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ESTIMATION OF EQUITABLE TOTAL COLORING OF SPLITTING ON DOUBLE WHEEL AND SUNLET GRAPHS

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ABSTRACT. A graph G is total colored if different colors are assigned to its elements, in the order of neighbouring vertices and edges are alloted with least diverse k-colors. If each of k-colors can be partitioned into color sets and differs by atmost one, then it becomes equitable. The minimum of k-colors required is known as equitably total chromatic number and symbolized by $\chi''_{=}(G)$. Further the splitting graph is formed by including a new vertex v' which is linked to every vertex that is adjoining to v in G. In this paper, $\chi''_{=}[S(DW_n)]$ and $\chi''_{=}[S(G_n)]$ are obtained by proper allocation of colors.

1. INTRODUCTION

In graph theory, the approach of various types of coloring in graphs have been introduced and studied during different periods of time. The idea of equity in total or complete coloring on splitting of graphs is a recent approach.

A graph is represented with $G(V,E)$ and its maximum degree by Δ. In total coloring all possible adjacent or incident vertices and edges must be assigned different colors for a graph G. Therefore it requires atleast $\Delta+1$ colors for any graph G. The well known conjecture[TCC] [1] of graphs states that $\Delta(G)+1 \leq$

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$\chi''(G) \leq \Delta(G) + 2$. Also graphs having $\chi''(G) = \Delta + 1$ and $\chi''(G) = \Delta + 2$ comes under type 1 and type 2 graphs respectively. The graphs considered in this paper are of type 1.

Meyer [2] introduced equitable coloring in 1973 and speculated that equitable chromatic number for a connected graph G to be atmost $\Delta(G)$. Hung-lin Fu first presented equitability in total coloring in 1994 and conjectured that $\chi''_{=}(G) \leq \Delta + 2$.

In 1980, Sampathkumar and Walikar [3] introduced splitting graph concept which is framed by attaching a new node v' to each of the node v and has the same adjacency pattern of v in G. Recently, Veninstine Vivik et.al [7] investigated the equitable total chromatic number of some families of Helm graphs. We construct the splitting graph for double wheel and sunlet graphs.

In real time situations many problems in networks, assignment and scheduling problems, etc can be optimally solved using total coloring with equity. In this paper, the proper allocation of colors to all the vertex and edges are made equitable for the splitting graph of DW_n and S_n.

2. Preliminaries

Definition 2.1. *If the elements (vertices and edges) in graph G could be sub-divided into r independent domains $T_1, T_2, \ldots, T_r$ and satisfies $||T_i| - |T_j|| \leq 1, \forall\, i \neq j$, then G is completely r- total colorable and becomes equitable. The minimum of such r is known as the equitable total chromatic number (ETCN), also represented by $\chi''_{=}(G)$.*

Definition 2.2. *[3] For graphs with node $v \in V(G)$, add a new node v'. Connect v' with all points of G neighbouring to v. The resulting graph $S(G)$ is termed as Splitting graph of G.*

Definition 2.3. *[6] For any integer $n \geq 4$, the wheel graph W_n consists of n vertices is obtained by joining the center vertex v with every other $n-1$ vertices $\{v_1, v_2, \ldots, v_n\}$ of the cycle graph C_{n-1}.*

Definition 2.4. *[5] A double-wheel graph DW_n of size n consists of $2C_n + K_1$, which contains two cycles of size n, where all the vertices of two cycles are attached to a common hub.*

Definition 2.5. *[6] The sunlet graph composed of $2n$ vertices is acquired by adding n pendant edges to the cycle C_n and is represented by S_n.*

Conjecture 2.1. *[4] [ETCC] For any graph G, the ETCC declare that $\chi''_{=}(G) \leq \Delta(G) + 2$.*

In the following section, the ETCN of Splitting of DW_n and S_n are determined.

3. Equitability in complete coloring for splitting of Double Wheel and Sunlet graphs

Theorem 3.1. *For $\kappa \geq 6$, the ETCN of splitting of Double Wheel graph is $\chi''_{=}(S(DW_\kappa)) = 4\kappa - 3$.*

Proof. The Double Wheel graph DW_κ consists of $2\kappa - 1$ vertices and $4(\kappa - 1)$ edges. Let $V(DW_\kappa) = \{v\} \bigcup \{v_h : 1 \leq h \leq \kappa - 1\} \bigcup \{u_h : 1 \leq h \leq \kappa - 1\}$ and $E(DW_\kappa) = \{p_h^{(1)} : 1 \leq h \leq \kappa - 1\} \bigcup \{p_h^{(2)} : 1 \leq h \leq \kappa - 2\} \bigcup \{p_{\kappa-1}^{(2)}\} \bigcup \{q_h^{(1)} : 1 \leq h \leq \kappa - 1\} \bigcup \{q_h^{(2)} : 1 \leq h \leq \kappa - 2\} \bigcup \{q_{\kappa-1}^{(2)}\}$, where $p_h^{(1)}$ is the edge between $vv_h \, (1 \leq h \leq \kappa - 1)$, $p_h^{(2)}$ is the edge $v_h v_{h+1} \, (1 \leq h \leq \kappa - 2)$, $p_{\kappa-1}^{(2)}$ is the edge $v_{\kappa-1}v_1$. Similarly $q_h^{(1)}$ is the edge between $vu_h \, (1 \leq h \leq \kappa - 1)$, $q_h^{(2)}$ is the edge $u_h u_{h+1} \, (1 \leq h \leq \kappa - 2)$, $q_{\kappa-1}^{(2)}$ is the edge $u_{\kappa-1}u_1$.

By the splitting graph of double wheel graph it extends to a new graph, with $4\kappa - 2$ vertices and $12(\kappa - 1)$ edges. Let $V(S(DW_\kappa)) = \{v\} \bigcup \{v'\} \bigcup \{v_h : 1 \leq h \leq \kappa - 1\} \bigcup \{v'_h : 1 \leq h \leq \kappa - 1\} \bigcup \{u_h : 1 \leq h \leq \kappa - 1\} \bigcup \{u'_h : 1 \leq h \leq \kappa - 1\}$.

The edges $E(S(DW_\kappa))$ are classfied in Table 1:

Table 1: Classification of edges of $S(DW_\kappa)$

Edges	Ranges of h	links between the vertices
$p_h^{(1)}$	$1 \leq h \leq \kappa - 1$	vv_h
$p_h^{(2)}$	$1 \leq h \leq \kappa - 2, h = \kappa - 1$	$v_h v_{h+1}$,$v_{\kappa-1}v_1$
$p_h^{(3)}$	$1 \leq h \leq \kappa - 1$	vv'_h
$p_h^{(4)}$	$1 \leq h \leq \kappa - 1$	$v'v_h$
$p_h^{(5)}$	$1 \leq h \leq \kappa - 2$	$v_h v'_{h+1}$
$p_h^{(6)}$	$2 \leq h \leq \kappa - 1$	$v_h v'_{h-1}$
$p_h^{(7)}$	$h = 1, h = 2$	$v_1 v'_{\kappa-1}, v_{\kappa-1}v'_1$

$q_h^{(1)}$	$1 \leq h \leq \kappa - 1$	vu_h
$q_h^{(2)}$	$1 \leq h \leq \kappa - 2, h = \kappa - 1$	$u_h u_{h+1}, u_{\kappa-1} u_1$
$q_h^{(3)}$	$1 \leq h \leq \kappa - 1$	vu'_h
$q_h^{(4)}$	$1 \leq h \leq \kappa - 1$	$v' u_h$
$q_h^{(5)}$	$1 \leq h \leq \kappa - 2$	$u_h u'_{h+1}$
$q_h^{(6)}$	$2 \leq h \leq \kappa - 1$	$u_h u'_{h-1}$
$q_h^{(7)}$	$h = 1, h = 2$	$u_1 u'_{\kappa-1}, u_{\kappa-1} u'_1$

The total coloring of $(S(DW_\kappa))$ is defined as $\phi : T\,(S(DW_\kappa)) \to C$, where $T = V[(S(DW_\kappa))] \cup V[(S(DW_\kappa))]$ and $C = \{1, 2, \ldots, 4\kappa - 3\}$. While coloring, the value $\mathrm{mod}\kappa = 0$ is replaced with κ. The coloring of edges are as follows:

$$\phi\left(p_h^{(1)}\right) = h{+}1,\ 1 \leq h \leq \kappa{-}1, \qquad \phi\left(p_h^{(2)}\right) = \begin{cases} h + 3(\mathrm{mod}\kappa),\ 1 \leq h \leq \kappa - 2, \\ 3, h = \kappa - 1, \end{cases}$$

$$\phi\left(p_h^{(3)}\right) = h + \kappa,\ 1 \leq h \leq \kappa - 1, \qquad \phi\left(p_h^{(4)}\right) = h + \kappa,\ 1 \leq h \leq \kappa - 1,$$

$$\phi\left(p_h^{(5)}\right) = \begin{cases} \kappa + 3,\ h = 1, \\ \kappa,\ h = 2, \\ h - 1,\ 3 \leq h \leq \kappa - 2, \end{cases} \qquad \phi\left(p_h^{(6)}\right) = \begin{cases} h + \kappa + 2,\ 2 \leq h \leq \kappa - 3, \\ \kappa + 1,\ h = \kappa - 2, \\ \kappa + 2,\ h = \kappa - 1, \end{cases}$$

$$\phi\left(p_h^{(7)}\right) = \begin{cases} \kappa - 1,\ h = 1, \\ 2, h = 2, \end{cases} \qquad \phi\left(q_h^{(1)}\right) = h + 2\kappa - 1,\ 1 \leq h \leq \kappa - 1,$$

$$\phi\left(q_h^{(2)}\right) = \begin{cases} h + 2\kappa + 1,\ 1eqh \leq \kappa - 3, \\ 1, h = \kappa - 2, \\ 2\kappa + 1, h = \kappa - 1, \end{cases} \qquad \phi\left(q_h^{(3)}\right) = h + 3\kappa - 2,\ 1 \leq h \leq \kappa - 1,$$

$$\phi\left(q_h^{(4)}\right) = h{+}3\kappa{-}2,\ 1 \leq h \leq \kappa{-}1, \quad \phi\left(q_h^{(5)}\right) = \begin{cases} 3\kappa + 1,\ h = 1, \\ 3\kappa - 2,\ h = 2, \\ h + 2\kappa - 3,\ 3 \leq h \leq \kappa - 2, \end{cases}$$

$$\phi\left(q_h^{(6)}\right) = \begin{cases} h + 3\kappa,\ 2 \leq h \leq \kappa - 3, \\ 3\kappa,\ h = \kappa - 2, \\ 3\kappa - 1,\ h = \kappa - 1, \end{cases} \qquad \phi\left(q_h^{(7)}\right) = \begin{cases} 3\kappa - 1,\ h = 1 \\ 3\kappa, h = 2. \end{cases} .$$

The coloring of vertices are formulated as follows:

$$\phi(v) = 1, \phi(v_1) = \kappa, \phi(v_h) = h, \qquad 2 \le h \le \kappa - 1$$
$$\phi(v') = 1, \phi(v'_1) = 2\kappa - 1, \phi(v'_h) = h + \kappa - 1,\ 2 \le h \le \kappa - 1$$
$$\phi(u_1) = 3\kappa - 2, \phi(u_h) = h + 2\kappa - 2,\ 2 \le h \le \kappa - 1$$
$$\phi(u'_1) = 4\kappa - 3, \phi(u'_h) = h + 3\kappa - 3,\ 2 \le h \le \kappa - 1.$$

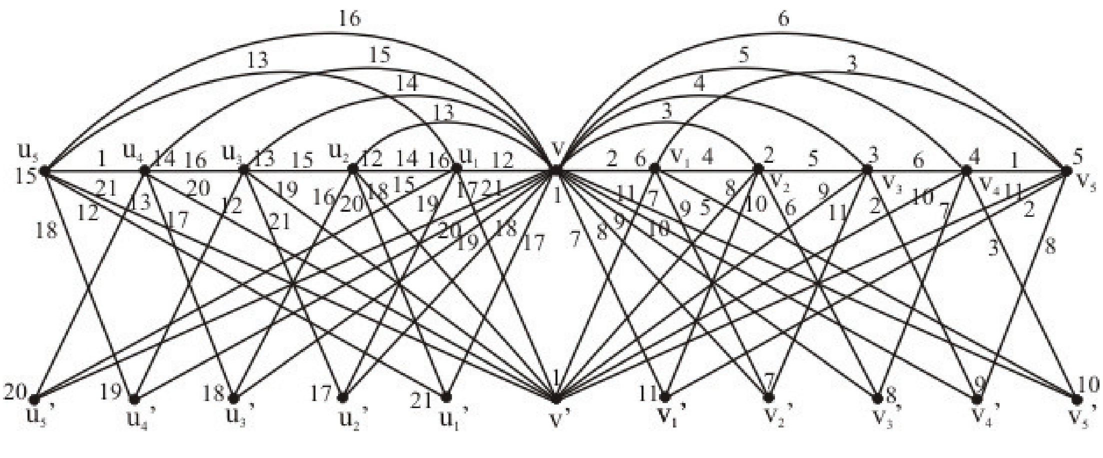

FIGURE 1. Splitting of $S(DW_6)$.

All the vertices and edges of this splitting graph is assigned colors by the above process with $4\kappa - 3$ different colors. The color classes of $S(DW_\kappa)$ are classified as $T\left(S(DW_\kappa)\right) = \{T_1, T_2, \ldots, T_{4\kappa-3}\}$, clearly these sets are independent and assures the inequality $||T_i| - |T_j|| \le 1$, for any $i \ne j$. For example, consider the case $\kappa = 6$ (see Figure 1) it is evident that it is equitably total colored with $4\kappa - 3$ colors and is true for all other values of $\kappa \ge 6$. Hence $\chi''_{=}\left(S(DW_\kappa)\right) = 4\kappa - 3$. □

Theorem 3.2. *For $\xi \ge 6$, the ETCN of splitting of Sunlet graph is $\chi''_{=}\left[S(G_\xi)\right] = 7$.*

Proof. The Sunlet graph S_ξ contains 2ξ vertices and 2ξ edges. Let $V\left(S_\xi\right) = \{v_g\} \bigcup \{u_g\}, 1 \le g \le \xi$ and $E\left(S_\xi\right) = \{x_g : 1 \le g \le \xi - 1\} \bigcup \{x_\xi\} \bigcup \{x'_g : 1 \le g \le \xi\}$, where x_ξ is the edge $v_g v_{g+1}\,(1 \le g \le \xi - 1)$, x_ξ is the edge $v_\xi v_1$ and x'_g is the edge $v_g u_g\,(1 \le g \le \xi)$.

The construction of splitting sunlet graph consists of 4ξ vertices and 5ξ edges.

Let $V\left(S(S_\xi)\right) = \{v_g\} \bigcup \{v'_g\} \bigcup \{u_g\} \bigcup \{u'_g\}, 1 \le g \le \xi$ and the edges $E\left(S(S_\xi)\right)$ are grouped in Table 2:

TABLE 2. Classification of edges of $S(S_\xi)$

Edges	Ranges of g	links between the vertices
x_g	$1 \leq g \leq \xi - 1$,$g = \xi$	$v_g v_{g+1}$,$v_\xi v_1$
x'_g	$1 \leq g \leq \xi$	$v_g u_g$
y_g	$1 \leq g \leq \xi - 1$,$g = \xi$	$v_g v'_{g+1}$,$v_\xi v'_1$
y'_g	$1 \leq g \leq \xi - 1$,$g = \xi$	$v_{g+1} v'_g$,$v_1 v'_\xi$
y''_g	$1 \leq g \leq \xi$	$v_g u'_g$

The total coloring of splitting graph of Sunlet graph is defined as $\phi : T \to C$, where $T = V[(S(S_\xi))] \cup V[(S(S_\xi))]$ and color set $C = \{1, 2, \ldots, 7\}$. The coloring of this graph is splitted into seven cases $\xi \equiv \gamma(\mathrm{mod}7)$, where $0 \leq \gamma \leq 6$. In the process of coloring, suppose the value $\mathrm{mod}7 = 0$ then it should be considered as 7. The vertices and edges are colored by the following process:

$\phi\left(v_g\right) = g + 2(\mathrm{mod}7),\ 1 \leq g \leq \xi,$ for $\gamma = 0, 1, 2, 4, 5, 6$

For $\gamma = 3$,

$$\phi\left(v_g\right) = \begin{cases} 6,\ g = 1 \\ g + 2(\mathrm{mod}7),\ 2 \leq g \leq \xi \end{cases}.$$

$\phi\left(v'_g\right) = g(\mathrm{mod}7),\ 1 \leq g \leq \xi,$ for $0 \leq \gamma \leq 6$

$\phi\left(u_g\right) = g(\mathrm{mod}7),\ 1 \leq g \leq \xi,$ for $0 \leq \gamma \leq 4\ \&\ \gamma = 6$

For $\gamma = 5$,

$$\phi\left(u_g\right) = \begin{cases} g(\mathrm{mod}7),\ 1 \leq g \leq \xi - 1 \\ 6,\ g = \xi \end{cases}$$

$\phi\left(u'_g\right) = g + 1(\mathrm{mod}7),\ 1 \leq g \leq \xi,$ for $\gamma = 0, 1, 6$

For the cases

$$\gamma = 2, \phi\left(u'_g\right) = \begin{cases} 6,\ g = 1 \\ g + 1(\mathrm{mod}7),\ 2 \leq g \leq \xi \end{cases}$$

$$\gamma = 3, \phi\left(u'_g\right) = \begin{cases} g + 4, g = 1, 2 \\ g + 1(\mathrm{mod}7),\ 3 \leq g \leq \xi \end{cases}$$

$$\gamma = 4, \phi\left(u'_g\right) = \begin{cases} g + 5, g = 1, 2 \\ g + 1(\mathrm{mod}7),\ 3 \leq g \leq \xi \end{cases}$$

$$\gamma = 5, \phi\left(u'_g\right) = \begin{cases} 7, \ g = 1 \\ g + 1(\mathrm{mod} 7), \ 2 \leq g \leq \xi \end{cases}$$

$$\phi\left(x_g\right) = g(\mathrm{mod} 7), \ 1 \leq g \leq \xi - 1, \text{for } 0 \leq \gamma \leq 6$$

$$\phi\left(x_\xi\right) = \begin{cases} 7, \gamma = 0 \\ 6, \gamma = 1, 3, 5, 6 \\ 2, \gamma = 2 \\ 4, \gamma = 4 \end{cases}$$

$$\phi\left(x'_g\right) = g + 1(\mathrm{mod} 7), \ 1 \leq g \leq \xi, \text{for } \gamma = 0, 1, 3, 4, 6$$

For $\gamma = 2$,

$$\phi\left(x'_g\right) = \begin{cases} 6, g = 1 \\ g + 1(\mathrm{mod} 7), 2 \leq g \leq \xi \end{cases}$$

and for $\gamma = 5$,

$$\phi\left(x'_g\right) = \begin{cases} g + 1(\mathrm{mod} 7), \ 1 \leq g \leq \xi - 1 \\ 5, g = \xi \end{cases}$$

$$\phi\left(y_g\right) = g + 4(\mathrm{mod} 7), \ 1 \leq g \leq \xi - 1, \text{for } 0 \leq \gamma \leq 6$$

$$\phi\left(y_\xi\right) = \begin{cases} 4, \gamma = 0 \\ 5, \gamma = 1 \\ 6, \gamma = 2, 3 \\ 1, \gamma = 4 \\ 2, \gamma = 5 \\ 3, \gamma = 6, \end{cases} \qquad \phi\left(y'_g\right) = g + 6(\mathrm{mod} 7), \ 1 \leq g \leq \xi - 1, \text{for } 0 \leq \gamma \leq 6$$

$$\phi\left(y'_\xi\right) = \begin{cases} 6, \gamma = 0 \\ 7, \ 1 \leq \gamma \leq 6, \end{cases} \qquad \phi\left(y''_g\right) = g + 3(\mathrm{mod} 7), \ 1 \leq g \leq \xi, \text{for } 0 \leq \gamma \leq 6$$

For $\gamma = 3$,

$$\phi\left(y''_g\right) = \begin{cases} g + 3(\mathrm{mod} 7), \ 1 \leq g \leq \xi - 1 \\ 7, \ g = \xi \end{cases}$$

By this method of coloring all the elements of splitting the sunlet graph is assigned with 7 colors. The color classes are independent sets such as $T\left(S(S_\xi)\right) = \{T_1, T_2, \ldots, T_7\}$ and observed that $||T_i| - |T_j|| \leq 1$ for $i \neq j$, satisfying equitablility condition. For example, consider $\xi = 7$ (see Figure 2) it is inferred that it

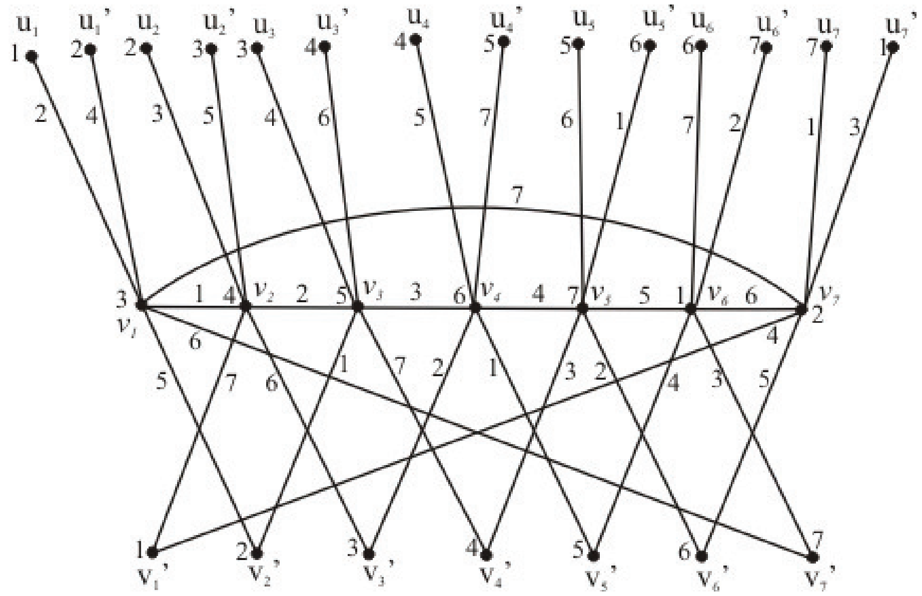

FIGURE 2. Splitting of $S(S_7)$.

requires 7 colors to be equitably total colored and is true for $\xi \geq 6$. Therefore $\chi''_{=}(S(S_\xi)) = 7$. □

4. CONCLUSION

The equitably total chromatic number for splitting of Double Wheel DW_n and Sunlet graph S_n are obtained and proved in this paper. The proof enables the optimal allotment of colors to all the rudiments of such structural networks or graphs in an equitable way. This kind of coloring in the field of splitting graphs is a recent approach and it can be extended to other families of graphs.

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