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A STATE-DEPENDENT INTEGRAL EQUATION OF FRACTIONAL ORDER

A. M. A. El-Sayed[1] and H. H. G. Hashem

ABSTRACT. Here, a state-dependent (self-reference) functional integral equation of fractional order is considered. The existence of solutions will be proved.The continuous dependence of the unique solution is studied. Some applications are considered.

1. INTRODUCTION

The state-dependent equations are one of the recent kind of functional equations ([1] and [2]- [5]). Consider the state-dependent, $\phi(t) < t,$ (self-reference, $\phi(t) = t,$) fractional order ($\beta \in (0, 1]$) integral equation

$$x(t) = x_o + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s, x(x(\phi(s))) \, ds, \; t \in [0, T]. \tag{1.1}$$

The existence of solutions $x \in C[0, T]$ and the continuous dependence of the unique solution on x_o and the function f is proved. As applications we study the existence of solutions of the initial value problem of the Riemann-Liouville fractional order differential equation

$$^{R}D^{\beta} \, x(t) = f(t, x(x(\phi(t))), \; t \in (0, T] \text{ and } x(0) = 0 \tag{1.2}$$

[1]*corresponding author*

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and the mild solution of the problem of the Caputo fractional order differential equation

(1.3) $$^{C}D^{\beta}\, x(t) \;=\; f(t, x(x(\phi(t)))),\ t \in (0,T] \text{ and } x(0) \;=\; x_o.$$

Also the absolutely continuous solution $x \in AC[0,T]$ of the problem

(1.4) $$\frac{dx(t)}{dt} \;=\; f(t, x(x(\phi(t)))),\ t \in (0,T] \text{ and } x(0) \;=\; x_o.$$

2. Main Results

(1) Let $\phi : [0,T] \to [0,T]$ such that $|\phi(t) - \phi(s)| \leq |t - s|,\ \phi(0) \;=\; 0.$

(2) Let $f : [0,T] \times [0,T] \to R$ satisfies Carathèodory condition (i.e. measurable in t for all $x \in [0,T]$ and continuous in x for all $t \;\in [0,T]$). There exist a bounded and measurable function $a : [0,T] \to [0,T],\ \sup|a(t)| \leq a$ and a constant b such that $|f(t,x)| \leq a(t) \;+\; b\,|x|\ \forall\ (t,x) \in [0,T] \times [0,T]$.

Remark 2.1.

(i) *For* $x(t) \in [0,T],\ \phi(t) \in [0,T]$ *we can get* $|x(t)| \leq T,\ |x(x(t))| \leq T,\ |x(x(\phi(t)))| \leq T$ *and*

$$||x|| \;=\; \sup_{t\in[0,T]} |x(t)| \;=\; \sup_{x(t)\in[0,T]} |x(x(t))| \;=\; \sup_{x(\phi(t))\in[0,T]} |x(x(\phi(t)))|.$$

(ii) *It must be noticed that assumption (1) implies that* $\phi(t) \leq\ t,\ t \in [0,T]$.
Define the subset $S \;\subset\; C[0,T]$ *by*

$$S = \{x \in C[0,T] : |x(t_2) - x(t_1)| \leq L|t_2 - t_1|^{\beta}, t_1,\ t_2 \in [0,T]\ \},$$

$$L \;=\; \tfrac{2(a+bT)}{\beta}.$$

2.1. **Existence of solutions.**

Theorem 2.1. *Let the assumptions (1)-(3) be satisfied. Then (1.1) has a solution* $x \in C[0,T]$.

Proof. Define the operator F by

$$Fx(t) \;=\; x_o \;+\; \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\, f(s, x(x(\phi(s)))\, ds,\ t \in [0,T].$$

Let $x \in S_L,$ then from assumptions (1)-(3) we have ($\Gamma(\beta) \geq 1$)

$$
\begin{aligned}
|Fx(t)| &\leq |x_o| + \int_0^t (t-s)^{\beta-1}\,(a(s) + b|x(x(\phi(s)))|)\,ds \\
&\leq |x_o| + \int_0^t (t-s)^{\beta-1}\,(a + bT)\,ds \\
&\leq |x_o| + \frac{(a + bT)\,T^\beta}{\beta} = |x_o| + \frac{2(a + bT)\,T^\beta}{\beta} \\
&= |x_o| + L\,T^\beta.
\end{aligned}
$$

Then $F : S_L \to S_L$ and the class $\{Fx\}$ is uniformly bounded on S_L.

Now, let $t_1,\ t_2 \in [0,T],\ t_1 < t_2,\ |t_2 - t_1| \leq \delta,$ then

$$
\begin{aligned}
&|Fx(t_2) - Fx(t_1)| \\
&\leq |\int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\,f(s,x(x(\phi(s)))\,ds - \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\,f(s,x(x(\phi(s)))\,ds| \\
&= |\int_{t_1}^{t_2} (t_2-s)^{\beta-1}\,f(s,x(x(\phi(s)))\,ds + \int_0^{t_1} (t_2-s)^{\beta-1}\,f(s,x(x(\phi(s)))\,ds- \\
&\quad - \int_0^{t_1} (t_1-s)^{\beta-1}\,f(s,x(x(\phi(s)))\,ds| \\
&= |\int_{t_1}^{t_2} (t_2-s)^{\beta-1}\,f(s,x(x(\phi(s)))\,ds \\
&\quad + \int_0^{t_1} ((t_2-s)^{\beta-1} - (t_1-s)^{\beta})\,f(s,x(x(\phi(s)))\,ds| \\
&\leq (a + b\,T) \int_{t_1}^{t_2} (t_2-s)^{\beta-1}\,ds \\
&\quad + (a + b\,T) \int_0^{t_1} ((t_1-s)^{\beta-1} - (t_2-s)^{\beta-1})\,ds \\
&\leq \frac{2(a + b\,T)}{\beta}\,(t_2-t_1)^\beta = L\,(t_2-t_1)^\beta.
\end{aligned}
$$

Then $\{Fx\}$ is equicontinuous on S_L. Let $\{x_n(t)\} \in S_L,\ x_n(t) \to x_o(t),$ then

$$
\begin{aligned}
&|x_n(x_n(\phi(t))) - x_o(x_o(\phi(t)))| \\
&= |x_n(x_n(\phi(t))) - x_n(x_o(\phi(t))) + x_n(x_o(\phi(t))) - x_o(x_o(\phi(t)))| \\
&\leq L\,|x_n(\phi(t)) - x_o(\phi(t))| + |x_n(x_o(\phi(t))) - x_o(x_o(\phi(t)))| \leq L\,\epsilon_1^\beta + \epsilon_2 = \epsilon.
\end{aligned}
$$

Applying Lebesgue dominated theorem [6] we can get

$$\lim_{n\to\infty} Fx_n(t) = x_o + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s, \lim_{n\to\infty} x_n(x_n(\phi(s))) \, ds$$

$$= x_o + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s, \lim_{n\to\infty} x_o(x_o(\phi(s))) \, ds = Fx_o(t).$$

Then F is continuous and (see [6]) (1.1) has a solution $x \in S_L \subset C[0,T]$. □

2.2. **Uniqueness and continuous dependence of the solution.**

Let the function f satisfies the Lipshitz condition

(3*) $|f(t,x) - f(t,y)| \leq b\,|x-y|,\ \forall\, (t,x),(t,y) \in [0,T]\times[0,T],\ \sup|f(t,0)| \leq a.$

Assumption (3*) implies assumption (3).

Theorem 2.2. *Let the assumptions (1), (2) and (3*) be satisfied. If* $b\,(L+1)\,\frac{T^\beta}{\beta} < 1$, *then the solution of the functional integral equation (1.1) is unique.*

Proof. Let x, y be two solutions of the functional integral equation (1.1), then

$$|x(t) - y(t)|$$

$$= b \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} (|x(x(\phi(s))) - y(y(\phi(s)))|)ds$$

$$\leq b \int_0^t (t-s))^{\beta-1}\, |x(x(\phi(s))) - y(y(\phi(s))) - x(y(\phi(s))) + x(y(\phi(s)))|ds$$

$$\leq b \int_0^t (t-s))^{\beta-1}\, (|x(x(\phi(s))) - x(y(\phi(s)))| + |y(y(\phi(s))) - x(y(\phi(s)))|)ds$$

$$\leq b \int_0^t (t-s))^{\beta-1}\, (L\,|x(\phi(s))) - y(\phi(s)))|$$

$$+ \ ||x-y||)\, ds \leq \ b\,(L+1)\,\frac{T^\beta}{\beta}\,||x-y||.$$

$$\Rightarrow \ (1 - b\,(L+1)\,\frac{T^\beta}{\beta})||x-y|| \ \leq \ 0$$

from which we obtain $||x-y|| \ \leq \ 0 \ \Rightarrow \ x(t) \ = \ y(t).$ □

Definition 2.1. *The solution of the functional integral equation (1.1) depends continuously on* x_o *and on the function* f, *if* $\forall\epsilon > 0$, *there exist* $\delta_1 > 0$ *and* $\delta_2 > 0$ *such that*

$$|x_o - x_o^*| \leq \delta_1,\ |f - f^*| \leq \delta_2 \ \Rightarrow \ ||x - x^*|| \leq \epsilon,$$

$$x^*(t) \; = \; x_o^* \; + \; \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \, f^*(s, x^*(x^*(\phi(s))) \, ds, \; t \in [0,T], \tag{2.1}$$

and the function f^* satisfies the assumptions of Theorem 2.2.

Theorem 2.3. *Let the assumptions of Theorem 2.2 be satisfied for the two functions f and f^*. Then the solution of the fractional order functional integral equation (1.1) depends continuously on x_o and f.*

Proof. Let x and x^* be the two solution of (1.1) and (2.1), then

$$|x^*(t) - x(t)| \leq \; |x_o^* - x_o| \; + \; I^\beta \; |f^*(s, x^*(x^*(\phi(s))) - f(s, x(x(\phi(s)))|,$$

$$\begin{aligned}
&|f^*(s, x^*(x^*(\phi(s))) - f(s, x(x(\phi(s)))| \\
&\leq |f^*(s, x^*(x^*(\phi(s))) - f^*(s, x^*(x(\phi(s)))| + |f^*(s, x^*(x(\phi(s))) - f^*(s, x(x(\phi(s)))| \\
&\quad + |f^*(s, x(x(\phi(s))) - f(s, x(x(\phi(s)))|| \\
&\leq b \, |x^*(x^*(\phi(s)) - x^*(x(\phi(s))| \; + \; b \, |x^*(x(\phi(s)) - x(x(\phi(s))| \; + \; \delta_2 \\
&\leq bL \, |x^*(\phi(s)) - x(\phi(s))| \; + \; b \, ||x^* - x|| \; + + \, \delta_2 \; \leq\leq \; b(L \; + 1) \, ||x^* - x|| \; + \; \delta_2,
\end{aligned}$$

and

$$||x^* - x|| \; \leq \; |x_o^* - x_o| \; + \; b(L \; + 1) \; ||x^* - x|| \; \frac{T^\beta}{\beta} + \; \delta_2 \; \frac{T^\beta}{\beta}.$$

Hence

$$||x^* - x|| \; \leq \; \frac{1}{(1 - b(L+1)\frac{T^\beta}{\beta})} \; \delta_1 \; + \; \delta_2 \; \frac{T^\beta}{\beta} \; \leq \; \epsilon.$$

□

3. Applications

(I) Consider the problem (1.2) which is equivalent to the integral equation (1.1).

Corollary 3.1. *Let the assumptions of Theorem 2.1 be satisfied, then the problem (1.2) has at least on solution $x \in C[0,T]$. Moreover, if the assumptions of Theorem 2.2 is satisfied, then this solution is unique and depends continuously on the function f.*

(II) Consider now the initial value problem (1.3). It is known that the corresponding integral equation of the problem (1.3) is the functional integral equation (1.1).

Corollary 3.2. *Let the assumptions of Theorem 2.1 be satisfied, then the problem (1.3) has at least one mild solution* $x \in C[0,T]$. *Moreover, if if the assumptions of Theorem 2.2 are satisfied, then this mild solution is unique and depends continuously on* x_o *and* f.

(III) Here we relax the assumptions in [1] and prove the existence of at least one (and exactly one) absolutely continuous solution $x \in AC[0,T]$ of (2.1) Now, let $\beta \to 1$ in (1.1). Then the functional integral equation (1.1) will be the integer order one

$$x(t) \;=\; x_o \;+ \int_0^t \; f(s, x(x(\phi(s)))\; ds,\; t \in [0,T]. \tag{3.1}$$

Consider the problem (1.4) which is equivalent to the integral equation (3.1).

Corollary 3.3. *Let the assumptions of Theorem 2.1 be satisfied and* $\beta \to 1,$ *then (1.4) has at least one absolutely continuous solution* $x \in AC[0,T]$. *Moreover, if if the assumptions of Theorem 2.2 is satisfied this solution is unique and depends continuously on* x_o *and* f.

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Faculty of Science
Alexandria University
Alexandria, Egypt
Email address: amasayed@alexu.edu.eg

Faculty of Science
Qassim University
Buraidah, Saudi Arabia
Email address: 3922@qu.edu.sa