t)v(t)dt,$$

where $\mu(t) = e^{\int p(t)dt}$. Using the operator $L(\cdot) = \frac{d}{dt}(\cdot)$ and p is any constant, the equation (3.1) is rewritten as,

$$L(y) + py = v(t),$$

or,

$$(3.2) \quad y = \frac{v(t)}{p} - \frac{L(y)}{p}.$$

Let the solution of the equation (3.1) be written in the series representation, so $y(t) = \sum_{i=0}^{\infty} y_i$. Then, the equation (3.2) become

$$(3.3) \quad \sum_{i=0}^{\infty} y_i = \frac{v(t)}{p} - \frac{1}{p}L\left(\sum_{i=0}^{\infty} y_i\right).$$

Compare both sides of equation (3.3), resulting

$$\begin{aligned} y_0 &= \frac{v(t)}{p} \\ y_1 &= -\frac{1}{p}L(y_0) = -\frac{1}{p}L\left(\frac{v(t)}{p}\right) \\ y_2 &= -\frac{1}{p}L(y_1) = (-1)^2 \frac{1}{p^2}L^2\left(\frac{v(t)}{p}\right) \\ y_3 &= -\frac{1}{p}L(y_2) = (-1)^3 \frac{1}{p^3}L^3\left(\frac{v(t)}{p}\right) \\ &\vdots \end{aligned}$$

The general form is

$$(3.4) \quad y_i = (-1)^i \frac{1}{p^i}L^i\left(\frac{v(t)}{p}\right),$$

where $L^i(\cdot) = \frac{d^i}{dt^i}(\cdot)$, $i = 1, 2, \dots$ is i -fold differential operator. The convergence of the ADM was discussed by Cherruault [1, 3].

4. SHEHU TRANSFORM COMPUTATION BY ADOMIAN DECOMPOSITION METHOD

Shehu transform definition is an improper integral over time. The more complicated function, the more complicated integral. In this work, Adomain decomposition method is applied to solve this problem. Start with equation (3.1) and suppose that $p = -\frac{s}{u}$ for $s > 0, u > 0$, this will give

$$(4.1) \quad \int \exp\left(-\frac{st}{u}\right) v(t) dt = \exp\left(-\frac{st}{u}\right) y(t).$$

Integrating from zero to infinity of equation (4.1) makes the left hand side of this equation become the Shehu transform of $v(t)$. Decompose y as in equation (3.4) to obtain

$$(4.2) \quad \mathbb{S}[v(t)] = \int_0^\infty \exp\left(-\frac{st}{u}\right) v(t) dt = \left[\exp\left(-\frac{st}{u}\right) \left(\sum_{i=0}^\infty y_i \right) \right]_0^\infty.$$

The examples of using this method are functions, $v(t) = 1, t, \exp(at), t^n, \sin(at)$, and $\cosh(at)$.

Example 1. Let $v(t) = 1$ then by the Shehu transform of equation (4.2), we have

$$\mathbb{S}[1] = \int_0^\infty \exp\left(-\frac{st}{u}\right) dt = \left[\exp\left(-\frac{st}{u}\right) \left(\sum_{i=0}^\infty y_i \right) \right]_0^\infty.$$

According to scheme (3.4), we have

$$y_0 = \frac{v(t)}{p(t)} = -\frac{u}{s}$$

$$y_1 = y_2 = \dots = 0.$$

Therefore,

$$\mathbb{S}[1] = \left[\exp\left(-\frac{st}{u}\right) \left(-\frac{u}{s} \right) \right]_0^\infty = \frac{u}{s}.$$

Example 2. Let $v(t) = t$, from equation (4.2) we have

$$\mathbb{S}[t] = \int_0^\infty \exp\left(-\frac{st}{u}\right) t dt = \left[\exp\left(-\frac{st}{u}\right) \left(\sum_{i=0}^\infty y_i \right) \right]_0^\infty.$$

From scheme (3.4), we have

$$\begin{aligned} y_0 &= \frac{v(t)}{p} = -\frac{ut}{s} \\ y_1 &= -\frac{1}{p}L\left(\frac{v(t)}{p}\right) = -\frac{u^2}{s^2} \\ y_2 &= y_3 = \dots = 0. \end{aligned}$$

Therefore,

$$\mathbb{S}[t] = \left[\exp\left(-\frac{st}{u}\right) \left(-\frac{ut}{s} - \frac{u^2}{s^2}\right) \right]_0^\infty = \frac{u^2}{s^2}.$$

Example 3. Let $v(t) = \exp(at)$, from equation (4.2) we have

$$\mathbb{S}[\exp(at)] = \int_0^\infty \exp\left(-\frac{st}{u}\right) \exp(at) dt = \left[\exp\left(-\frac{st}{u}\right) \left(\sum_{i=0}^\infty y_i\right) \right]_0^\infty.$$

From scheme (3.4), we have

$$\begin{aligned} y_0 &= \frac{v(t)}{p} = -\frac{ue^{at}}{s} \\ y_1 &= -\frac{1}{p}L\left(\frac{v(t)}{p}\right) = -\frac{u^2ae^{at}}{s^2} \\ y_2 &= -(-1)^2\frac{1}{p^2}L^2\left(\frac{v(t)}{p}\right) = -\frac{u^3a^2e^{at}}{s^3} \\ y_3 &= (-1)^3\frac{1}{p^3}L^3\left(\frac{v(t)}{p}\right) = -\frac{u^4a^3e^{at}}{s^4} \\ &\vdots \\ y_n &= -\frac{u^{n+1}a^ne^{at}}{s^{n+1}}, \quad n = 0, 1, 2, 3, \dots \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{S}[e^{at}] &= \left[\exp\left(-\frac{st}{u}\right) \left(-\frac{ue^{at}}{s} - \frac{u^2ae^{at}}{s^2} - \frac{u^3a^2e^{at}}{s^3} - \frac{u^4a^3e^{at}}{s^4} - \dots\right) \right]_0^\infty \\ &= \frac{u}{s} + \frac{u^2a}{s^2} + \frac{u^3a^2}{s^3} + \frac{u^4a^3}{s^4} + \dots + \frac{u^{n+1}a^n}{s^{n+1}} + \dots \\ &= \frac{u}{s-au}, \quad s > au. \end{aligned}$$

Example 4. Let $v(t) = t^n$, from the Shehu transform (4.2) we have

$$\mathbb{S}[t^n] = \int_0^\infty \exp\left(-\frac{st}{u}\right) t^n dt = \left[\exp\left(-\frac{st}{u}\right) \left(\sum_{i=0}^\infty y_i \right) \right]_0^\infty.$$

From scheme (3.4), we have

$$\begin{aligned} y_0 &= \frac{v(t)}{p} = -\frac{ut^n}{s} \\ y_1 &= -\frac{1}{p}L\left(\frac{v(t)}{p}\right) = -\frac{u^2nt^{n-1}}{s^2} \\ y_2 &= (-1)^2\frac{1}{p^2}L^2\left(\frac{v(t)}{p}\right) = -\frac{u^3n(n-1)t^{n-2}}{s^3} \\ y_3 &= (-1)^3\frac{1}{p^3}L^3\left(\frac{v(t)}{p}\right) = -\frac{u^4n(n-1)(n-2)t^{n-3}}{s^4} \\ &\vdots \\ y_n &= -\frac{u^{n+1}}{s^{n+1}}n!. \end{aligned}$$

We can see that, $y_{n+1} = y_{n+2} = \dots = 0$. So, we get the following

$$\begin{aligned} \mathbb{S}[t^n] &= \left[\exp\left(-\frac{st}{u}\right) \left(-\frac{ut^n}{s} - \frac{u^2nt^{n-1}}{s^2} - \frac{u^3n(n-1)t^{n-2}}{s^3} - \dots - \frac{u^{n+1}}{s^{n+1}}n! \right) \right]_0^\infty \\ &= \left(\frac{u}{s} \right)^{n+1} n!. \end{aligned}$$

Example 5. Let $v(t) = \sin(at)$, from the Shehu transform (4.2) we have

$$\mathbb{S}[\sin(at)] = \int_0^\infty \exp\left(-\frac{st}{u}\right) \sin(at) dt = \left[\exp\left(-\frac{st}{u}\right) \left(\sum_{i=0}^\infty y_i \right) \right]_0^\infty.$$

From equation (3.4), we have

$$\begin{aligned}
y_0 &= \frac{v(t)}{p(t)} = -\frac{u}{s} \sin(at) \\
y_1 &= -\frac{1}{p} L \left(\frac{v(t)}{p} \right) = -\frac{u^2}{s^2} \cos(at)a \\
y_2 &= (-1)^2 \frac{1}{p^2} L^2 \left(\frac{v(t)}{p} \right) = \frac{u^3}{s^3} \sin(at)a^2 \\
y_3 &= (-1)^3 \frac{1}{p^3} L^3 \left(\frac{v(t)}{p} \right) = \frac{u^4}{s^4} \cos(at)a^3 \\
&\vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{S}[\sin(at)] &= \left[\exp \left(-\frac{st}{u} \right) \left(-\frac{u}{s} \sin(at) - \frac{u^2}{s^2} \cos(at)a + \frac{u^3}{s^3} \sin(at)a^2 \right. \right. \\
&\quad \left. \left. + \frac{u^4}{s^4} \cos(at)a^3 - \dots \right) \right]_0^\infty \\
&= \frac{u^2 a}{s^2} - \frac{u^4 a^3}{s^4} + \frac{u^6 a^5}{s^6} - \dots = \frac{u^2 a}{s^2 + u^2 a}.
\end{aligned}$$

Example 6. Let $v(t) = \cosh(at)$, from the Shehu transform (4.2) we have

$$\mathbb{S}[\cosh(at)] = \int_0^\infty \exp \left(-\frac{st}{u} \right) \cosh(at) dt = \left[\exp \left(-\frac{st}{u} \right) \left(\sum_{i=0}^\infty y_i \right) \right]_0^\infty.$$

From equation (3.4), we have

$$\begin{aligned}
y_0 &= \frac{v(t)}{p} = -\frac{u}{s} \cosh(at) \\
y_1 &= -\frac{1}{p} L \left(\frac{v(t)}{p} \right) = -\frac{u^2}{s^2} \sinh(at)a \\
y_2 &= (-1)^2 \frac{1}{p^2} L^2 \left(\frac{v(t)}{p} \right) = -\frac{u^3}{s^3} \cosh(at)a^2 \\
y_3 &= (-1)^3 \frac{1}{p^3} L^3 \left(\frac{v(t)}{p} \right) = -\frac{u^4}{s^4} \sinh(at)a^3 \\
&\vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
 \mathbb{S}[\cosh(at)] &= \left[\exp\left(-\frac{st}{u}\right) \left(-\frac{u}{s} \cosh(at) - \frac{u^2}{s^2} \sinh(at)a - \frac{u^3}{s^3} \cosh(at)a^2 \right. \right. \\
 &\quad \left. \left. - \frac{u^4}{s^4} \sinh(at)a^3 - \dots \right) \right]_0^\infty \\
 &= \frac{u}{s} + \frac{u^3 a^2}{s^3} + \frac{u^5 a^4}{s^5} + \dots \\
 &= \frac{us}{s^2 - u^2 a^2}.
 \end{aligned}$$

5. CONCLUSION

The decomposition method is a simple method for solving the order differential equation. In this work, this method has been applied to compute Shehu transform for some functions. It has been observed that the suggested method is simplicity for Shehu transform. Using definition of Shehu transform, the improper integral has to compute while using ADM, it requires only simple differentiation.

ACKNOWLEDGMENT

The authors are grateful to the reviewers for valuable suggestions in improving the quality of the paper. This work was financially supported by Faculty of Science, Burapha University (Grant no. SC01/2562).

REFERENCES

- [1] K. ABBAOU, Y. CHERRUAULT: *Convergence of adomian's method applied to differential equations*, Computers and Mathematics with Applications, **28**(5) (1994), 103–109, 1994.
- [2] E. BABOLIAN, J. BIAZAR, A. VAHIDI: *A new computational method for laplace transforms by decomposition method*, Applied Mathematics and Computation, **150**(3) (2004), 841–846.
- [3] Y. CHERRUAULT: *Convergence of adomian's method*, Kybernetes, **18**(2) (1989), 31–38.
- [4] S. MAITAMA, W. ZHAO: *New integral transform: Shehu transform a generalization of sumudu and laplace transform for solving differential equations*, International Journal of Analysis and Applications, **17**(2) (2019), 167–190.

- [5] P. MORADWEYSI, R. ANSARI, K. HOSSEINI, F. SADEGHI: *Application of modified adomian decomposition method to pull-in instability of nano-switches using nonlocal timoshenko beam theory*, Applied Mathematical Modelling, **54** (2018), 594–604.
- [6] J. SAELAO, N. YOKCHOO: *The solution of klein–gordon equation by using modified adomian decomposition method*, Mathematics and Computers in Simulation, 171:94–102, 2020. International Conference in Mathematics and Applications, held in Bangkok, Thailand, on December 16-18, 2018.
- [7] M. TATARI, M. DEGHAN, M. RAZZAGHI: *Application of the adomian decomposition method for the fokker–planck equation*, Mathematical and Computer Modelling, **45**(5) (2007), 639–650.

DEPARTMENT OF MATHEMATICS
BURAPHA UNIVERSITY
MUENG, CHONBURI, THAILAND
Email address: duangkamolp@buu.ac.th

DEPARTMENT OF MATHEMATICS
BURAPHA UNIVERSITY
MUENG, CHONBURI, THAILAND
Email address: sineenart@buu.ac.th