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EXACT SOLUTION FOR SAWADA–KOTERA EQUATION USING BACKLAND TRANSFORMATIONS AND TRAVELLING WAVE SOLUTIONS

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ABSTRACT. Nonlinear phenomena are very important in a variety of scientific fields. Finding solutions of nonlinear partial differential equations is one of the most difficult problems in mathematics and physics.

A precise solution is derived for these non-linear evolution equations (NLEEs), which has been illustrated by the solitary wave solutions for the fifth degree equations, the Sawada-Kotera equation, and then we apply the Backland transformation, which shows a new special precise solution.

1. INTRODUCTION

In the work of Buckland [1, 2], Z. Yan [10] and M. Gharib [5], surface transformations are illustrated, then P. Bracken [3] and D. Demskoi [4] develop the theory of this subject.

This type of study has been applied to non-linear partial differential equations. Then, an infinite series of solutions is generated using algebraic and mathematical structures. While most of the equations with non-linear development are

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solved using inverse scattering, such as the KDV and mKDV equations which represent the pseudo-spherical surface (pss).

In the work of Khater and others [6–8], the function $u\ (x,\ t)$ representing pss has been derived from it; some advanced nonlinear equations have been derived under special conditions using certain methods in mathematics to arrive at a new solution for the nonlinear equations, which appears in the research, Ablowitz, Kaup, Newell, Sequr (SKNS) system [5, 9]; this is very important for physics in various fields, engineering and applied mathematics.

2. Definition of the Open Problem

Nonlinear evolution equations (NLEEs) arise throughout Nonlinear phenomena. NLEEs are very useful to describe various nonlinear phenomena of physics, chemistry, biology and others. A NLEE is represented by a nonlinear partial differential equation (NPDE), and it is known that finding the exact solution of NPDE is not easy but the difficulty increases when we deal with NPDEs of high orders.

Recently, NLEEs attract the attention of many mathematicians and physicists, because of it's several applications. Great efforts have been paid to develop method for solving NLEEs. The implementation of the geometrical properties and Backlund transformations (BTs) is one of the modern and useful method in this area.

We discuss in this study the following certain NLEEs.

The **Sawada–Kotera equation**. It is defined as fifth order and the form of:

$$u_t = u_{5x} + 5uu_{3x} + 5u_x u_{xx} + 5u^2 u_x \tag{2.1}$$

In this work, we obtain the BTs for (2.1) equations by geometrical method. Also, we apply the BTs to their solutions and generated new travelling wave solutions.

3. Nonlinear Evolution Equations (NLEEs) and Some Examples

The advanced nonlinear equations contain the dimensions of time and space, and thus serve the various fields of physics, chemistry, engineering and other sciences. They are represented by partial differential equations (PDE) with their marginal and elementary conditions, to facilitate application processes on them in various fields.

During the last few years several new nonlinear evolution equations have been formulated depending on the physical phenomenon to be described.

Some of these NLEE are discussed below.

In this work, we consider evolution equations of the form as follows:

$$\frac{\partial u}{\partial t} = F\left(u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^n u}{\partial^n x}\right),$$

for a function $u(x,t)$.

It will be convenient to use the following notation for the derivatives of u:

$$u_{ix} = \frac{\partial^i u}{\partial^i x}, \qquad 0 \le i \le n.$$

Some examples for the NLEEs:

(a) the Korteweg-de-Vries (KdV) equation

KdV equation is the simplest wave model that combines dispersion with non linearity. A model to surface water waves. A simple generalization of KdV equation is:

$$u_t + uu_x + u_{xxx} = 0, \qquad -\infty < x < \infty, \;\; 0 \le t < \infty,$$

where u is the wave amplitude; x, t are space and time variables respectively.

It is a nonlinear dispersive partial differential equation of third order. In this equation, the first term is the time evolution term, second term is the nonlinear term and the third term introduces dispersion.

(b) Sine Gordon (SG) equation

The typical type of SG equation is given by

$$u_{tt} + u_{xx} + \sin u = 0,$$

where u is the real scalar field. It was first studied in the course of investigation of surfaces of constant Gaussian curvature. It has a wide range of applications in physics, solid state physics, mechanical transmission lines, crystal dislocation, shape waves et c.

(c) modified KdV (mKdV) equation

The general form of mKdV equations is

$$u_t + \left(au + bu^2 + cu^3\right) u_x + \beta u_{3x} + \frac{d}{2t} u = 0.$$

(d) fifth-order evolution equations

The general form of fifth-order evolution equations is

$$u_t = u_{5x} + G\left(u, u_x, \ldots, u_{4x}\right). \tag{3.1}$$

As particular case of (3.1) the following the fifth-order KdV equation

$$u_t = u_{5x} + 10uu_{3x} + 20u_x u_{xx} + 30u^2 u_x.$$

(e) the Sawada–Kotera equation

$$u_t = u_{5x} + 5uu_{3x} + 5u_x u_{xx} + 5u^2 u_x$$

4. Backlund Transformations for The Sawada-Kotera Equation

$$u_t = u_{5x} + 5uu_{3x} + 5u_x u_{2x} + 5u^2 u_x \tag{4.1}$$

Consider the following AKNS eigenvalues problem:

$$\begin{aligned} \Phi_X &= P\Phi, \\ \Phi_t &= Q\Phi, \end{aligned} \tag{4.2}$$

where $\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$, P and Q are 2×2 null-trace matrices

$$P = \begin{bmatrix} \eta & q \\ r & -\eta \end{bmatrix}, \qquad Q = \begin{bmatrix} A & B \\ C & -A \end{bmatrix}$$

and η is a parameter independent of x and t, while r, A, B and C are assumed to be functions of differentiable functions f_{ij} depending on u and its derivatives considered as:

$$\begin{aligned} A = \frac{f_{22}}{2}, \qquad B = \frac{f_{12} - f_{32}}{2}, \qquad C = \frac{f_{11} + f_{32}}{2}, \\ q = u = \frac{f_{11} - f_{31}}{2}, \qquad r = \frac{f_{11} + f_{31}}{2}. \end{aligned} \tag{4.3}$$

Equation (4.3) will be reduced to:

$$P = \begin{bmatrix} \eta & \frac{f_{11}-f_{31}}{2} \\ \frac{f_{11}+f_{31}}{2} & -\eta \end{bmatrix} Q = \begin{bmatrix} \frac{f_{22}}{2} & \frac{f_{12}-f_{32}}{2} \\ \frac{f_{11}+f_{32}}{2} & -\frac{f_{22}}{2} \end{bmatrix}. \tag{4.4}$$

From the Equations (4.1) and (4.4), we get the following scattering problem:

$$\begin{aligned} \frac{\partial \phi_1}{\partial x} - \eta\phi_1 &= q\phi_2 \\ \frac{\partial \phi_2}{\partial x} + \eta\phi_2 &= \frac{f_{11}+f_{31}}{2}\phi_1, \end{aligned} \tag{4.5}$$

in which eigen functions ϕ_1 and ϕ_2 evolve in time according to

$$\frac{\partial \phi_1}{\phi t} = \frac{f_{22}}{2}\phi_1 + \frac{f_{12}-f_{32}}{2}\phi_2 \tag{4.6}$$

$$\frac{\partial \phi_2}{\phi t} = \frac{f_{11}+f_{32}}{2}\phi_1 - \frac{f_{22}}{2}\phi_2. \tag{4.7}$$

The integrability condition reads

$$P_t - Q_x + PQ - QP = 0, \tag{4.8}$$

or it is in component form

$$\begin{aligned} A_x + uC - rB &= 0 \\ u_t - B_x - 2Aq + 2\eta B &= 0 \\ r_t - C_x - 2\eta C + 2Ar &= 0. \end{aligned}$$

Konno and Wadati and Shimizu, introduced the function

$$\Gamma = \frac{\phi_1}{\phi_2},$$

differentiating equation (4.2) with respect to x and t, respectively. Using Equations (4.8), (4.5),(4.6), (4.7), (4.3), then, Eqs. (4) are reduced to the Riccati equations:

$$\begin{aligned} \frac{\partial \Gamma}{\partial x} &= 2\partial\Gamma + q - r\Gamma^2 \\ \frac{\partial \Gamma}{\partial t} &= B + 2A\Gamma - C\Gamma^2. \end{aligned} \tag{4.9}$$

Consider

$$
\begin{aligned}
A &= u_{3x} - uu_x - \frac{1}{2}\eta(2u_{xx} - u^2)\\
B &= u_{xx} + \frac{1}{2}u^2,\\
C &= -u + \frac{1-\eta^2}{4} - u_{4x} + \eta u_{3x} + \frac{1}{2}\left(-6u - 1 - \eta^2\right)u_{2x} - u_x^2\\
&\quad + \eta u u_x - u^3 + \frac{u^2}{2}\left(\frac{1-u^2}{2} - 1\right)\\
r &= \frac{1}{2}.
\end{aligned}
$$

Then by equation (4.9), we obtain

$$\frac{\partial \Gamma}{\partial x} = 2\eta\Gamma + \frac{u-2}{2} + \frac{1}{2}\Gamma^2. \tag{4.10}$$

Choosing Γ' and u' as follows:

$$\Gamma' = \frac{1}{\Gamma},$$

$$u' = -\frac{1}{2} + \left[\frac{2\Gamma_x}{1+\Gamma^2}\right], \tag{4.11}$$

or

$$u' = -\frac{1}{2} + 2\frac{\partial}{\partial x}\tan^{-1}\Gamma, \tag{4.12}$$

then G' and u' satisfy equation (4.10) and u' is a new solution. Equation (4.12) is called BT equation **Constructing a Second Solution from a Known Travelling Wave** Solution

$$u(x,t) = -\frac{1}{3} + \frac{2}{1+\cosh\partial}, \tag{4.13}$$

where $\partial = x - t, k = 1$. So,

(4.14)

$$\mathrm{A} = \frac{\left(\begin{array}{c} 25\eta + 24\sinh\rho + 76\mathrm{a}\cosh\rho + 192\cosh\rho\sinh\rho - 216\sinh^3\rho + 78\mathrm{a}\cosh^2\rho \\ +28\eta\cosh^3\rho\mathrm{h}^2\rho + 168\cosh^2\rho\sinh\rho - 72\eta\cosh\rho\sinh^2\rho \end{array}\right)}{72\cosh\rho + 108\cosh^2\rho + 72\cosh^3\rho + 18\cosh^4\rho + 18}$$

$$\mathrm{B} = \frac{\cosh^3\rho - 45\cosh^2\rho - 21\cosh\rho + 72\sinh^2\rho + 25}{54\cosh\rho + 54\cosh^2\rho + 18\cosh^3\rho + 18} \tag{4.15}$$

(4.16)

$$\mathrm{C}=\frac{\begin{pmatrix} 428\cosh^2\rho-288\mathrm{a}\sinh\rho-51\eta^2-20\cosh\rho-508\cosh^3\rho-560\cosh^4\rho\\ +32\cosh^5\rho-2268\sinh^2\rho-2592\sinh^4\rho-111\eta^2\cosh\rho-648\eta\sinh^3\rho\\ +864\cosh\rho\sinh^2\rho+864\eta\cosh^2\rho\sinh\rho-648\eta\cosh\rho\sinh^3\rho+576\eta\cosh^3\rho\sinh\rho\\ -42\eta^2\cos\mathrm{h}^2\rho+30\eta^2\cos\mathrm{h}^3\rho\rho\sinh^2\rho-216\eta^2\cosh\rho\sinh^2\rho\mathrm{inh}^2\rho-364 \end{pmatrix}}{270\cosh\rho+540\cosh^2\rho+540\cosh^3\rho+270\cosh^4\rho+54\cosh^5\rho+54}$$

(4.17)
$$r=\frac{1}{2},\qquad q=\frac{u-2}{2}$$

are a solution of the generalized Sawada-Kotera equatipon equation (4.1). Thus, we get

$$\frac{\mathrm{r}}{\mathrm{C}+\mathrm{r}}=\frac{-135\cosh\rho-270\cosh^2\rho-270\cosh^3\rho-135\cosh^4\rho-27\cosh^5\rho-27}{\begin{pmatrix} 288\eta\sinh\rho-115\cosh\rho+51\eta^2-698\cosh^2\rho+238\cosh^3\rho+425\cosh^4\rho\\ -59\cosh^5\rho+2268\sinh^2\rho+2592\sinh^4\rho+111\eta^2\cosh\rho+648\eta\sinh^3\rho\\ -\mathrm{sh}^2\rho\sinh\rho+648\eta\cosh\rho\sinh^3\rho-0\eta^2\cosh^3\rho+3\eta^2\cosh^4\rho+15 \end{pmatrix}}$$

and

(4.18)

$$\xi=1+\int\frac{-135\cosh\rho-270\cosh^2\rho-270\cosh^3\rho-135\cosh^4\rho-27\cosh^5\rho-27}{\begin{pmatrix} 288\eta\sinh\rho-115\cosh\rho+51\eta^2-698\cosh^2\rho+238\cosh^3\rho+425\cosh^4\rho\\ -59\cosh^5\rho+2268\sinh^2\rho+2592\sinh^4\rho+111\eta^2\cosh\rho+648\eta\sinh^3\rho\\ -\mathrm{sh}^2\rho\sinh\rho+648\eta\cosh\rho\sinh^3\rho-\eta^2\cosh^3\rho+3\eta^2\cosh^4\rho+15 \end{pmatrix}}$$

$$\delta=\mathrm{C}\frac{(36\sinh^2\rho-36\cosh^2\rho-11\cosh^3\rho-39\cosh\ \rho-14)}{27\cosh\ \rho+27\cosh^2\rho+9\cosh^3\rho+9}$$

$$+\frac{\begin{pmatrix} (25\eta-53\cosh\rho+24\sinh\rho+76\mathrm{a}\cosh\rho+192\cosh\rho\sinh\rho-75\cosh^2\rho\\ -47\cosh^3\rho-11\cosh^4\rho+36\sinh^2\rho-216\sinh^3\rho+78\mathrm{a}\cosh^2\rho+28\eta\cosh^3\mathrm{z}\\ +\mathrm{a}\cosh^4\mathrm{z}-72\eta\sinh^2\rho+\mathrm{h}^2\rho+168\cosh^2\rho\sinh\ \rho-72\eta\cosh\rho\sinh^2\sigma-14) \end{pmatrix}}{72\cosh\ \rho+108\cosh^2\rho+72\cosh^3\rho+18\cosh^4\rho+18}$$

Substituting equations (4.14),(4.16), (4.17),(4.18), we have following equation

$$\Gamma = (C + kr)^{-1} \left[(A + k\eta) + \omega \coth\omega \left(\zeta + c_2 \right) \right], \tag{4.19}$$

where $\omega^2 = \delta$

Then, substituting equation (4.19) into BTs equation (4.11) to find a new solution $u'$of the generalized Sawada-Kotera, equation (4.1) corresponding to the known solution equation (4.13):

$$u' = -\frac{1}{2} + \left[\frac{2G_x}{1 + G^2} \right]$$

5. Future Studies

In this work, we introduce some open problems that appeared in our study of Backlund transformations to nonlinear evolution equations which describe pseudo spherical surfaces.

These problems can be summarized as follow:

Problem 1: How can we find the functions f_{ij} which defined in equations describe pseudo spherical surfaces definition?

Problem 2: How to find exact solution of NLEE for simple nonconstant and travelling wave cases?

Problem 3: How can derive Backlund transformations for the fifth order NLEE?

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