

A CORRESPONDING CULLEN INTEGRAL THEOREM FOR A SPHERICALLY REGULAR FUNCTION IN SPHERICAL COORDINATES

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ABSTRACT. In this paper, we consider the fundamental elements and results of the regular functions of one quaternionic variable. Using the geometric properties of quaternions, we express the spherical coordinates of the Cullen form, introduced in 1965, for the quaternion. As the vector part of the quaternion can be parameterized in three-dimensional spherical coordinates, the regularity and harmonicity of a quaternionic function defined in the Cullen form is provided as the expression of the spherical coordinate system. Based on the regularity of the quaternion expressed in spherical coordinates, the Cullen integral formula that can be applied to the quaternionic function expressed in spherical coordinates within the three dimensions is newly expressed and proved.

1. INTRODUCTION

Let $\mathbb{H}$ be the algebra of the quaternions denoted by

$$\mathbb{H} = \{q \mid q = t + ix + jy + kz, \quad t, x, y, z \in \mathbb{R}\},$$

where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. Quaternions are non-commutative and associative with respect to the product by the rules between the bases i, j, k . There have been many attempts to define

2020 *Mathematics Subject Classification.* 32G35, 32W50, 32A99, 11E88.

Key words and phrases. quaternions, Cullen form, Cauchy–Riemann system, spherical coordinates, regular functions.

Submitted: 23.01.2021; *Accepted:* 07.02.2021; *Published:* 14.02.2021.

the regularity of the quaternion-valued function f of a quaternionic variable q . By using the exterior differential calculus, Sudbery [14] provided simple proofs of the main theorems and clarified the relation between quaternionic and complex analyses. Mandic et. al [13] introduced a convenient way to calculate derivatives of regular functions conforming to the corresponding solution in the complex domain based on the isomorphism with real vectors and the use of quaternion involutions. From the non-commutative product for the base of the quaternion, Kim et. al [8, 10] structured the form of the quaternion into a scalar part and a vector part to define a hyperholomorphic function in the modified form of the quaternion system and investigated the properties of hyperholomorphic functions in the quaternion system. Kim and Shon [9] have given hyperholomorphic functions on the quaternions and proposed the Cauchy–Riemann equations corresponding to the polar form of the quaternion system by presenting the expression of the structure of the quaternion. Nguyen [15] dealt with the initial value problem involving a linear first-order operator in quaternionic analysis and a regular function taking values in quaternionic algebra. Nguyen proved the necessary and sufficient conditions on the coefficients of a linear first-order operator under which a linear first-order operator is associated with the Cauchy–Fueter operator of quaternionic analysis. We recall the Fueter operator as follows:

$$\frac{\partial}{\partial t} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}.$$

Fueter [4, 5] proposed the holomorphicity condition as the Cauchy–Riemann–Fueter equation and defined a function that satisfies the following Cauchy–Riemann–Fueter equation as a regular function:

$$\frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = 0.$$

Gentili and Struppa [6, 7] showed that the Cauchy–Riemann–Fueter equation is not satisfied with general polynomial functions; thus, there is a limitation in defining regularity with the Cauchy–Riemann–Fueter equation, and they proposed local regularity by using the Cullen form of quaternions to solve this limitation. This is an attempt to treat quaternions as a complex number structure as a complex-like form proposed by Collen [3]. This is useful to show that the polynomials and power series have Cullen regularity. Defining polynomials and power series for quaternionic functions with regularity was generalized by

Laville and Ramadanoff [11, 12]. Based on their proposals, to define the regularity for the Cullen form, the unit spherical in the vector part of a quaternion is expressed as

$$\mathbb{S} := \{q = ix + jy + kz \mid x, y, z \in \mathbb{R}, \quad x^2 + y^2 + z^2 = 1\}.$$

If $I \in \mathbb{S}$, then $I^2 = -1$. For any non-real quaternion $q \in \mathbb{H} \setminus \mathbb{R}$, there exists a unique $t, r \in \mathbb{R}$ with $r > 0$ such that $q = t + Ir$, where

$$I = \frac{Ve(q)}{|Ve(q)|} \quad \text{and} \quad r = |Ve(q)|.$$

In particular, for $I \in \mathbb{S}$, it can be parametrized by spherical coordinates such that

$$I = i \cos \alpha \sin \beta + j \sin \alpha \sin \beta + k \cos \beta.$$

We examine the representation of spherical coordinates. Cartesian unit vectors are considered fixed in space and act as constants, whereas spherical coordinate unit vectors are a function of the direction (θ, ϕ) of the point under consideration. Another important point is that the unit vector always points in the direction in which its coordinates increase. The Cartesian coordinates are the simplest type of coordinate system wherein the reference axes are orthogonal to each other. Most applications such as drawing a graph or reading a map use the principles of the Cartesian coordinate system. In these situations, the exact and unique location of each data point or map reference is defined as a pair of (x, y, z) in three dimensions. However, Cartesian coordinates are difficult to use in some applications, including curves, surfaces, and spaces. Systems derived from circular shapes such as polar and spherical coordinate systems should be used. Three-dimensional relatives are used in a wide range of applications from engineering and aviation to computer animation and architecture. Points in Cartesian coordinates are defined as pairs (x, y, z) in radial coordinates in the same manner as points in Cartesian coordinates are defined as a pair of coordinates (r, θ, φ) . Polar coordinates are used in any situation where circular or spherical symmetry is presented in the form of a physical object or some type of circular or orbital vibrating motion. Physically curved forms or structures include disks, cylinders, globes, or domes. These could be anything from pressure vessels containing liquefied gases to dome structures in ancient and modern architectural masterpieces. Physicists and engineers use polar coordinates when working with the curved trajectory of a moving object and when its movement

vibrates or rotates. Examples include the orbital motion of planets and satellites, orbital motions such as shaking pendulums, or mechanical vibrations. In an electrical context, polar coordinates are used in the design of applications using alternating current. In this paper, by converting the x, y, z variables of the vector part of the quaternion into spherical coordinates, we introduce the Fueter operator, expressed in spherical coordinates. Subsequently, we define the regularity and harmonicity of the quaternionic function expressed in spherical coordinates. We propose that Cullen's integral theorem be expressed in a form suitable for spherical coordinates and prove that the integral theorem is satisfied in a state expressed in spherical coordinates.

2. RESULTS

A set $\Omega = \{q \in \mathbb{H} \mid |q| = \sqrt{t^2 + r^2} < R \text{ for } R > 0\}$ is called a domain in $\mathbb{H}$.

Definition 2.1. [7] Let Ω be a domain in $\mathbb{H}$. A function $f : \Omega \rightarrow \mathbb{H}$ is said to be Cullen-regular if, for $I \in \mathbb{S}$, its restriction f_I to the complex line $L_I = \mathbb{R} + I\mathbb{R}$ such that $f_I : \Omega \cap L_I \rightarrow \mathbb{H}$ is holomorphic.

This is equivalent to the function $f : \Omega \rightarrow \mathbb{H}$ is regular and that function f satisfies $\overline{\partial}_I f(t + Ir) = 0$ for

$$\overline{\partial}_I = \frac{\partial}{\partial t} + I \frac{\partial}{\partial r}.$$

We consider the complex-like form, the so-called Cullen form of a quaternion. For a quaternion $q = t + ix + jy + kz$, it comprises the scalar part t denoted by $Sc(q) = t$ and the vector part $ix + jy + kz$ denoted by $Ve(q) = ix + jy + kz$. We can express q as $q = t + Ir$, where

$$I = i \frac{x}{r} + j \frac{y}{r} + k \frac{z}{r} \quad \text{and} \quad r = \sqrt{x^2 + y^2 + z^2},$$

satisfying $I^2 = -1$. As the vector part of the quaternion number has a vector form in three dimensions, it can be expressed in spherical coordinates. A quaternion $q = t + ix + jy + kz$ can be parametrized by spherical coordinates $q = t + (i \cos \alpha \sin \beta + j \sin \alpha \sin \beta + k \cos \beta)r$. When dealing with the Fueter operator and complex-like quaternionic functions, the coordinate system based on the variables (t, r, α, β) can be more easily calculated and be more concise compared to the Cartesian coordinates (t, x, y, z) .

Consider the Fueter operator given by

$$D = \frac{\partial}{\partial t} + i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}.$$

The Fueter operator in this coordinate system has the form

$$D = \frac{\partial}{\partial t} + I \frac{\partial}{\partial r} - \frac{1}{r} \left(I_\alpha^{-1} \frac{\partial}{\partial \alpha} + I_\beta^{-1} \frac{\partial}{\partial \beta} \right).$$

where I_α and I_β represent the derivatives of I with respect to α and β , respectively, for $q = t + ir \cos \alpha \sin \beta + jr \sin \alpha \sin \beta + kr \cos \beta$. It was introduced by Solarz [1, 2] in 2008. When representing the quaternion in spherical coordinates, the defined function f can be expressed as $f = u + Iv$ for $I = i \cos \alpha \sin \beta + j \sin \alpha \sin \beta + k \cos \beta$, where u and v are real functions with t and r as variables. Solarz [1, 2], however, has applied the differential operator to functions u and v , including α and β as variables. In fact, as u and v are the real-valued functions of two variables t and r , respectively, the partial derivatives of u and v with respect to each x and y are zero. Thus, $\frac{\partial}{\partial \alpha}$ and $\frac{\partial}{\partial \beta}$ are sufficient to apply only to I .

We review the process of constructing spherical coordinates. For $x = r \cos \alpha \sin \beta$, $y = r \sin \alpha \sin \beta$, and $z = r \cos \beta$,

$$(2.1) \quad \frac{\partial}{\partial x} = \cos \alpha \sin \beta \frac{\partial}{\partial r} - \frac{1}{r} \frac{\sin \alpha}{\sin \beta} \frac{\partial}{\partial \alpha} + \frac{1}{r} \cos \alpha \cos \beta \frac{\partial}{\partial \beta},$$

$$(2.2) \quad \frac{\partial}{\partial y} = \sin \alpha \sin \beta \frac{\partial}{\partial r} + \frac{1}{r} \frac{\cos \alpha}{\sin \beta} \frac{\partial}{\partial \alpha} + \frac{1}{r} \sin \alpha \cos \beta \frac{\partial}{\partial \beta},$$

$$(2.3) \quad \frac{\partial}{\partial z} = \cos \beta \frac{\partial}{\partial r} - \frac{1}{r} \sin \beta \frac{\partial}{\partial \beta}.$$

We correspond to the Fueter operator by rearranging equations (2.1)-(2.3), we obtain

$$\begin{aligned} \overline{D}_s &= \frac{\partial}{\partial t} + (i \cos \alpha \sin \beta + j \sin \alpha \sin \beta + k \cos \beta) \frac{\partial}{\partial r} \\ &\quad - \frac{1}{r} \left(i \frac{\sin \alpha}{\sin \beta} - j \frac{\cos \alpha}{\sin \beta} \right) \frac{\partial}{\partial \alpha} \\ &\quad - \frac{1}{r} (-i \cos \alpha \cos \beta - j \sin \alpha \cos \beta + k \sin \beta) \frac{\partial}{\partial \beta}. \end{aligned}$$

From the expression of I , we denote I in spherical coordinates:

$$\begin{aligned} I &= i \cos \alpha \sin \beta + j \sin \alpha \sin \beta + k \cos \beta, \\ I_\alpha &:= \frac{\partial I}{\partial \alpha} = -i \sin \alpha \sin \beta + j \cos \alpha \sin \beta, \\ I_\beta &:= \frac{\partial I}{\partial \beta} = i \cos \alpha \cos \beta + j \sin \alpha \cos \beta - k \sin \beta, \\ I_\alpha^{-1} &= i \frac{\sin \alpha}{\sin \beta} - j \frac{\cos \alpha}{\sin \beta} \\ I_\beta^{-1} &= -i \cos \alpha \cos \beta - j \sin \alpha \cos \beta + k \sin \beta. \end{aligned}$$

Hence, we obtain

$$\overline{D}_s = \frac{\partial}{\partial t} + I \frac{\partial}{\partial r} - \frac{1}{r} I_\alpha^{-1} \frac{\partial}{\partial \alpha} - \frac{1}{r} I_\beta^{-1} \frac{\partial}{\partial \beta},$$

and

$$D_s = \frac{\partial}{\partial t} - I \frac{\partial}{\partial r} + \frac{1}{r} I_\alpha^{-1} \frac{\partial}{\partial \alpha} + \frac{1}{r} I_\beta^{-1} \frac{\partial}{\partial \beta}.$$

Let $\mathcal{S}$ be a set of unit pure quaternions, denoted by

$$\mathcal{S} = \{q = ix + jy + kz \mid x, y, z \in \mathbb{R}, x^2 + y^2 + z^2 = 1\}.$$

By the definition of $\mathcal{S}$, all elements of $\mathcal{S}$ have appropriate α and β , so it can be expressed in spherical coordinates. For any non-real quaternion q , there uniquely exists $t, r \in \mathbb{R}$ with $r > 0$, and $I \in \mathcal{S}$, expressed by spherical coordinates such that $q = t + Ir$. Let Λ_I be a set similar to the complex plane, and it is represented by

$$\Lambda_I = \{p = t + Ir \mid t, r \in \mathbb{R}, I = i \cos \alpha \sin \beta + j \sin \alpha \sin \beta + k \cos \beta\}.$$

Referring to [6, 7], we define the spherically left regularity of $f : \Omega \rightarrow \mathbb{H}$, denoted by for a quaternion q in spherical coordinates

$$f(q) = f(t + Ir) = u(t, r) + Iv(t, r),$$

where $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Definition 2.2. Let Ω be an open set in $\mathbb{H}$. A real differentiable function $f : \Omega \rightarrow \mathbb{H}$ is said to be spherically left-regular if, for every $I \in \mathcal{S}$ in spherical coordinates, its restriction $f_\Lambda(q) = u(t, r) + Iv(t, r)$ to the set Λ_I is holomorphic on $\Omega \cap \Lambda_I$. That is, f is said to be spherically left-regular if the following two conditions are satisfied:

- (i) u, v are continuously real differential functions,

(ii) f satisfies that $\overline{D}_s f = 0$.

Remark 2.1. In Definition 2.2, (ii) is calculated to

$$\begin{aligned}\overline{D}_s f &= \left(\frac{\partial}{\partial t} + I \frac{\partial}{\partial r} - \frac{1}{r} I_\alpha^{-1} \frac{\partial}{\partial \alpha} - \frac{1}{r} I_\beta^{-1} \frac{\partial}{\partial \beta} \right) (u(t, r) + Iv(t, r)) \\ &= \frac{\partial u}{\partial t} + I \frac{\partial u}{\partial r} + I \frac{\partial v}{\partial t} - \frac{\partial v}{\partial r} - \frac{1}{r} I_\alpha^{-1} \frac{\partial I}{\partial \alpha} v - \frac{1}{r} I_\beta^{-1} \frac{\partial I}{\partial \beta} v, \\ &= \frac{\partial u}{\partial t} - \frac{\partial v}{\partial r} - \frac{2v}{r} + I \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial t} \right),\end{aligned}$$

and is equivalent to the following equations for every q in $\Omega \cap \Lambda_I$,

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial v}{\partial r} = \frac{2v}{r}, \\ \frac{\partial u}{\partial r} = -\frac{\partial v}{\partial t}, \end{cases},$$

called the spherical Cauchy–Riemann equations over $\Omega \cap \Lambda_I$.

We consider the spherical Laplacian operator over $\Omega \cap \Lambda_I$ denoted by Δ_s ,

$$\Delta_s := D_s \overline{D}_s = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{1}{\sin^2 \beta} \frac{\partial^2}{\partial \alpha^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \beta^2} + \frac{1}{r^2} \frac{\cos \beta}{\sin \beta} \frac{\partial}{\partial \beta}.$$

Definition 2.3. Let Ω be an open set in $\mathbb{H}$. A function $f : \Omega \rightarrow \mathbb{H}$ is said to be spherically harmonic if, for every $I \in \mathcal{S}$ in spherical coordinates, its restriction $f_\Lambda(q) = u(t, r) + Iv(t, r)$ to the set Λ_I satisfies $\Delta_s f_\Lambda = 0$; that is, $f = u + Iv$ satisfies the equations

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} = 0, \\ \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial r^2} - \frac{2v}{r^2} + \frac{2}{r} \frac{\partial v}{\partial r} = 0, \end{cases}$$

over $\Omega \cap \Lambda_I$.

Remark 2.2. In fact, for a quaternion $q \in \Omega \cap \Lambda_I$ in spherical coordinates, we have

$$\overline{D}_s f = \frac{\partial u}{\partial t} - \frac{\partial v}{\partial r} - \frac{2v}{r} + I \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial t} \right).$$

It can then be calculated as follows:

$$\begin{aligned}
 0 &= D_s(\overline{D_s}f) \\
 &= \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 v}{\partial t \partial r} - \frac{2v}{r} + \frac{\partial^2 v}{\partial t \partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \left(\frac{\partial v}{\partial t} + \frac{\partial u}{\partial r} \right), \\
 &\quad + I \left(\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 u}{\partial t \partial r} - \frac{\partial^2 u}{\partial r \partial t} + \frac{\partial^2 v}{\partial r^2} - \frac{2v}{r^2} + \frac{2}{r} \frac{\partial v}{\partial r} \right), \\
 &= \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + I \left(\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial r^2} - \frac{2v}{r^2} + \frac{2}{r} \frac{\partial v}{\partial r} \right).
 \end{aligned}$$

Hence, the conditions that define a spherically harmonic function are given by the following equations:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} = 0, \\ \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial r^2} - \frac{2v}{r^2} + \frac{2}{r} \frac{\partial v}{\partial r} = 0. \end{cases}$$

As all elements of Λ_I satisfy the associative rule, the spherical Laplacian operator Δ_s can also be derived to be used in the equation defining the harmonicity of a function in $\Omega \cap \Lambda_I$ by using the operation result of $D_s \overline{D_s} (= \overline{D_s} D_s)$. In detail, we have

$$\begin{aligned}
 D_s \overline{D_s} &= \frac{\partial^2}{\partial t^2} + I \frac{\partial^2}{\partial t \partial r} - \frac{1}{r} I_\alpha^{-1} \frac{\partial^2}{\partial t \partial \alpha} - \frac{1}{r} I_\beta^{-1} \frac{\partial^2}{\partial t \partial \beta}, \\
 &\quad - I \left(\frac{\partial^2}{\partial r \partial t} + I \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} I_\alpha^{-1} \frac{\partial}{\partial \alpha} - \frac{1}{r} I_\alpha^{-1} \frac{\partial^2}{\partial r \partial \alpha} + \frac{1}{r^2} I_\beta^{-1} \frac{\partial}{\partial \beta} - \frac{1}{r} I_\beta^{-1} \frac{\partial^2}{\partial r \partial \beta} \right), \\
 &\quad + \frac{1}{r} I_\alpha^{-1} \left(\frac{\partial^2}{\partial \alpha \partial t} + I_\alpha \frac{\partial}{\partial r} + I \frac{\partial^2}{\partial \alpha \partial r} - \frac{1}{r} \frac{\partial I_\alpha^{-1}}{\partial \alpha} - \frac{\partial}{\partial \alpha} \right), \\
 &\quad - \frac{1}{r} I_\alpha^{-1} \frac{\partial^2}{\partial \alpha^2} - \frac{1}{r} \frac{\partial I_\beta^{-1}}{\partial \alpha} \frac{\partial}{\partial \beta} - \frac{1}{r} I_\beta^{-1} \frac{\partial^2}{\partial \alpha \partial \beta} \Big), \\
 &\quad + \frac{1}{r} I_\beta^{-1} \left(\frac{\partial^2}{\partial \beta \partial t} + I_\beta \frac{\partial}{\partial r} + I \frac{\partial^2}{\partial \beta \partial r} - \frac{1}{r} \frac{\partial I_\alpha^{-1}}{\partial \beta} \frac{\partial}{\partial \alpha} \right), \\
 &\quad - \frac{1}{r} I_\alpha^{-1} \frac{\partial^2}{\partial \beta \partial \alpha} - \frac{1}{r} \frac{\partial I_\beta^{-1}}{\partial \beta} \frac{\partial}{\partial \beta} - \frac{1}{r} I_\beta^{-1} \frac{\partial^2}{\partial \beta^2} \Big).
 \end{aligned}$$

That is, if we rearrange the above equations, we obtain

$$\begin{aligned} D_s \overline{D_s} &= \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r^2} \left(II_\alpha^{-1} + I_\beta^{-1} \frac{\partial I_\alpha^{-1}}{\partial \beta} + I_\alpha^{-1} \frac{\partial I_\alpha^{-1}}{\partial \alpha} \right) \frac{\partial}{\partial \alpha} \\ &\quad - \frac{1}{r^2} \left(II_\beta^{-1} + I_\alpha^{-1} \frac{\partial I_\beta^{-1}}{\partial \alpha} + I_\beta^{-1} \frac{\partial I_\beta^{-1}}{\partial \beta} \right) \frac{\partial}{\partial \beta} - \frac{1}{r^2} (I_\alpha^{-1})^2 \frac{\partial^2}{\partial \alpha^2} - \frac{1}{r^2} (I_\beta^{-1})^2 \frac{\partial^2}{\partial \beta^2} \\ &= \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\cos \beta}{\sin \beta} \frac{\partial}{\partial \beta} + \frac{1}{r^2} \frac{1}{\sin^2 \beta} \frac{\partial^2}{\partial \alpha^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \beta^2}. \end{aligned}$$

Proposition 2.1. *Let Ω be an open set in $\mathbb{H}$. If a function $f = u + Iv$ is spherically left-regular in Ω , then each component function u and v of f is harmonic in $\Omega \cap \Lambda_I$.*

Proof. Because u and v have two variables t and r , expressed by $u = u(t, r)$ and $v = v(t, r)$, the functions u and v can be calculated as follows:

$$\begin{aligned} \Delta_s u &= \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial r} \frac{\partial v}{\partial t} - \frac{2}{r} \frac{\partial v}{\partial t} \\ &= \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial r} - \frac{2v}{r} \right) = 0. \end{aligned}$$

In addition, the function v is obtained through a process similar to the case of u . □

Consider that the spherically left-regular functions satisfy the corresponding integral theorem in $\Omega \cap \Lambda_I$.

Theorem 2.1. *Let Ω be a smooth, simple closed hypersurface in the quaternionic space except the real axis, and let Ω° be the interior of Ω . Let a function $f = u + Iv$ be spherically left-regular in Ω . If*

$$\omega := d\check{t} + d\check{r} - d\check{\alpha} + d\check{\beta},$$

where each $d(\check{\cdot})$ is the $d(\cdot)$ -removed form on $dV = dt \wedge dr \wedge d\alpha \wedge d\beta$. If ωf is the quaternion product in spherical coordinates of the form ω on the function f , then for any domain $U \subset \Omega^\circ$ with a smooth simple boundary bU , satisfying

$$\int_{bU} \omega f = 0.$$

Proof. By the setting of ω , we have

$$\begin{aligned} \omega f &= u \, dr \wedge d\alpha \wedge d\beta + u \, dt \wedge d\alpha \wedge d\beta - u \, dt \wedge dr \wedge d\beta + u \, dt \wedge dr \wedge d\alpha \\ &\quad + Iv \, dr \wedge d\alpha \wedge d\beta + Iv \, dt \wedge d\alpha \wedge d\beta - Iv \, dt \wedge d\alpha \wedge d\beta + Iv \, dt \wedge dr \wedge d\alpha, \end{aligned}$$

and

$$\begin{aligned}
 D_s(\omega f) &= \frac{\partial u}{\partial t} dV + I \frac{\partial u}{\partial r} dV + I \frac{\partial v}{\partial t} dV - \frac{\partial v}{\partial r} dV, \\
 &\quad - \frac{1}{r} I_\alpha^{-1} I_\alpha v dV - \frac{1}{r} I_\beta^{-1} I_\beta v dV \\
 &= \left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial r} \right) dV + I \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial t} \right) dV - \frac{2}{r} v dV.
 \end{aligned}$$

As f is spherically left-regular in Ω , the spherical Cauchy–Riemann equations 2.1 are satisfied, and we obtain $D_s(\omega f) = 0$. Thus, we have

$$\int_{bU} \omega f = \int_U D_s(\omega f) = 0.$$

□

3. CONCLUSION

As the vector part of the quaternion represents the three-dimensional orthogonal coordinates, we examine the expression of the three-dimensional spherical coordinates. It is expressed in three-dimensional spatial coordinates to maintain the Cullen form, and the coordinate variable is expressed as an angle between the spatial coordinate axis. By expressing the Fueter differential operator in spatial coordinates, the spherical coordinates of the differential operator in the structure of the Cullen form are expressed. This makes it easy to analyze the vector part of the quaternion in space; moreover, as the sine and cosine are circulated via a differential operation, the complexity of the calculation is reduced, and the flexibility of expression increases.

Based on these properties, the regularity and harmonicity of the quaternionic function are defined, and the relation between a regular function and a harmonic function in space is defined in spherical coordinates. In addition, the integral theorem was proposed and proved by expressing the Cullen integral formula in spherical coordinates. In fact, physics and engineering fields are interested in spherical and spatial motion. Therefore, if the differential operator of the quaternionic function is defined in spherical coordinates and the regularity and harmonicity of the function and the integral formula are expressed in spherical coordinates, it can be useful in analyzing motion in space.

ACKNOWLEDGMENT

This study was supported by the Dongguk University Research Fund 2020 and the National Research Foundation of Korea (NRF) (2017R1C1B5073944).

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