ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **10** (2021), no.2, 819–839
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.2.13

BOUNDS FOR THE ZEROS OF POLYNOMIALS BASED ON CERTAIN MATRIX INEQUALITIES

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ABSTRACT. The Frobenius companion matrix plays an important link between matrix analysis and polynomials. In this paper, we apply some matrix inequalities involving spectral norm, spectral radius, numerical radius, and partitioned matrices to types of Frobenius companion matrix of monic polynomials to derive further new upper bounds for the zeros of polynomials. Using this method, we found several main results which can be referred to in Theorems 2.1-2.6 showing that some of them are better than some known results such as Fujii and Kubo [5], Cauchy [4], Kittaneh [11], and Linden [14] by giving an example and comparing with them. In addition, we prove new numerical radius inequalities for 2×2 matrices, which are shown in Propositions 2.1-2.3. Then we applied some of these inequalities to the Frobenius matrix and got the new upper bounds of the zeros of the polynomials as shown in Theorems 2.8-2.11 then we compared it with some results in the example, and it turns out that it is better.

1. INTRODUCTION

The problem of Locating the zeros of polynomials has attracted the attention of many mathematicians, including famous ones like Cauchy and Montel. In addition to the classical complex analysis methods, matrix analysis techniques

2020 *Mathematics Subject Classification.* 26C10, 30C15, 15A60.
Key words and phrases. Frobenius companion matrix, zeros of polynomials, bounds.
Submitted: 23.12.2020; *Accepted:* 07.01.2021; *Published:* 17.02.2021.

have been used to obtain upper and lower bounds for the zeros of polynomials. In the same context, The Frobenius companion matrix plays an important link between matrix analysis and the geometry of polynomials. It has been used for the location of the zeros of polynomials by matrix methods (see, e.g., [1, 2, 4, 5, 10–14], and references therein).

Suppose that $p(z) = z^n + a_n z^{n-1} + \cdots + a_2 z + a_1$ is a complex monic polynomial with $n \geq 2$ and $a_1 \neq 0$. Let $z_1, z_2, z_3, \ldots, z_n$ be the zeros of p arranged in such a way that $|z_1| \geq |z_1| \geq \cdots \geq |z_n|$. The Frobenius companion matrix C_p of p is defined as

$$C_p = \begin{bmatrix} -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

It is well-known that the characteristic polynomial of C_p is p itself. Thus, the zeros of p are exactly the eigenvalues of C_p (see, e.g., [7, p. 316]).

Let $p_1(z) = (z - a_n)p(z) = z^{n+1} - b_n z^{n-1} - b_{n-1} z^{n-2} - \cdots - b_2 z - b_1$. Then $z_1, z_2, z_3, \ldots, z_n$ and a_n are the zeros of p_1. The corresponding Frobenius companion matrix C_{p_1} of p_1 is given by

$$C_{p_1} = \begin{bmatrix} 0 & b_n & b_{n-1} & \cdots & b_2 & b_1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

We have

$$C_{p_1}^2 = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_2 & b_1 & 0 \\ 0 & b_n & \cdots & b_3 & b_2 & b_1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix},$$

and

$$C_{p_1}^3 = \begin{bmatrix} b_{n-1} & b_n^2 + b_{n-2} & \cdots & b_n b_4 & b_n b_3 & b_n b_2 & b_n b_1 \\ b_n & b_{n-1} & \cdots & b_3 & b_2 & b_1 & 0 \\ 0 & b_n & \cdots & b_4 & b_3 & b_2 & b_1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 \end{bmatrix},$$

where $b_j = a_n a_j - a_{j-1}\ for\ j = 1, 2, \ldots, n$,with $a_0 = 0$. Let $p_2(z) = (z^2 - a_n z + a_n^2 - a_{n-1})p(z) = z^{n+2} - c_n z^{n-1} - \cdots - c_2 z - c_1$. The corresponding Frobenius companion matrix C_{p_2} of p_2 is given by

$$C_{p_2} = \begin{bmatrix} 0 & 0 & c_n & c_{n-1} & c_{n-2} & \cdots & c_2 & c_1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

We get

$$C_{p_2}^2 = \begin{bmatrix} 0 & c_n & c_{n-1} & \cdots & c_2 & c_1 & 0 \\ 0 & 0 & c_n & \cdots & c_3 & c_2 & c_1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix},$$

where $b_j = a_n a_j - a_{j-1}$ and $c_j = -a_n b_j + a_{n-1} a_j - a_{j-2}\ for\ j = 1, 2, \ldots, n$, with $a_0 = a_{-1} = 0$. Let $p_3(z) = (z^3 - a_n z^2 + (a_n^2 - a_{n-1})\, z - a_n^3 + 2a_n a_{n-1} - a_{n-2})\, p(z)$ $= z^{n+3} - d_n z^{n-1} - \cdots - d_2 z - d_1$. The corresponding Frobenius companion matrix C_{p_3} of p_3 is given by

$$C_{p_3}=\begin{bmatrix} 0 & 0 & 0 & d_n & d_{n-1} & \cdots & d_2 & d_1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

where $b_j = a_n a_j - a_{j-1}, c_j = -a_n b_j + a_{n-1}a_j - a_{j-2}$, and $d_j = -a_n c_j - a_{n-1}b_{j-1} + a_{n-2}a_j - a_{j-3}\ for\ j = 1, 2, \ldots, n$, with $a_0 = a_{-1} = a_{-2} = 0$.

In fact, it should be mentioned here that the zeros of p are contained in the zeros of p_1, p_2, and p_3. So, any upper bound for the zeros of p_1, p_2, or p_3 can be considered as an upper bound for the zeros of p.

Let $\mathbb{M}_n(\mathbb{C})$ denote the algebra of all $n\times n$ complex matrices. The eigenvalues of A are denoted by $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$, and are arranged so that $|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)|$. The singular values of A (i.e., the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$) are denoted by $s_1(A)$, $s_2(A), \ldots, s_n(A)$, and arranged so that $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. Recall that $s_j^2(A) = \lambda_j(A^*A) = \lambda_j(A\,A^*)$ for $j = 1, 2, \ldots, n$. For $A \in \mathbb{M}_n(\mathbb{C})$, let $r(A)$, $w(A)$, and $\|A\|$ denote the spectral radius, the numerical radius, and the spectral norm of A, respectively. Recall that $w(A) = \max_{\|x\|=1} |\langle Ax, x\rangle|$. Now, If z any zero of p, then

$$|z| \leq r(A) \leq w(A) \leq \|A\| = s_1(A)$$

(see, e.g., [8]).

2. Main results

In this section, we employ various matrix inequalities involving the spectral norm, the spectral radius, and the numerical radius to the companion matrices C_p, C_{p_1}, $C_{p_1}^2$, $C_{p_1}^3$ C_{p_2}, $C_{p_2}^2$, and C_{p_3} to obtain new bounds for the zeros of p.

The following lemma can be found in [7, p. 175].

Lemma 2.1. *Let* $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ *be a Hermitian matrix written in partitioned form as*

$$A = \begin{bmatrix} \tilde{A} & x \\ x^* & a_{nn} \end{bmatrix},$$

where $x \in \mathbb{C}^{n-1}$ *and* $\tilde{A} \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$\det A = a_{nn} \, \textit{det} \, \tilde{A} - x^* \left(adj \ \tilde{A} \right) x,$$

where $adj \ \tilde{A}$ *is the adjugate (classical adjoint) of* $\tilde{A}$.

The following Lemma is used in the proof of Theorem 2.1.

Lemma 2.2. *Let* $M = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$. *Then*

$$\|M\|^2 = \alpha + 1,$$

where $\alpha = \sum_{j=1}^{n-1} |a_j|^2$.

Proof. The characteristic polynomials of MM^* is determinant of the partitioned matrix

$$tI - MM^* = \begin{bmatrix} t-\alpha & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \\ \overline{a_{n-1}} & t-1 & 0 & \cdots & 0 & 0 \\ \overline{a_{n-2}} & 0 & t-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{a_2} & 0 & 0 & \cdots & t-1 & 0 \\ \overline{a_1} & 0 & 0 & \cdots & 0 & t-1 \end{bmatrix}.$$

Now using Lemma 2.1, we get

$$\det (tI - MM^*) = (t-1) \det \tilde{A}_1 - |a_{n-2}|^2 (t-1)^{n-2},$$

where

$$\tilde{A}_1 = \begin{bmatrix} t-\alpha & a_{n-1} & a_{n-2} & \cdots & a_2 \\ \overline{a_{n-1}} & t-1 & 0 & \cdots & 0 \\ \overline{a_{n-2}} & 0 & t-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{a_2} & 0 & 0 & \cdots & t-1 \end{bmatrix}.$$

Applying Lemma 2.1 to $\tilde{A}_1$ again, we get

$$\det \tilde{A}_1 = (t-1)\det \tilde{A}_2 - |a_{n-3}|^2 (t-1)^{n-3},$$

where

$$\tilde{A}_2 = \begin{bmatrix} t-\alpha & a_{n-1} & a_{n-2} & \cdots & a_3 \\ \overline{a_{n-1}} & t-1 & 0 & \cdots & 0 \\ \overline{a_{n-2}} & 0 & t-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{a_3} & 0 & 0 & \cdots & t-1 \end{bmatrix}.$$

Continue this process to get

$$\det (tI - MM^*) = (t-1)^{n-2}\left(t^2 - (\alpha+1)t\right).$$

Since $s_j^2(C_{p_1}) = \lambda_j\left(C_{p_1}C_{p_1}^*\right)$, it follows that

$$\begin{aligned} s_1(M) &= \|M\| = \sqrt{\alpha+1}, \\ s_n(M) &= 0, \end{aligned}$$

and

$$s_j(M) = 1 \text{ for } j = 2, \ldots, n-1.$$

□

Now we are in a position to derive new bound for the zeros of p.

Theorem 2.1. *Ler z be any zero of p, then*

$$|z| \le \sqrt{\sum_{j=1}^{n-1} |a_j|^2 + 1} + \sqrt{\sum_{j=1}^{n} |a_j - a_{j-1}|^2}$$

Proof. Let $A = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$ and

$$B = \begin{bmatrix} -a_n + a_{n-1} & -a_{n-1} + a_{n-2} & \cdots & -a_2 + a_1 & -a_1 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then $C_p = A + B$. So, by the triangle inequality, we have $\|C_p\| \leq \|A\| + \|B\|$. By using Lemma 2.2, we have

$$\|A\| = \sqrt{\sum_{j=1}^{n-1} |a_j|^2 + 1}$$

and by compute $\|B\| = \sqrt{\sum_{j=1}^{n} |a_j - a_{j-1}|^2}$. Consequently

$$\|C_p\| \leq \sqrt{\sum_{j=1}^{n-1} |a_j|^2 + 1} + \sqrt{\sum_{j=1}^{n} |a_j - a_{j-1}|^2},$$

which yields the desired inequality. □

Theorem 2.2. *If z is any zero of p_1, then*

$$|z| \leq \left(1 + 2\sum_{j=1}^{n} |b_j|^2\right)^{\frac{1}{4}}.$$

Proof. First write the companion matrix of p_1 as $C_{p_1}^2 = L + N + F$, where

$$L = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad N = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & b_n & \cdots & b_2 & b_1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and

$$F=\begin{bmatrix} 0 & 0 \\ I_{n-1} & 0 \end{bmatrix},$$

where I_{n-1} is the identity matrix of order $n-1$. Then note that

$$L^*N=L^*F=N^*L=N^*F=F^*L=F^*N=0.$$

Since

$$\|L^*L\|=\|N^*N\|=\sum_{j=1}^{n}|b_j|^2$$

and$\|F^*F\|=1$, it follows by the triangle inequality that

$$\begin{aligned}\left|\left|C_{p_1}^2\right|\right|^2 &= \|L^*L+N^*N+F^*F\|\le\|L^*L\|+\|N^*N\|+\|F^*F\| \\ &= 1+2\sum_{j=1}^{n}|b_j|^2,\end{aligned}$$

and so

$$\left|\left|C_{p_1}^2\right|\right|\le\left(1+2\sum_{j=1}^{n}|b_j|^2\right)^{\frac{1}{2}}.$$

Then the result follows from the fact that $|z|\le\left|\left|C_{p_1}^2\right|\right|^{\frac{1}{2}}$. □

Now, using an argument similar to that used in the proof of The following theorem 2.2 and the fact $|z|\le\left|\left|C_{p_1}^3\right|\right|^{\frac{1}{3}}$ for every zero z of p_1, we have the following related bound for the zeros of p_1.

Theorem 2.3. *If z is any zer of p_1, then*

$$|z|\le\left(1+\gamma+2\sum_{j=1}^{n}|b_j|^2\right)^{\frac{1}{6}},$$

where $\gamma=|b_{n-1}|^2+|b_n^2+b_{n-2}|^2+|b_n\,b_{n-1}+b_{n-3}|^2+|b_n|^2\left(\sum_{j=1}^{n-2}|b_j|\right)$.

The following lemma can be found in [9].

Lemma 2.3. *Let* $A\in\mathbb{M}_n(\mathbb{C})$ *be partitioned as*

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{ij} *is an* $n_i \times n_j$ *matrix* $for\ i, j = 1, 2$ *with* $n_1 + n_2 = n$. *If*

$$\tilde{A} = \left[\begin{array}{cc} \|A_{11}\| & \|A_{12}\| \\ \|A_{21}\| & \|A_{22}\| \end{array} \right],$$

then

$$r(A) \leq r\left(\bar{A}\right).$$

In the following two Theorems we use some parttitions of $C_{p_1}^2$ and $C_{p_2}^2$ to estimate $r\left(C_{p_1}^2\right)$ and $r\left(C_{p_2}^2\right)$ to derive new bounds for the zeros of p_1 and p_2, respectively.

Theorem 2.4. *If* z *is any zer of* p_1, *then*

$$|z| \leq \left(\frac{1}{2}\left[1 + \lambda + \sqrt{(1-\lambda)^2 + 4\mu}\right]\right)^{\frac{1}{2}},$$

where

$$\lambda = \left(\frac{1}{2}\left[|b_{n-1}|^2 + 2|b_n|^2 + \sqrt{|b_{n-1}|^2\left(|b_{n-1}|^2 + 4|b_n|^2\right)}\right]\right)^{\frac{1}{2}}$$

and

$$\mu = \left(\frac{1}{2}\left[\xi + \sqrt{\xi^2 - 4\left(\sum_{j=1}^{n-1}|b_j|^2\sum_{j=1}^{n-2}|b_j|^2 - \left|\sum_{j=1}^{n-1}b_j\overline{b_{j+1}}\right|^2\right)}\right]\right)^{\frac{1}{2}},$$

where $\xi = |b_{n-1}|^2 + 2\sum_{j=1}^{n-2}|b_j|^2$.

Proof. By applying Lemma 2.3 to $C_{p_1}^2$, partitioned as

$$C_{p_1}^2 = \left[\begin{array}{cc} S_{11} & S_{12} \\ S_{21} & S_{22} \end{array} \right],$$

where $S_{11} = \begin{bmatrix} b_n & b_{n-1} \\ 1 & b_n \end{bmatrix}$, $S_{12} = \begin{bmatrix} b_{n-2} & \cdots & b_1 & 0 \\ b_{n-1} & \cdots & b_2 & b_1 \end{bmatrix}$, $S_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$, and

$S_{22} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix}$, we have

$$r\left(C_{p_1}^2\right) \leq r\left(\begin{bmatrix} ||S_{11}|| & ||S_{12}|| \\ ||S_{21}|| & ||S_{22}|| \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(||S_{11}|| + ||S_{22}|| + \sqrt{\left(||S_{11}|| - ||S_{22}||\right)^2 + 4\,||S_{12}||\,||S_{21}||}\right).$$

Since $||S_{11}|| = \lambda$, $||S_{12}|| = \mu$, and $||S_{21}|| = ||S_{22}|| = 1$, it follows that

$$r\left(C_{p_1}^2\right) \leq \frac{1}{2}\left[1 + \lambda + \sqrt{(1-\lambda)^2 + 4\mu}\right].$$

Now the desired bound follows from the fact $|z| \leq r\left(C_{p_1}^2\right)^{\frac{1}{2}}$. □

The following lemma can be found in [13].

Lemma 2.4. *Let* $G = \begin{bmatrix} c_n & c_{n-1} & \cdots & c_3 & c_2 & c_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix}$, *with* $n \geq 4$. *Then*

$$||G||^2 = \frac{1}{2}\left(1 + \alpha + \sqrt{(1+\alpha)^2 - 4\left(|c_1|^2 + |c_2|^2\right)}\right),$$

where $\alpha = \sum_{j=1}^{n} |c_j|^2$.

Our first theorem is related to the result in [13].

Theorem 2.5. *If z is any zero of p_2 and $n \geq 4$, then*

$$|z| \leq \left(\frac{1}{2} \left[\tau + \sqrt{\tau^2 + 4\sqrt{\sum_{j=1}^{n} |c_j|^2}} \right] \right)^{\frac{1}{2}},$$

where

$$\tau = \left(\frac{1}{2} \left[1 + \sum_{j=1}^{n} |c_j|^2 + \sqrt{\left(1 + \sum_{j=1}^{n} |c_j|^2 \right)^2 - 4\left(|c_1|^2 + |c_2|^2 \right)} \right] \right)^{\frac{1}{2}}.$$

Proof. Let

$$\begin{aligned} C_{11} &= [0], \; C_{12} = \begin{bmatrix} c_n & c_{n-2} & \cdots & c_1 & 0 \end{bmatrix}, \\ C_{21} &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ and } C_{22} = \begin{bmatrix} 0 & c_n & \cdots & c_3 & c_2 & c_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Then

$$C_{p_2}^2 = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}.$$

Using Lemma 2.3, we get

$$r\left(C_{p_2}^2\right) \leq r\left(\begin{bmatrix} ||C_{11}|| & ||C_{12}|| \\ ||C_{21}|| & ||C_{22}|| \end{bmatrix} \right)$$

$$= \frac{1}{2}\left(||C_{11}|| + ||C_{22}|| + \sqrt{(||C_{11}|| - ||C_{22}||)^2 + 4\,||C_{12}||\,||C_{21}||} \right).$$

According to Lemma 2.4, we have that $||C_{22}|| = \tau$. By simple computations,

$$\begin{aligned} ||C_{11}|| &= 0, \\ ||C_{12}|| &= \sqrt{\sum_{j=1}^{n} |c_j|^2}, \end{aligned}$$

and

$$||C_{21}|| = 1.$$

Consequently,

$$r\left(C_{p_2}^2\right) \leq \frac{1}{2}\left(\tau + \sqrt{\tau^2 + 4\sqrt{\sum_{j=1}^{n} |c_j|^2}}\right).$$

Recalling that $|z| \leq r\left(C_{p_2}^2\right)^{\frac{1}{2}}$. □

Now, in the following Theorem we use the fact $r(A) = r(D^{-1}AD)$ for any invertiable matrix D to get generalized bound for the zeros of p_1.

Theorem 2.6. *If z is any zero of p_1 and $n \geq 4$, then*

$$|z| \leq \frac{1}{2}\left(\max\left\{\frac{r_1}{r_2}, \frac{r_2}{r_1}|b_n|\right\} + \alpha \right.$$
$$\left. + \sqrt{\left(\max\left\{\frac{r_1}{r_2}, \frac{r_2}{r_1}|b_n|\right\} - \alpha\right)^2 + 4\frac{r_2}{r_3}\sqrt{\sum_{j=1}^{n-1}\left(\frac{r_{j+2}}{r_1}\right)^2 |b_j|^2}}\right),$$

where $\alpha = \max\left\{\frac{r_3}{r_4}, \frac{r_4}{r_5}, \ldots, \frac{r_{n-1}}{r_n}\right\}$

Proof. Let $D =$ diagonal $(r_1, r_{2,\ldots,}r_n)$, where $r_1, r_{2,\ldots,}r_n$ are postive real numbers. Then

$$D^{-1}C_{p_1}D = \begin{bmatrix} 0 & \frac{r_2}{r_1}b_n & \cdots & \frac{r_{n-1}}{r_1}b_{n-2} & \frac{r_n}{r_1} \\ \frac{r_1}{r_2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{r_2}{r_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \frac{r_{n-1}}{r_n} & 0 \end{bmatrix}.$$

and

$$r\left(C_{p_1}\right) = r\left(D^{-1}C_{p_1}D\right) \leq r\left(\begin{bmatrix} \|N_{11}\| & \|N_{12}\| \\ \|N_{21}\| & \|N_{22}\| \end{bmatrix}\right),$$

where

$$N_{11} = \begin{bmatrix} 0 & \frac{r_2}{r_1}b_n \\ \frac{r_1}{r_2} & 0 \end{bmatrix}, \; N_{12} = \begin{bmatrix} \frac{r_3}{r_1}b_{n-1} & \frac{r_4}{r_1}b_{n-2} & \cdots & \frac{r_n}{r_1}b_1 \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$N_{21} = \begin{bmatrix} 0 & \frac{r_2}{r_3} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \text{ and } N_{22} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \frac{r_3}{r_4} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \frac{r_{n-1}}{r_n} & 0 \end{bmatrix}.$$

Applying Lemma 2.3, we have

$$\begin{aligned} r\left(C_{p_1}\right) &\leq r\left(\begin{bmatrix} \|N_{11}\| & \|N_{12}\| \\ \|N_{21}\| & \|N_{22}\| \end{bmatrix}\right) \\ &= \frac{1}{2}\left(\|N_{11}\| + \|N_{22}\| + \sqrt{\left(\|N_{11}\| - \|N_{22}\|\right)^2 + 4\|N_{12}\|\|N_{21}\|}\right). \end{aligned}$$

By tedious computations, one can show that $\|N_{11}\| = \max\left\{\frac{r_1}{r_2}, \frac{r_2}{r_1}|b_n|\right\}$, $\|N_{12}\| = \sqrt{\sum_{j=1}^{n-1}\left(\frac{r_{j+2}}{r_1}\right)^2 |b_j|^2}$, $\|N_{21}\| = \frac{r_2}{r_3}$, and $\|N_{22}\| = \alpha$, it follows that

$$\begin{aligned} r\left(C_{p_1}\right) \leq \frac{1}{2}&\left(\left(\max\left\{\frac{r_1}{r_2}, \frac{r_2}{r_1}|b_n|\right\} + \alpha\right)\right. \\ &\left. + \sqrt{\left(\max\left\{\frac{r_1}{r_2}, \frac{r_2}{r_1}|b_n|\right\} - \alpha\right)^2 + 4\frac{r_2}{r_3}\sqrt{\sum_{j=1}^{n-1}\left(\frac{r_{j+2}}{r_1}\right)^2 |b_j|^2}}\right). \end{aligned}$$

Now the desired bound follows from the fact that $|z| \leq r\left(C_{p_1}\right)$. □

The following two lemmas are well-known and they can be found in [15] and [16, p. 133], respectively. The first lemma gives a useful formulation of the numerical radius.

Lemma 2.5. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$w(A) = \max_{\theta \in R} \left|\left| \text{ Re } \left(e^{i\theta}A\right)\right|\right|.$$

Lemma 2.6. *Let T_n be the $n \times n$ tridiagonal matrix given by*

$$T_n = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \frac{1}{2} \\ 0 & 0 & \cdots & \frac{1}{2} & 0 \end{bmatrix}.$$

Then the eigenvalues of T_n are

$$\lambda_j = \cos\frac{j\pi}{n+1} \textit{ for } j = 1, 2, \ldots, n.$$

Proposition 2.1. *Let* $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \max\left(w\left(\operatorname{Re} A\right), w\left(\operatorname{Re} D\right)\right) + \frac{1}{2}w\left(e^{i\theta}B + e^{-i\theta}C^*\right).$$

Proof. Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Using Lemma 2.5, we have

$$\begin{aligned} \left|\left|\operatorname{Re}\left(e^{i\theta}M\right)\right|\right| &= r\left(\operatorname{Re}\left(e^{i\theta}M\right)\right) \\ &= \frac{1}{2}r\left(e^{i\theta}\begin{bmatrix} A & B \\ C & D \end{bmatrix} + e^{-i\theta}\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}\right) \\ &= \frac{1}{2}r\left(\begin{bmatrix} e^{i\theta}A + e^{-i\theta}A^* & e^{i\theta}B + e^{-i\theta}C^* \\ e^{i\theta}C + e^{-i\theta}B^* & e^{i\theta}D + e^{-i\theta}D^* \end{bmatrix}\right). \end{aligned}$$

By applying the triangle inequality of the numerical radius norm and Lemma 2.5, we get

$$\begin{aligned} w\left(M\right) &\leq \frac{1}{2}\left(w\left(\begin{bmatrix} e^{i\theta}A + e^{-i\theta}A^* & 0 \\ 0 & e^{i\theta}D + e^{-i\theta}D^* \end{bmatrix}\right)\right. \\ &\qquad \left. + w\left(\begin{bmatrix} 0 & e^{i\theta}B + e^{-i\theta}C^* \\ e^{i\theta}C + e^{-i\theta}B^* & 0 \end{bmatrix}\right)\right) \\ &= \max\left(w\left(\operatorname{Re} A\right), w\left(\operatorname{Re} D\right)\right) + \frac{1}{2}w\left(e^{i\theta}B + e^{-i\theta}C^*\right). \end{aligned}$$

The following lemma 2.7 gives a bound for the spectral radii of matrices . □

The following lemma can be found in [12].

Lemma 2.7. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$\begin{aligned} r\left(AB\right) \leq \frac{1}{4}&\left(\|AB\| + \|BA\|\right. \\ &\left. + \sqrt{\left(\|AB\| - \|BA\|\right)^2 + 4\min\left(\|A\|\,\|BAB\|, \|B\|\,\|ABA\|\right)}\right). \end{aligned}$$

Proposition 2.2. *Let* $B, C \in \mathbb{M}_n(\mathbb{C})$. *If* $M = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, *Then*

$$w\left(M\right) \leq \frac{1}{8}\left(\|BC^*\| + \|CB^*\| + \sqrt{\sigma^2 + 4\min\left(\|B\|\,\|C^*BC\|, \|C\|\,\|B^*CB\|\right)}\right),$$

where $\sigma = \|BC^*\| - \|CB^*\|$.

Proof. Let $M = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. Using Lemma 2.5, we have

$$\begin{aligned} \left\| \operatorname{Re}\left(e^{i\theta}M\right)\right\| &= r\left(\operatorname{Re}\left(e^{i\theta}M\right)\right) \\ &= \frac{1}{2}r\left(e^{i\theta}\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} + e^{-i\theta}\begin{bmatrix} 0 & C^* \\ B^* & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}r\left(\begin{bmatrix} 0 & e^{i\theta}B + e^{-i\theta}C^* \\ e^{i\theta}C + e^{-i\theta}B^* & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}r\left(\begin{bmatrix} 0 & e^{i\theta}B \\ e^{i\theta}C & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta}B^* & 0 \\ 0 & e^{-i\theta}C^* \end{bmatrix}\right). \end{aligned}$$

Using a commutativity property of the spectral radius, we have

$$\begin{aligned} \left\| \operatorname{Re}\left(e^{i\theta}M\right)\right\| &= \frac{1}{2}r\left(\begin{bmatrix} e^{-i\theta}B^* & 0 \\ 0 & e^{-i\theta}C^* \end{bmatrix}\begin{bmatrix} 0 & e^{i\theta}B \\ e^{i\theta}C & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}r\left(\begin{bmatrix} 0 & B^*B \\ C^*C & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}r\left(\begin{bmatrix} 0 & |B|^2 \\ |C|^2 & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}\sqrt{r\left(|B|^2|C|^2\right)} = \frac{1}{2}r\left(|B|\,|C|\right). \end{aligned}$$

By applying Lemma 2.7 and Lemma 2.5, we get

$$\begin{aligned} w(M) \le \frac{1}{8}&\left(\|BC^*\| + \|CB^*\|\right. \\ &\left.+\sqrt{\sigma^2 + 4\min\left(\|B\|\,\||C|\,|B|\,|C|\|, \|C\|\,\||B|\,|C|\,|B|\|\right)}\right). \end{aligned}$$

Using the fact $\||C|\,|B|\,|C|\| = \|C^*BC\|$ and $\||B|\,|C|\,|B|\| = \|B^*CB\|$, we have

$$w(M) \le \frac{1}{8}\left(\|BC^*\| + \|CB^*\| + \sqrt{\sigma^2 + 4\min\left(\|B\|\,\|C^*BC\|, \|C\|\,\|B^*CB\|\right)}\right).$$

□

Now, we are in a position to derive a new bound for the zeros of p_3.

Theorem 2.7. *If z is any zero of p_3, then*

$$|z| \le \cos\frac{\pi}{n+3} + \cos\frac{\pi}{n+3} + \frac{1}{2}w(\beta),$$

where $\beta = \left[\begin{array}{cccccccc} e^{-i\theta} & 0 & 0 & e^{i\theta}d_n & e^{i\theta}d_{n-1} & \cdots & e^{i\theta}d_2 & e^{i\theta}d_1 \end{array} \right]$.

Proof. Partition C_{p_3} as

$$C_{p_3} = \left[\begin{array}{cc} L_{11} & L_{12} \\ L_{21} & L_{22} \end{array} \right],$$

where

$$\begin{aligned} L_{11} &= [0]\,,\ L_{12} = \left[\begin{array}{cccccc} 0 & 0 & 0 & d_n & \cdots & d_1 \end{array} \right], \\ L_{21} &= \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} \right], \text{ and } L_{22} = \left[\begin{array}{ccccc} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array} \right]. \end{aligned}$$

Since $w\left(\operatorname{Re} L_{11}\right) = 0$, and since by Lemma 2.6, we have $w\left(\operatorname{Re} L_{22}\right) = \cos\frac{\pi}{n+3}$. Using Proposition 2.1, we get

$$w\left(C_{p_3}\right) \leq \cos\frac{\pi}{n+3} + \frac{1}{2}w\left(e^{i\theta}L_{12} + e^{-i\theta}L_{21}\right),$$

where $e^{i\theta}L_{12} + e^{-i\theta}L_{21} = \beta$.

Now the desired bound follows from the fact that $|z| \leq w\left(C_{p_3}\right)$. □

The following lemma can be found in [3].

Lemma 2.8. *Let* $T = \left[\begin{array}{cc} 0 & X \\ Y & 0 \end{array} \right]$ *with* $X \in \mathbb{M}_{k\times m}(\mathbb{C})$ *and* $Y \in \mathbb{M}_{m\times k}(\mathbb{C})$. *Then*

$$w^4(T) \leq \frac{1}{16}\|P\|^2 + \frac{1}{4}w^2(XY) + \frac{1}{8}w(XYP + PXY),$$

where $P = |X^*|^2 + |Y|^2$.

We use a numerical radius inequality in the following Proposition to establish a new bound for the zeros of p_3.

Proposition 2.3. *Let* $B \in \mathbb{M}_{k\times m}(\mathbb{C})$, $C \in \mathbb{M}_{m\times k}(\mathbb{C})$, *and* $D \in \mathbb{M}_m(\mathbb{C})$, *and let* $M = \left[\begin{array}{cc} 0 & B \\ C & D \end{array} \right]$. *Then*

$$w\left(\left[\begin{array}{cc} 0 & B \\ C & D \end{array} \right]\right) \leq w(D) + \left(\frac{1}{16}\|N\|^2 + \frac{1}{4}w^2(BC) + \frac{1}{8}w(BCN + NBC)\right)^{\frac{1}{4}},$$

where $N = BB^* + C^*C$.

Proof. Let $U = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$ and $V = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. Then $M = U + V$. Using the triangle inequality and Lemma 2.8, we get

$$\begin{aligned} w(M) &= w(U+V) \\ &\leq w(U) + w(V) \\ &\leq w(D) + \left(\frac{1}{16}\|N\|^2 + \frac{1}{4}w^2(BC) + \frac{1}{8}w(BCN+NBC)\right)^{\frac{1}{4}}, \end{aligned}$$

as required. □

The following lemma can be found in [6, pp. 8-9].

Lemma 2.9. *Let L_n be the $n \times n$ matrix given by*

$$L_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Then

$$w(L_n) = \cos\frac{\pi}{n+1}.$$

Theorem 2.8. *If z is any zero of p, then*

$$|z| \leq |a_n| + \cos\frac{\pi}{n} + \frac{1}{\sqrt{2}}\left(\frac{1}{4}(\psi+1)^2 + |a_{n-1}|(|a_{n-1}| + \psi + 1)\right)^{\frac{1}{4}},$$

where $\psi = \sum_{j=1}^{n-1} |a_j|^2$.

Proof. Let

$$\begin{aligned} M_{11} &= [-a_n], \; M_{12} = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}, \\ M_{21} &= \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ and } M_{22} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \end{aligned}$$

Then

$$C_p = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

So, by the triangle inequality, we have

$$w(C_p) \leq w\left(\begin{bmatrix} M_{11} & 0 \\ 0 & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\right).$$

By computation, $w\left(\begin{bmatrix} M_{11} & 0 \\ 0 & 0 \end{bmatrix}\right) = w(M_{11}) = |a_n|$ and by using Proposition 2.3, we get

$$w\left(\begin{bmatrix} 0 & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\right) \leq w(M_{22})$$
$$+ \left(\frac{1}{16}\|N\|^2 + \frac{1}{4}w^2(M_{12}M_{21}) + \frac{1}{8}w(M_{12}M_{21}N + NM_{12}M_{21})\right)^{\frac{1}{4}},$$

where $N = M_{12}M_{12}^* + M_{21}^*M_{21}$. Using Lemma 2.9, we have $w(M_{22}) = \cos\frac{\pi}{n}$. But, by computations, $w(M_{12}M_{21}) = |a_{n-1}|$, $w(M_{12}M_{21}N + NM_{12}M_{21}) = 2|a_{n-1}|$ $(\psi + 1)$, and $\|N\| = (\psi + 1)$. Consequently,

$$w(C_p) \leq |a_n| + \cos\frac{\pi}{n} + \frac{1}{\sqrt{2}}\left(\frac{1}{4}(\psi + 1)^2 + |a_{n-1}|(|a_{n-1}| + \psi + 1)\right)^{\frac{1}{4}}.$$

Recalling that $|z| \leq w(C_p)$, the result follows. □

Theorem 2.9. *If z is any zero of p_1, then*

$$|z| \leq \cos\frac{\pi}{n+1} + \frac{1}{\sqrt{2}}\left(\frac{1}{4}\xi^2 + |b_n|\xi + |b_n|^2\right)^{\frac{1}{4}},$$

where $\xi = 1 + \sum_{j=1}^{n}|b_j|^2$.

Proof. Partition C_{p_1} as

$$C_{p_1} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix},$$

where

$$D_{11} = [0], \; D_{12} = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_2 & b_1 \end{bmatrix},$$

$$D_{21} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ and } D_{22} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Then By using Proposition 2.3, we get

$$w(C_{p_1}) \leq w(D_{22}) + \left(\frac{1}{16} \|A\|^2 + \frac{1}{4} w^2 (D_{12} D_{21}) + \frac{1}{8} w (D_{12} D_{21} A + A D_{12} D_{21}) \right)^{\frac{1}{4}},$$

where $A = D_{12} D_{12}^* + D_{21}^* D_{21}$. Using Lemma 2.9, we have $w(D_{22}) = \cos \frac{\pi}{n+1}$. But, by using simple computations, we have $w(D_{12} D_{21}) = |b_n|, w(D_{12} D_{21} A + A D_{12} D_{21}) = 2|b_n| \xi$, and $\|A\| = \xi$. Consequently,

$$w(C_{p_3}) \leq \cos \frac{\pi}{n+1} + \frac{1}{\sqrt{2}} \left(\frac{1}{4} \xi^2 + |b_n| \xi + |b_n|^2 \right)^{\frac{1}{4}}.$$

Recalling that $|z| \leq w(C_{p_1})$. □

The following two theorems, however, can be proved similarly as in Theorem 2.9.

Theorem 2.10. *If z is any zero of p_2, then*

$$|z| \leq \cos \frac{\pi}{n+2} + \frac{1}{2} \left(1 + \sum_{j=1}^{n} |c_j|^2 \right)^{\frac{1}{2}}.$$

Theorem 2.11. *If z is any zero of p_3, then*

$$|z| \leq \cos \frac{\pi}{n+3} + \frac{1}{2} \left(1 + \sum_{j=1}^{n} |d_j|^2 \right)^{\frac{1}{2}}.$$

Example 1. *Consider the polynomial* $p(z) = z^3 + \frac{1}{2}z^2 + z + 1$. *Then the upper bounds for the zeros of this polynomial* $p(z)$ *estimated by different mathematicians are as shown in the following table*

Bound	***Value***
Fujii and Kubo [5]	1.7071
Cauchy [4]	2
Kittaneh [11]	1.9652
Linden [14]	1.8333

But if z *is a zero of the polynomial* $p(z) = z^3 + \frac{1}{2}z^2 + z + 1$, *then Theorem 2.2 gives* $|z| \leq 1.3296$*, Theorem 2.9 gives* $|z| \leq 1.6508$*, Theorem 2.10 gives* $|z| \leq 1.6951$, *and Theorem 2.11 gives* $|z| \leq 1.6621$ *which are better than all the estimates mentioned above.*

Finally, we remark that lower bound counterparts of the upper bounds obtained in this paper can be derived by considering the polynomial $\frac{z^n}{a_1} p\left(\frac{1}{z}\right)$ whose zeros are the reciprocals of those of p. This enables us to describe annuli in the complex plane containing all the zeros of p. Moreover, for $k < n$, compression matrix inequalities may be applied to C_p^k in order to obtain further bounds for the zeros of p.

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