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$\delta-$IDEALS IN PSEUDO-COMPLEMENTED ALMOST DISTRIBUTIVE FUZZY LATTICES

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ABSTRACT. The concept of $\delta-$ideals is introduced in a Pseudo-Complemented Almost Distributive Fuzzy Lattice (PCADFL) and some important properties of these ideals are derived. PCADFL are characterized in terms of $\delta-$ideals. In addition, prime ideals also verified in PCADFL. Finally, some properties of $\delta-$ideals are studied with respect to fuzzy lattice homomorphism.

1. INTRODUCTION

The theory of pseudo-complementation was introduced and extensively studied in semi-lattices and particularly in distributive lattices by O. Frink [3] and G. Birkhoff [1]. SG. Karpagavalli and A. Nasreen Sultana [7] introduced Pseudo-Complementation on Almost Distributive Fuzzy Lattices (PCADFL) and proved that it is equationally definable on ADFL by using properties of pseudo-complem -entation on almost distributive lattice using the fuzzy partial order relation and fuzzy lattice defined by I. Chon [8]. In [6], N. Rafi, Ravi Kumar Bandaru and S. N. Rao introduced $\delta-$ideals in Pseudo-complemented Almost Distributive Lattices and some important properties are derived. In this paper, the concept of

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$\delta-$ideals is introduced in a Pseudo-Complemented Almost Distributive Fuzzy Lattices (PCADFL) in terms of pseudo-complementation and filters. We derive a set of equivalent conditions for the class of all $\delta-$ideals to become a fuzzy lattice of all ideals, which leads to a characterization of PCADFL.

2. Preliminaries

In this section, we recall certain basic definitions and results required.

Definition 2.1. *[4] Let L be an ADFL and I be any non empty subset of R. Then I is said to be an ideal of an ADFL L, if it satisfies the following axioms:*

(1) $a, b \in I$ implies that $a \vee b \in I$,
(2) $a \in I, b \in R$ implies that $a \wedge b \in I$.

Definition 2.2. *[5] A prime ideal of L is called a minimal prime ideal if it is a minimal element in the set of all prime ideals L ordered by set inclusion.*

Theorem 2.1. *[5] Let L be an ADL. Then a prime ideal P is minimal if and only if for any $x \in P$, there exist an element $y \notin P$ such that $x \wedge y = 0$.*

Definition 2.3. *[6] Let L be a pseudo-complemented ADL. Then for any filter F of L, define the set $\delta(F) = \{x \in L | x^* \in F\}$.*

Definition 2.4. *[2] An element x of a pseudo-complemented lattice L is called dense if $x^* = 0$ and the set $D(L)$ of all dense element of L forms a filter of L.*

3. $\delta-$ Ideals in Pseudo-Complemented Almost Distributive Fuzzy Lattices

In this section, the concept of $\delta-$ideals is extended in Pseudo-Complemented Almost Distributive Fuzzy Lattice (PCADFL). Throughout this paper (R, A) stands for a PCADFL $(R, \vee, \wedge, ^*, 0, 1)$.

Definition 3.1. *Let (R, A) be a PCADFL, then for any filter $\mathcal{F}$ of R, the set $\delta(\mathcal{F})$ is defined as follows: $A(\delta(\mathcal{F}), a) > 0,$ for $a \in R, a^* \in \mathcal{F}$.*

Theorem 3.1. *Let (R, A) be a PCADFL with maximal elements. Then for any filter $\mathcal{F}$ of R, $\delta(\mathcal{F})$ is an ideal of R.*

Proof. Since $0^* \in \mathcal{F}$, we get that $0 \in \delta(\mathcal{F})$. Let $a, b \in \delta(\mathcal{F})$. Then $a^*, b^* \in \mathcal{F}$, which implies $a^* \wedge b^* \in \mathcal{F}$. Since $\mathcal{F}$ is a filter of R. Therefore $a^* \wedge b^* = (a \vee b)^*$ where $(a \vee b)^* \in \mathcal{F}$. Hence $A(a^* \wedge b^*, (a \vee b)^*) > 0$. Now, let $a \in \delta(\mathcal{F})$ and $r \in R$. Then $a^* \in \mathcal{F}$, that implies $a^* \vee r^* \in \mathcal{F}$, where $a^* \vee r^* = (a \wedge r)^*$ such that $a^* = a^{***}$ where $a = a \wedge r$ then

$$\begin{aligned} A(a^*, a^{***}) &= A((a \wedge r)^*, (a \wedge r)^{***}) \\ &= A((a \wedge r)^*, (a^* \vee r^*)^{**}) \\ &= A((a \wedge r)^*, (a^{**} \wedge r^{**})^*) \\ &= A((a \wedge r)^*, (a^{***} \vee r^{***})) \\ &= A((a \wedge r)^*, (a^* \vee r^*)) \\ &= A((a \wedge r)^*, (a \wedge r)^*) \\ &= 1 > 0. \end{aligned}$$

Hence $(a \wedge r)^* \in \mathcal{F}$. So that $(a \wedge r) \in \delta(\mathcal{F})$. Therefore $\delta(\mathcal{F})$ is an ideal of R. □

Definition 3.2. *Let (R, A) be a PCADFL. An ideal I of (R, A) is called a $\delta-$ideal of PCADFL if $A(I, \delta(\mathcal{F})) > 0$, for some filter $\mathcal{F}$ of R.*

Example 1. *Let $R = \{0, x, y, z, 1\}$. Define a fuzzy relation $A : R \times R \to [0, 1]$ and $a^* = 0$ if $a \neq 0$ and $0^* = x$. Clearly (R, A) is a fuzzy poset. Then $(R, \vee, \wedge, 0)$ is an ADFL with 0 and $a \to a^*$ is a PCADFL on (R, A) whose Hasse diagram is given below.*

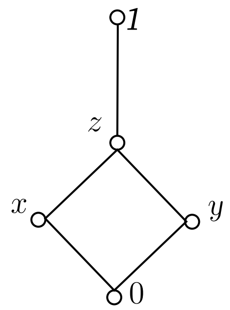

FIGURE 1. Hasse diagram of PCADFL $R = \{0, x, y, z, 1\}$

Now, let us consider $I = \{0, x\}$ and $\mathcal{F} = \{y, z, 1\}$. Clearly I is an ideal of R and $\mathcal{F}$ is a filter of R. By definition 3.2. which satisfies $A(I, \delta(\mathcal{F})) > 0$, for some filter $\mathcal{F}$ of R. Which implies $\delta(\mathcal{F}) = a|a^* \in \mathcal{F}$. Hence $\delta(\mathcal{F}) = \{0, x\}$ such that $\delta(\mathcal{F}) = I$. Therefore I is a $\delta-$ideal of R.

Theorem 3.2. *Let (R, A) be a PCADFL, then for each $a \in R, (a^*]$ is a $\delta-$ideal of R if and only if $A(\delta([a)), (a^*]) > 0$.*

Proof. Let $x \in (a^*]$. Then $x \wedge a = 0$ and $x^* \wedge a = a$, where $a \in [a)$. So that

$$A(x \wedge a, 0) = A(x \wedge x^* \wedge a, 0) = A(0 \wedge a, 0) = A(0, 0) = 1 > 0.$$

Thus, $x^* \in [a)$ which implies $x \in \delta([a))$, such that, $(a^*] \subseteq \delta([a))$. Conversely, if $x \in \delta([a))$ and $x^* \in [a)$ then $x^* \wedge a = a$. Thus $x \wedge a^* = x$. Therefore $x \in (a^*]$. Such that $\delta([a)) \subseteq (a^*]$ and hence $(a^*] \in \delta([a))$. Therefore $A((a^*], \delta([a)) > 0$. Since $\delta([a)) \leq (a^*]$. We have $A(\delta([a)), (a^*]) > 0$. Therefore $(a^*] = \delta([a))$ by anti-symmetry of A. Hence $A(\delta([a)), (a^*]) > 0$. Therefore $(a^*]$ is a $\delta-$ideal of R. □

Lemma 3.1. *Let (R, A) be a PCADFL. Every prime ideal without dense element is a $\delta-$ideal if and only if $A(P, \delta(R - P)) > 0$.*

Proof. Let $a \in P$ where P is a prime ideal of R without dense element and $a \wedge a^* = 0 \in P$. Then clearly, $A(a \wedge a^*, 0) = A(a^*, 0)$ since $a^* = 0$ and $A(0, 0) = 1 > 0$. If $a^* = 0$ then clearly it is a dense element of R and said to be $a \vee a^*$ which is not in P. Hence $a \vee a^* \notin P$, that implies $a^* \notin P$. Therefore $a^* \in (R - P)$. Thus $a \in \delta(R - P)$. Since $P \subseteq \delta(R - P)$ implies that $A(P, \delta(R - P)) > 0$. Conversely, suppose that $a \in \delta(R - P)$. Then $a^* \in (R - P)$ which implies that $a^* \notin P$. Therefore $\delta(R - P) \subseteq P$ such that $A(\delta(R - P), P) > 0$. Thus $P = \delta(R - P)$ by antisymmetry property of A. Hence $A(P, \delta(R - P)) > 0$. Therefore P is a $\delta-$ideal. □

Lemma 3.2. *Let (R, A) be a PCADFL. Every minimal prime ideal of R is a $\delta-$ideal if and only if $A(P \cap D(R), \phi) > 0$.*

Proof. Let (R, A) be a PCADFL and P be a minimal prime ideal of R. If $a \in P \cap D(R)$. Then $a \in P$ and $a \in D(R)$ only if $a^* = 0$. A Prime ideal P is minimal if and only if to each $a \in P$ there exists $b \notin P$ such that $a \wedge b = 0$ and $a^* \wedge b = b$. Thus $A(a^* \wedge b, b) = A(0 \wedge b, b) = A(0, b)$ suppose that $b = 0$, $A(0, 0) = 1 > 0$. Therefore, if $b = 0 \in P$, which is a contradiction. Thus $P \cap D(R) = \phi$ by antisymmetry property of A. Hence $A(P \cap D(R), \phi) > 0$. Therefore P is a $\delta-$ideal. □

Lemma 3.3. *Let (R, A) be a PCADFL. A proper $\delta-$ideal contains no dense element if and only if $A(\delta(\mathcal{F}) \cap D(R), \phi) > 0$.*

Proof. Let I be a proper $\delta-$ideal of PCADFL then $A(I, \delta(\mathcal{F})) > 0$, for some filter $\mathcal{F}$ of R. Suppose that $a \in \delta(\mathcal{F}) \cap D(R)$. If $a \in D(R)$, we have $a^* = 0 \in \mathcal{F}$ which is a contradiction. Therefore $\delta(\mathcal{F}) \cap D(R) = \phi$. Hence $A(\delta(\mathcal{F}) \cap D(R), \phi) > 0$. □

Let us denote the set of all $\delta-$ideals of R by $\mathcal{J}^\delta(R)$. We can observe clearly from example (1) that $\mathcal{J}^\delta(R)$ is not a sublattice of $\mathcal{J}(R)$ of all ideals of R. Consider $\mathcal{F} = \{y, z, 1\}$ and $\mathcal{G} = \{x, z, 1\}$. Clearly $\mathcal{F}$ and $\mathcal{G}$ are filters of R. Now $\delta(\mathcal{F}) = \{0, x\}$ and $\delta(\mathcal{G}) = \{0, y\}$. But $\delta(\mathcal{F}) \vee \delta(\mathcal{G}) = \{0, x, y, z\}$ is not a $\delta-$ideal of R, because $z \in \delta(\mathcal{F}) \vee \delta(\mathcal{G})$ is a dense element. In the following theorem we prove that $\mathcal{J}^\delta(R)$ forms a complete almost distributive fuzzy lattice.

Theorem 3.3. *Let (R, A) be a PCADFL. Then the set $\mathcal{J}^\delta(R)$ forms a complete almost distributive fuzzy lattice.*

Proof. Suppose that (R, A) be a PCADFL and $\mathcal{F}$ and $\mathcal{G}$ be any two filters of R, define two binary operations $\sqcap$ and $\sqcup$ as follows:

$$\delta(\mathcal{F}) \sqcap \delta(\mathcal{G}) = \delta(\mathcal{F} \sqcap \mathcal{G}) \text{ and } \delta(\mathcal{F}) \sqcup \delta(\mathcal{G}) = \delta(\mathcal{F} \vee \mathcal{G}).$$

It is clear that $\delta(\mathcal{F} \sqcap \mathcal{G})$ is the infimum of $\delta(\mathcal{F})$ and $\delta(\mathcal{G})$ in $\mathcal{J}^\delta(R)$. Also $\delta(\mathcal{F}) \sqcup \delta(\mathcal{G})$ is a $\delta-$ideal of R. Suppose that $\delta(\mathcal{F}), \delta(\mathcal{G}) \subseteq \delta(\mathcal{F} \vee \mathcal{G}) = \delta(\mathcal{F}) \sqcup \delta(\mathcal{G})$. Hence $A(\delta(\mathcal{F}) \sqcap \delta(\mathcal{G}), \delta(\mathcal{F} \sqcap \mathcal{G})) > 0$ and $A(\delta(\mathcal{F}) \sqcup \delta(\mathcal{G}), \delta(\mathcal{F} \vee \mathcal{G})) > 0$. Let $\delta(\mathcal{H})$ be a $\delta-$ideal of R such that $\delta(\mathcal{F}) \subseteq \delta(\mathcal{H})$ implies that $A(\delta(\mathcal{F}), \delta(\mathcal{H})) > 0$ and $\delta(\mathcal{G}) \subseteq \delta(\mathcal{H})$ which implies that $A(\delta(\mathcal{G}), \delta(\mathcal{H})) > 0$ by antisymmetry property of A, where $\mathcal{H}$ is a filter of R. Now we claim that $\delta(\mathcal{F} \vee \mathcal{G}) \subseteq \delta(\mathcal{H})$. Thus $A(\delta(\mathcal{F} \vee \mathcal{G}), \delta(\mathcal{H})) > 0$. Let $a \in \delta(\mathcal{F} \vee \mathcal{G})$. Then $a^* \in \mathcal{F} \vee \mathcal{G}$. Hence $a^* = f \wedge g$ for some $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Since $f \in \mathcal{F}$ and $g \in \mathcal{G}$, we get that $f^* \in \delta(\mathcal{F}) \subseteq \delta(\mathcal{H})$ and $g^* \in \delta(\mathcal{G}) \subseteq \delta(\mathcal{H})$. Then, for $f^* \in \delta(\mathcal{H})$ and $g^* \in \delta(\mathcal{H})$ which implies $f^* \vee g^* \in \delta(\mathcal{H})$.

$$\begin{aligned}
A(f^* \vee g^*, a) = A((f \wedge g)^*, a) &= A((f \wedge g)^{***}, a) \text{ since } (a^* = a^{***}) \\
&= A((f^* \vee g^*)^{**}, a) = A((f^{**} \wedge g^{**})^*, a) \\
&= A((a^*)^*, a) = A(a^{**}, a) \text{ since } (a^{**} = a) \\
&= A(a, a) = 1 > 0.
\end{aligned}$$

Since $a^* = f \wedge g$ which implies $f = f^{**}$ and $g = g^{**}$ where $(a^*)^* = (f^{**} \wedge g^{**})^*$. Since $f^* \vee g^* \in \delta(\mathcal{H})$ which implies $(f^{**} \wedge g^{**})^* \in \delta(\mathcal{H})$. Therefore $a^{**} \in \delta(\mathcal{H})$. Hence $a \in \delta(\mathcal{H})$. Thus $\delta(\mathcal{F}) \sqcup \delta(\mathcal{G}) = \delta(\mathcal{F} \vee \mathcal{G})$ is the supremum of both $\delta(\mathcal{F})$ and $\delta(\mathcal{G})$ in $\mathcal{J}^\delta(R)$. Therefore $A(\delta(\mathcal{F}) \sqcup \delta(\mathcal{G}), \delta(\mathcal{F} \vee \mathcal{G})) > 0$. Hence $(\mathcal{J}^\delta(R), \sqcap, \sqcup)$ is a fuzzy lattice. Clearly, $\mathcal{J}^{\delta}(R)$ is a fuzzy partially ordered set with respect to set inclusion. Then by the extension of the property, we can obtain that $\mathcal{J}^{\delta}(R)$ is a complete fuzzy lattice. Therefore $\mathcal{J}^{\delta}(R)$ is a complete almost distributive fuzzy lattice. □

Theorem 3.4. *Let (R, A) be a PCADFL. $\mathcal{B}^*(R)$ is a fuzzy sublattice of the lattice $\mathcal{J}^{\delta}(R)$ of all $\delta-$ideals of R and hence is a Boolean fuzzy algebra. Moreover, the mapping $a \to (a^*]$ is a dual homomorphism from R onto $\mathcal{B}^*(R)$.*

Proof. Suppose that (R,A) be a PCADFL. Let $(a^*], (b^*] \in \mathcal{B}^*(R)$ for any $a, b \in R$. Then $(a^*] \sqcap (b^*] \in \mathcal{B}^*(R)$.

$$\begin{aligned} A((a^*] \sqcup (b^*], ((a \wedge b)^*]) &= A(\delta([a)) \sqcup \delta([b)), ((a \wedge b)^*]) \\ &= A(\delta([a) \vee [b)), ((a \wedge b)^*]) \\ &= A(\delta([a \wedge b)), ((a \wedge b)^*]) \\ &= A((a \wedge b)^*], ((a \wedge b)^*]) \\ &= 1 > 0. \end{aligned}$$

Hence $((a \wedge b)^*] \in \mathcal{B}^*(R)$. Therefore $\mathcal{B}^*(R)$ is a fuzzy sublattice of $\mathcal{J}^{\delta}(R)$ and it is an almost distributive fuzzy lattice. Such that $(0^{**}]$ and $(0^*]$ are the least and greatest elements of $\mathcal{B}^*(R)$. Now for any $a \in R, A((a^*] \sqcap (a^{**}], (0]) > 0$ and similarly

$$\begin{aligned} A((a^*] \sqcup (a^{**}], \delta(R)) &= A(\delta([a)) \sqcup \delta([a^*)), \delta(R)) \\ &= A(\delta([a) \sqcup [a^*)), \delta(R)) \\ &= A(\delta([a) \vee [a^*)), \delta(R)) \\ &= A(\delta([a \wedge a^*)), \delta(R)) \\ &= A(\delta([0)), \delta(R)) \\ &= A(\delta(R), \delta(R)) \\ &= 1 > 0. \end{aligned}$$

Since $\delta(R) = R$. Hence $(a^{**}]$ is the complement of $(a^*]$ in $\mathcal{B}^*(R)$. Therefore $(\mathcal{B}^*(R), \sqcap, \sqcup, 0)$ is a bounded almost distributive fuzzy lattice in which every element is complemented. The remaining part can be proved easily. □

Lemma 3.4. *Let (R, A) be a PCADFL. Every proper $\delta-$ideal is contained in a minimal prime ideal.*

Proof. Suppose that (R, A) be a PCADFL. Let I be a proper $\delta-$ideal of R. Then $A(I, \delta(\mathcal{F})) > 0$, for some filter $\mathcal{F}$ of R. Clearly $\delta(\mathcal{F}) \cap D(R) = \phi$ which implies that $A(\delta(\mathcal{F}) \cap D(R), \phi) > 0$. Then there exists a prime ideal P of R such that $\delta(\mathcal{F}) \subseteq P$ and $P \cap D(R) = \phi$. Let $a \in P$ and $a \in P \cap D(R)$ then we always have $a \wedge a^* = 0$, suppose that $a^* \in P$. A Prime ideal P is minimal if and only if to each $a \in P$ there exists $a^* \notin P$ such that $a \wedge a^* = 0$. Thus which is a contradiction. Therefore P is a minimal prime ideal of R. □

Corollary 3.1. *Let (R, A) be a PCADFL. The minimal prime ideals of a PCADFL are maximal elements of the complete fuzzy lattice $\mathcal{J}^\delta(R)$. Clearly, it was observed that $\mathcal{J}^\delta(R)$ is not a fuzzy sublattice of the ideal fuzzy lattice $\mathcal{J}(R)$. Consequently, we prove some equivalent conditions for $\mathcal{J}^\delta(R)$ to become a fuzzy sublattice of $\mathcal{J}(R)$, which leads to a characterization of PCADFL as follows.*

Theorem 3.5. *Let (R, A) be a PCADFL with maximal elements. Then the following are equivalent:*

(1) (R, A) is a PCADFL
(2) For any $a, b \in R$, $A(a^* \vee b^*, (a \wedge b)^*) > 0$
(3) For any two filters $\mathcal{F}, \mathcal{G}$ of R, $A(\delta(\mathcal{F} \vee \mathcal{G}), \delta(\mathcal{F}) \vee \delta(\mathcal{G})) > 0$
(4) $\mathcal{J}^\delta(R)$ is a fuzzy sublattice of $\mathcal{J}(R)$.

Proof.
$(1) \implies (2)$: Assume that (R, A) is a PCADFL. Let $a, b \in R$.

$$\begin{aligned} A((a \wedge b)^*, a^* \vee b^*) &= A((a \wedge b)^{***}, a^* \vee b^*) \\ &= A((a^* \vee b^*)^{**}, a^* \vee b^*) \\ &= A((a^{**} \wedge b^{**})^*, a^* \vee b^*) \\ &= A((a^{***} \vee b^{***}), a^* \vee b^*) \\ &= A(a^* \vee b^*, a^* \vee b^*) \\ &= 1 > 0. \end{aligned}$$

Therefore $A((a \wedge b)^*, a^* \vee b^*) > 0$.

$(2) \implies (3)$: Assume the condition (2). Let $\mathcal{F}$ and $\mathcal{G}$ are the two filters of R. We always have $\delta(\mathcal{F}) \vee \delta(\mathcal{G}) \subseteq \delta(\mathcal{F} \vee \mathcal{G})$. Conversely, let $a \in \delta(\mathcal{F} \vee \mathcal{G})$. Then $a^* \in \mathcal{F} \vee \mathcal{G}$ which implies $a^* = f \wedge g$ for any $f \in \mathcal{F}$ and $g \in \mathcal{G}$. There exists $(a^*)^* = (f \wedge g)^*$, such that $a^{**} = f^* \vee g^*$. Therefore $f^* \vee g^* \in \delta(\mathcal{F}) \vee \delta(\mathcal{G})$. Thus $a \in \delta(\mathcal{F}) \vee \delta(\mathcal{G})$. Hence $\delta(\mathcal{F} \vee \mathcal{G}) \subseteq \delta(\mathcal{F}) \vee \delta(\mathcal{G})$. Therefore $A(\delta(\mathcal{F} \vee \mathcal{G}), \delta(\mathcal{F}) \vee \delta(\mathcal{G})) > 0$.

Since $(\delta(\mathcal{F}) \vee \delta(\mathcal{G}) \leq \delta(\mathcal{F} \vee \mathcal{G}))$. Such that $A(\delta(\mathcal{F}) \vee \delta(\mathcal{G}), \delta(\mathcal{F} \vee \mathcal{G})) > 0$ which implies $\delta(\mathcal{F}) \vee \delta(\mathcal{G}) = \delta(\mathcal{F} \vee \mathcal{G})$ by antisymmetry property of A. Therefore $A(\delta(\mathcal{F} \vee \mathcal{G}), \delta(\mathcal{F}) \vee \delta(\mathcal{G})) > 0$.

$(3) \implies (4)$: It is obvious.

$(4) \implies (1)$: Assume that $\mathcal{J}^{\delta}(R)$ is a fuzzy sublattice of $\mathcal{J}(R)$. By theorem 3.4, for any $a, b \in R$, let $(a^*], (b^*] \in \mathcal{B}^*(R)$. Such that $(0^{**}]$ and $(0^*]$ are the least and greatest elements of $\mathcal{B}^*(R)$. Now for any $a \in R, A((a^*] \sqcap (a^{**}], (0]) > 0$. Hence $((a \wedge b)^*] \in \mathcal{B}^*(R)$. Therefore $\mathcal{B}^*(R)$ is a fuzzy sublattice of $\mathcal{J}^{\delta}(R)$ and it is an almost distributive fuzzy lattice. Since $\delta(R) = R$. Hence $(a^{**}]$ is the complement of $(a^*]$ in $\mathcal{B}^*(R)$. Therefore $(\mathcal{B}^*(R), \sqcap, \sqcup, 0)$ is a bounded almost distributive fuzzy lattice in which every element is complemented. Clearly, it satisfies the conditions of PCADFL. Therefore (R, A) is a PCADFL. Hence proved. □

If f is a homomorphism of a PCADFL R with 0 into another PCADFL R' with $0'$. Such that $Kerf = \{a \in R | f(a) = 0\} = \{0\}$ and f is onto, then f is need not be an isomorphism which is observed by an example as follows.

Example 2. *Suppose that $R = \{0, x, y\}$ and $R' = \{0', z\}$ be two chains of PCADFLs. A mapping is defined as $f : R \to R'$ by $f(0) = 0', f(x) = z$ and $f(y) = z$. Therefore f is a fuzzy lattice homomorphism from $R \to R'$ and also f is onto. Hence $Kerf = \{0\}$. Since f is not one-one. Therefore f is not an isomorphism.*

Theorem 3.6. *Let R and R' be two PCADFLs with pseudo-complementation * and $f : R \to R'$ an onto homomorphism. If $Ker\ f = \{0\}$, then prove that $A(f(a^*), \{f(a)\}^*) > 0$ for all $a \in R$.*

Proof. Let R and R' be two PCADFLs with pseudo-complementation * and $a \in R$. We have $A(f(a) \wedge f(a^*), 0) = A(f(a \wedge a^*), 0) = A(f(0), 0)) = A(0, 0) = 1 > 0$. Suppose $f(a) \wedge f(s) = 0$ for some $s \in R$. Such that $A(f(a) \wedge f(s), 0) = A(f(a \wedge s), 0) = A(f(0), 0) = A(0, 0) = 1 > 0$. Hence $a \wedge s \in Kerf = \{0\}$, since $a \wedge s = 0$ and $a^* \wedge s = s$. So that $A(f(a^*) \wedge f(s), f(s)) = A(f(a^* \wedge s), f(s)) = A(f(s), f(s)) > 1 = 0$. Clearly, $f(a^*)$ is the pseudo-complement of $f(a)$ in R'. Which implies $A(f(a^*), \{f(a)\}^*) = 1$. Similarly, $A(\{f(a)\}^*, f(a^*)) = 1$. Therefore $A(f(a^*), \{f(a)\}^*) = A(\{f(a)\}^*, f(a^*)) = 1$. So, we get $A(f(a^*), \{f(a)\}^*) > 0$ for all $a \in R$. □

In the following theorem, we prove that the image of a $\delta-$ideal of R under the above fuzzy lattice homomorphism is again a $\delta-$ideal.

Theorem 3.7. *Let R and R' be two PCADFLs with pseudo-complementation * and $f : R \to R'$ an onto fuzzy lattice homomorphism such that $Ker\ f = \{0\}$, then prove that if I is a $\delta-$ideal of R, then $f(I)$ is a $\delta-$ideal of R'.*

Proof. Let I be a $\delta-$ideal of R. Then $A(I, \delta(\mathcal{G})) > 0$ for some filter $\mathcal{G}$ of R. Clearly $f(\mathcal{G})$ also a filter in R'. To show that $A(\delta\{f(\mathcal{G})\}, f\{\delta(\mathcal{G})\}) > 0$. Since $x \in f\{\delta(\mathcal{G})\}$. Then $x = f(a)$ for any $a \in \delta_f(\mathcal{G})$. Therefore $a^* \in \mathcal{G}$. Now, $A(f(a) \wedge f(a^*), 0) = A(f(a \wedge a^*), 0) = A(f(0), 0) = A(0, 0) = 1 > 0$. Therefore $A(f(a) \wedge f(a^*), 0) = A(f(a^*) \wedge f(a), 0) = 1$. Since $f(a^*) \in f(\mathcal{G})$ and $\{f(a)\}^* \in f(\mathcal{G})$ which implies $\{f(a)\}^* \wedge f(a^*) = f(a^*)$. Hence $x = f(a) \in \delta\{f(\mathcal{G})\}$. Thus $f\{\delta(\mathcal{G})\} \subseteq \delta\{f(\mathcal{G})\}$. Conversely, let $y \in \delta\{f(\mathcal{G})\}$. Since f is onto, let $a \in R$ such that $y = f(a)$. Then $\{f(a)\}^* \in f(\mathcal{G})$. Therefore $\{f(a)\}^* = f(x)$ for any $x \in \mathcal{G}$. So that,

$$\begin{aligned} A(f(a) \wedge \{f(a)\}^*, 0) &= A(f(a) \wedge f(x), 0) \\ &= A(f(a \wedge x), 0) \\ &= A(f(0), 0) \\ &= 1 > 0. \end{aligned}$$

Since $a \wedge x = 0$, such that $a \wedge x \in Kerf = \{0\}$, there exists $a^* \wedge x = x$ where $x \in \mathcal{G}$. Similarly, $a^* \in \mathcal{G}$ then $a \in \delta(\mathcal{G})$. Therefore $y = f(a) \in f\{\delta(\mathcal{G})\}$. Thus $\delta\{f(\mathcal{G})\} \subseteq f\{\delta(\mathcal{G})\}$ and hence which implies $A(\delta\{f(\mathcal{G})\}, f\{\delta(\mathcal{G})\}) > 0$. Since $f\{\delta(\mathcal{G})\} \leq \delta\{f(\mathcal{G})\}$, we get $A(f\{\delta(\mathcal{G})\}, \delta\{f(\mathcal{G})\} > 0$. So, we have $f\{\delta(\mathcal{G})\} = \delta\{f(\mathcal{G})\}$ by antisymmetry property of A. Therefore $A(\delta\{f(\mathcal{G})\}, f\{\delta(\mathcal{G})\}) > 0$. Hence proved. □

4. Conclusion

In this paper, some properties of $\delta-$ideals are studied and then proved that the set of all $\delta-$ideals of a PCADFL forms a complete almost distributive fuzzy lattice. Necessary and sufficient conditions for a pseudo-complemented ADL to become PCADFL were investigated. In future, we can develop $\delta-$ideals of PCADFL into $\delta-$filters of PCADFL and also which can be characterized in terms of filter congruences.

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