

CONNECTION BETWEEN FUZZY PROXIMITIES AND FUZZY UNIFORMITIES

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ABSTRACT. The present paper introduces the concept of fuzzy proximity and fuzzy uniformity. Moreover every fuzzy uniformity induces a fuzzy proximity and vice- versa. Here we shall study some connection between fuzzy uniformities and fuzzy uniformities induces fuzzy proximities in a canonical way and vice-versa.

1. INTRODUCTION

Many concepts of general topology were extended to fuzzy set theory after the papers of Zadeh and Chang. Fuzzy uniformities were introduced by Lowen and Hutton, [1]. The two approaches are quite difficult. The one proposed by Hutton suits in a better manner to Fuzzy set theory.

The concept of Fuzzy proximity till then was unsatisfactory. Its “Fuzzyness” was rather poor since these proximities were in a canonical one-one correspondence with the usual proximities.

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Moreover the open sets of the induced topologies are crisp and though every Lowen fuzzy uniformities induces a fuzzy proximity, this correspondence cannot work well since the two structures do not give the same fuzzy topology.

For the re reasons, another definition of fuzzy proximity was given by Artico and Moresco, [3] which enables to associate a topology in a completely different way. Moreover every fuzzy uniformity induces a fuzzy proximity and vice-versa.

1.1. Notations and preliminaries. $(L, \vee, \wedge, ')$ is a (complete) completely distributive lattice with order reversing involution $'$ (= complementation).

Given a set X , any element of L^X is called fuzzy set. If Y is a subset of X , we shall use the same letter Y to indicate the element of L^X . We define:

$$\begin{aligned} f(x) &= l \text{ if } x \in Y \\ f(x) &= 0 \text{ otherwise.} \end{aligned}$$

i.e. $a \in L, x \in X$; as denote the elements of L^X which takes the value 'a' at the point x and 0 elsewhere, ax is said to be a fuzzy point and x its support. Also $lx = x$. If $\mu \in L^X$. We say that ax belongs to μ or that ax is a fuzzy point of μ if $a \leq \mu(x)$.

L^X inherits a structure of Lattice with order reversing involution in a natural way by defining $\vee, \wedge, '$ point wise (same notation of L is used.)

If $f : X \rightarrow Y$ is a function and μ, v belong to L^X, L^Y respectively, are usual we put

$$\begin{aligned} f^{\leftarrow}(v)(x) &= v(f(x)) = (v \circ f)(x) \text{ for } x \in X \\ f(\mu)(y) &= \sup \{ \mu(x) : x \in X, f(x) = y \} \text{ for } y \in Y \\ f(f^{\leftarrow}(v)) &= v \wedge f(x) \text{ and } f(f(\mu)) \geq \mu \end{aligned}$$

Clearly $f f^{\leftarrow}(v) \in L^X, f(\mu) \in L^Y$.

Moreover $f^{\leftarrow}$ preserves complementation, arbitrary unions and arbitrary intersections and that:

$$f(V_{i \in I} \mu_i) = V_{i \in I} f(\mu_i)$$

A fuzzy topological spaces is a pair (X, τ) where $\tau \in L^X$ contains the constants $\underline{0}$ and $\underline{1}$ is closed under finite intersection and arbitrary unions. The elements of τ are called open and their complements closed.

Given a fuzzy topological space (X, τ) a fuzzy set $\mu \in L^X$ is said to be τ -nhd (or simply nhd) of ax if there exists $v \in \tau$ such that $ax \leq v \leq \mu$. Clearly a

fuzzy set is open iff it is a nhd of any of its points, interior and closure of fuzzy sets are defined in the usual way.

If (X, τ) and (Y, τ') are fuzzy topological spaces a function $f : X \rightarrow Y$ is said to be continuous if $f^{\leftarrow}(v) \in \tau$ for every $v \in \tau'$.

Now we use the definition of fuzzy uniform space given by Hutton. We denote the set of maps. $U : L^X \rightarrow L^X$ which satisfy

- (i) $U(\underline{0}) = \underline{0}$
- (ii) $U : L^X \rightarrow L^X$
- (iii) $U\left(\bigvee_{i \in I} \mu_i\right) = \bigvee_{i \in I} U(\mu_i)$ for $\mu, \mu_i \in L^X$

If U, V belongs to Z , we define $U \wedge V$ to be the infimum of U and V in Z which turns out to satisfy

$$(U \wedge V)(\mu) = \bigwedge_{\mu_1 \vee \mu_2 = \mu} (U(\mu_1) \vee V(\mu_2)) .$$

Moreover we define

$$U^{-1}(\mu) = \inf \{ \rho : U(\rho) \leq \mu \} ,$$

an element U such that $U = U^{-1}$ is called symmetric.

Definition 1.1. A fuzzy uniformity on X is a subset $\wp$ of Z such that

- (U₁) $\wp \neq \phi$
- (U₂) $U \in \wp$ and $U \leq V \in Z$ implies $V \in \wp$
- (U₃) $U, V \in \wp$ implies $U \wedge V \in \wp$
- (U₄) $U \in \wp$ implies there exists $V \in \wp$ such that $V \circ V \leq U$
- (U₅) $U \in \wp$ implies $U^{-1} \in \wp$

Subbasis and basis of a uniformity get the obvious significance. Clearly (U₅) may be replaced by: $\wp$ has a basis of symmetric elements.

Given a function $f : X \rightarrow Y$, for any $V : L^Y \rightarrow L^Y$, we define

$$f^{\leftarrow}(V) : L^X \rightarrow L^X \text{ by}$$

$$f^{\leftarrow}(V)(\mu) = f^{\leftarrow}(V(f(\mu)))$$

for any $\mu \in L^X$.

It is clear that V satisfies (i) to (iii), then $f^{\leftarrow}(V)$ satisfies (i) to (iii) too.

If $(X, \wp)$ and $(Y, \mathcal{U})$ are uniform spaces, a function $f : X \rightarrow Y$ is said to be a uniform map if for every $V \in \mathcal{U}$, the element $f^{\leftarrow}(V)$ belongs to $\wp$.

Hutton showed that any Fuzzy uniformity $\wp$ induces a fuzzy topology by putting $\mu \in \tau_{\wp}$ iff

$$\mu = \sup \{ \rho \in L^X : U(\rho) \leq \mu \text{ for some } U \in \wp \}$$

Moreover every uniform map from $(X, \wp)$ to $(Y, \mathcal{U})$ is continuous equipping X and Y with the induced fuzzy topologies.

2. SOME RESULTS

Proposition 2.1. *Let $(X, \wp)$ to $(Y, \mathcal{U})$ be uniform spaces $f : X \rightarrow Y$ a function and τ' a subbasis of $\mathcal{U}$. Then f is a uniform map iff $f^{\leftarrow}(S) \in \wp$ for every $S \in \tau'$.*

Proof. The ‘only if’ part is trivial. For the converse let us suppose that if S_1 and S_2 belong to τ' , then $f^{\leftarrow}(S_1 \wedge S_2)$ belongs to $\wp$; namely we show that $f^{\leftarrow}(S_1 \wedge S_2) = f^{\leftarrow}(S_1) \wedge f^{\leftarrow}(S_2)$.

First we observe that first number of the quality is less than or equal to the second one. For the other inequality we have for

$$\begin{aligned} \mu \in L^X \text{ and } x \in X \quad & (f^{\leftarrow}(S_1) \wedge f^{\leftarrow}(S_2))(\mu)(x) \\ &= \bigwedge_{\mu_1 \vee \mu_2 = \mu} (S_1(f(\mu_1)) \vee (S_2(f(\mu_2))) (f(x)) \\ \text{and } & f^{\leftarrow}(S_1 \wedge S_2)(\mu)(x) \\ &= (S_1 \wedge S_2)(f(\mu))(f(x)) \\ &= v_1 \vee v_2 (S_1(v_1) \vee S_2(v_2))(f(x)) \end{aligned}$$

we see that $\inf v_1 \vee v_2 = f(\mu)$, we then have

$$\begin{aligned} & (f^{\leftarrow}(v_1) \wedge \mu) \vee (f^{\leftarrow}(v_2) \wedge \mu) \\ &= f^{\leftarrow}(v_1) \wedge (f^{\leftarrow}(v_2) \wedge \mu) \\ &= f^{\leftarrow}(v_1 \vee v_2) \wedge \mu = f^{\leftarrow}(f(\mu)) \wedge \mu = \mu \end{aligned}$$

Moreover for $i = 1, 2$

$$\begin{aligned} f(f^{\leftarrow}(v_i) \wedge \mu)(y) &= \sup \{f^{\leftarrow}(v_i) \wedge \mu(x) : f(x) = y\} \\ &= \sup \{v_i(f(x) \wedge \mu(x)) : f(x) = y\} \\ &= v_i(y) \wedge \sup \{\mu(x) : f(x) = y\} \\ &= v_i(y) \wedge f(\mu)(y) = v_i(y) \end{aligned}$$

Hence if we take $\mu_i = f^{\leftarrow}(v_i) \wedge \mu$, we have $\mu_1 \vee \mu_2 = \mu$ and $f(\mu_i) = v_i$ and the conclusion follows. $\square$

Definition 2.1. A Fuzzy proximity on a set X is a function $\delta : L^X \times L^X \rightarrow \{0, 1\}$ which satisfies for any $\mu, v, \rho \in L^X$ the following conditions:

- (P₁) $\delta(\underline{0}, \underline{1}) = 0$
- (P₂) $\delta(\mu, \rho) = \delta(\rho, \mu)$
- (P₃) $\delta(\mu, \rho) \vee \delta(v, \rho) = \delta(\mu \vee v, \rho)$
- (P₄) if $\delta(\mu, \rho) = 0$ there exists $\gamma \in L^X$ Such that $\delta(\mu, \gamma) = 0, \delta(\rho, \gamma) = 0$
- (P₅) $\delta(\mu, \rho) = 0$, implies $\mu \leq \rho'$

The pair (X, δ) is said to be fuzzy proximity space.

If $\delta(\mu, \rho) = 0$ we say that μ and ρ are far, otherwise we say that they are proximal. (P₁ – P₄) are the natural extensions of classical case. (P₅) needs some comment since A. Katsaras formulated the analogous axiom in a different manner. In the case $L = \{0, 1\}$, (P₅) means exactly that if two subsets intersect then they are proximal. In the case $L = \{0, 1\} = 1$, (P₅) means that μ and ρ are proximal whenever exist $x \in X$ such that $\mu(x) + \rho(x) > 1$.

Definition 2.2. Let (X, δ) and (Y, η) be fuzzy proximity spaces. A function f is a proximity map if one of the following equivalent condition holds:

- (a) For every, $v, \sigma \in L^Y$, $\eta(v, \sigma) = 0$ implies $\delta(f^{\leftarrow}(v), f^{\leftarrow}(\sigma)) = 0$
- (b) For every $\mu, \rho \in L^X$, $\delta(\mu, \rho) = 1$ implies $\eta(f\mu, f\rho) = 1$

To see that conditions (a) and (b) are equivalent, we may use part (i) of the following Lemma.

Lemma 2.1. Let (X, δ) be a fuzzy proximity space. For every $\mu, \rho \in L^X$ and $\delta(\mu, \rho) = 1$

- (i) $\eta(f(\mu), f(\rho)) = 1$

(ii) If $\delta(\mu, \rho_i) = 0$ for $i = 1, \dots, n$, then $\delta\left(\bigwedge_{i=1\dots n} \mu_i, \bigvee_{i=1\dots n} \rho_i\right) = 0$.

Proof. One can use (P₃) to prove (i) and (P₃) to prove (ii). $\square$

Remark 2.1. Clearly the set of all proximities on a given set X can be equipped with a partial order by defining δ_1 finer than δ_2 (or δ_2 coarser than δ_1) if the identity of X is a proximity map from (X, δ_1) to (X, δ_2) .

We shall define the fuzzy topology induced by fuzzy proximity.

We take a proximity space (X, δ) and for any, $\mu \in L^X$ we put

$$\text{int}(\mu) = \sup \{ \rho : \delta(\rho, \mu') = 0 \},$$

and denote it by μ^0 or $\text{int}(\mu)$.

Theorem 2.1. The function $\text{int} : L^X \rightarrow L^X$ satisfies the interior axioms namely, we have for $\mu, \rho \in L^X$

$$(I_1) \text{ int}(\underline{1}) = \underline{1}$$

$$(I_2) \text{ int}(\mu) \leq \mu$$

$$(I_3) \text{ int}(\text{int}(\mu)) = \text{int} \mu$$

$$(I_4) \text{ int}(\mu \wedge \rho) = \text{int}(\mu) \wedge \text{int}(\rho)$$

Proof. (I₁) and (I₂) follow trivially from (P₁) and (P₅) respectively.

$$(I_3) \text{ clearly } \text{int}(\text{int}(\mu)) \leq \text{int}(\mu)$$

We now take ρ such $\delta(\rho, \mu') = 0$.

By (P₄) there exist γ such that $\delta(\rho, \gamma') = 0$ and $\delta(\gamma, \mu') = 0$; hence $\rho \leq \text{int}(\gamma)$, $\rho \leq \text{int}(\mu)$ and $\text{int}(\gamma) \leq \text{int}(\mu)$ because int is monotone, therefore $\gamma \leq \text{int}(\text{int}(\mu))$ for every ρ , such that $\delta(\rho, \mu') = 0$.

So that $\text{int}(\text{int} \mu) \geq \text{int}(\mu)$

$$(I_4) \text{ Trivially } \text{int}(\mu \wedge \rho) \leq \text{int}(\mu) \wedge \text{int}(\rho).$$

For the converse, we see that in a completely distributive lattice, the infinite distributive law holds, hence we have

$$\begin{aligned} \text{int}(\mu) \wedge \text{int}(\rho) &= \sup \{ v : \delta(v, \mu) = 0 \} \wedge \sup \{ \sigma : \delta(\sigma, \rho') = 0 \} \\ &= \sup \{ v \wedge \sigma : \delta(v, \mu') = 0 = \delta(\sigma, \rho') \} \\ &\leq \sup \{ t : \delta(t, \mu' \vee \rho') = 0 \} \\ &= \sup \{ t : \delta(t, \mu \wedge \rho)' = 0 \} \\ &= \text{int}(\mu \wedge \rho) \end{aligned}$$

$\square$

Definition 2.3. The fuzzy topology induced by fuzzy proximity δ is denoted by τ_δ and consists of all fuzzy sets $\mu \in L^X$ such that $\mu = \text{int}(\mu)$.

Clearly the closure of μ in the topology τ_δ denoted $Cl_{\tau_\delta}(\mu)$ or $Cl(\mu)$ is given by $(\text{int}(\mu'))'$.

Remark 2.2. (I) If $L = I$ then μ is a τ_δ -nhd of a iff for every $b < a$ we have

$$\delta(bx, I - \mu) = 0$$

(II) : If (X, δ) is a classical proximity space, for any $\mu \in L^X$, we put

$$\text{coz}(\mu) = (x \in X : \mu(x) > 0) \text{ and}$$

define $\hat{\delta}(\mu, \rho) = 0$ iff $\text{coz}(\mu) \delta \text{coz}(\rho)$.

Then $\hat{\delta}$ is fuzzy proximity and τ_δ open fuzzy sets are exactly the characteristic functions of the sets which are open in the topology induced by δ .

(III) The fuzzy proximity introduced by Katsaras [2] satisfy conditions $IP_1 - P_5$) and the δ of the example above is a Katsaras proximity. Furthermore, given a Katsaras proximity η , it is clear to prove that there exists a classical proximity δ such that $\hat{\delta} = \eta$; indeed for A, B subset of X . We put $A \delta B$ iff $A \eta B$.

To prove that δ is a usual proximity and $\hat{\delta} = \eta$, we consider the fact that for every $\mu, \rho \in I^X$ we have that the closure of μ introduced by Katsaras (denoted by $\bar{\mu}$ in this example) is a characteristic function and

$$\mu \eta \rho \text{ iff } \bar{\mu} \bar{\rho} \text{ iff } \text{coz}(\mu) \eta \text{coz}(\rho) \text{ iff } \text{coz}(\mu) \delta \text{coz}(\rho),$$

iff $\hat{\delta}(\mu, \rho) = 1$. Thus Katsaras proximity are in a canonical 1-1 correspondence with the usual proximities.

Proposition 2.2. Let $(X, \delta), (Y, \eta)$ be f. proximity spaces. If $f: X \rightarrow Y$ is proximity map, then it is continuous equipping X and Y with the induced fuzzy topologies.

Proof. Let v be τ_η -open set i.e. $v = \sup \{\sigma : \eta(\sigma, v') = 0\}$.

Hence $f^{\leftarrow}(v) = \sup \{f^{\leftarrow}(\sigma) : \eta(\sigma, v') = 0\}$

$$\leq \sup \{\rho : \delta(\rho, f^{\leftarrow}(v))' = 0\}$$

i.e. $f^{\leftarrow}(v) = \text{int}(f^{\leftarrow}(v))$ is a τ_δ -open set. □

Proposition 2.3. Let δ be a fuzzy proximity on X . Then,

(i) $\delta(\mu, \rho) = 0$ iff $\delta(\bar{\mu}, \rho) = 0$;

(ii) $\bar{\mu} = \sup \{v : \delta(\mu, \rho) = \delta(v, \rho) \text{ for every } \rho \in L^X\}$.

Proof. The 'if part' is trivial, for the converse let us take γ such that $\delta(\gamma', \mu) = 0 = \delta(\rho, \gamma)$. Hence $\gamma' \leq \text{int}(\mu')$ so that $\gamma \geq (\text{int}(\mu'))' = \bar{\mu}$ and $\delta(\rho, \mu) = 0$.

By (i) we get that $\bar{\mu} \leq \sup \{v : \delta(\mu, \rho) = \delta(v, \rho) \text{ for every } v \in L^X\}$.

We then take $v \leq \bar{\mu}$ such that $\gamma(\mu, \rho) = \delta(v, \rho)$ for every $\rho \in L^X$ and we put $t = \bar{\mu} \vee v$. We see that $t > \bar{\mu}$ and $\delta(t, \rho)$ for every $\rho \in L^X$.

Since $t' < (\bar{\mu})' = \text{int}(\mu')$; by the definition of int there exists $\sigma \leq t'$ such that $\delta(\mu, \sigma) = 0$ while (P5) implies $\delta(t, \sigma) = 1$ which is a contradiction. $\square$

3. CONNECTION BETWEEN FUZZY PROXIMITIES AND FUZZY UNIFORMITIES

Now we shall study some connection between fuzzy uniformities and fuzzy proximities. We shall show that any fuzzy uniformity induces a fuzzy proximity in a canonical way and vice-versa.

Let $\wp$ be a fuzzy uniformity and for $\mu, \rho \in L^X$ we define

$\delta_\wp(\mu, \rho) = 0$ iff there exists

$$U \in \wp \text{ s.t. } U(\mu) \leq \rho'.$$

Theorem 3.1. $\delta_\wp$ as defined above is a fuzzy proximity.

Proof. We shall verify properties (P1-P5)

(P₁) - is trivial.

(P₂) $\delta_\wp(\mu, \rho) = \delta_\wp(\rho, \mu)$, since for $U \in \wp$, $U(\mu) \leq \rho'$ iff $U^{-1}(\rho) \leq \mu$.

(P₃) It is sufficient to prove that

$$\delta_\wp(\mu, \rho) = 0 = \delta_\wp(v, \rho)$$

implies $\delta_\wp(\mu \vee v, \rho) = 0$ since the converse is trivial.

If $U(\mu) \leq \rho'$, $V(v) \leq \rho'$, we have $(U \wedge V)(\mu \vee v) \leq \rho'$ and then $\delta_\wp(\mu \vee v, \rho) = 0$.

(P₄) Let $\delta_\wp(\mu, \rho) = 0 \exists U \in \wp$ such that $U(\mu) \leq \rho'$. We take $V \in \wp$, then $V = V^{-1}$, and

$$V(V(\mu)) \leq \rho' \Rightarrow V(\rho) \leq (V(\mu))'.$$

Hence for $\gamma = V(\rho)$ we have $\delta_\wp(\mu, \gamma) = 0 = \delta_\wp(\rho, \gamma')$.

(P₅) Trivial. $\square$

Remark 3.1. We say that a fuzzy uniformity $\wp$ is separated if for given points ax , by such that $ax \leq (by)'$ there exists $U \in \wp$ such that:

$$U(ax) \leq (by)'.$$

Theorem 3.2. Let $\wp$ be a fuzzy uniformity, then $\wp$ and $\delta_\wp$ induce the same topology.

Proof. Given a fuzzy set μ , we see that $\{v : \exists U \in \wp \text{ such that } U(v) \leq \mu\} = \{v : \delta_\wp(v, \mu') = 0\}$ and the supremum of the first member of the equality is the interior of μ in the topology induced by $\wp$, while the supremum of the second one is the interior of μ in the topology induced by $\delta_\wp$. $\square$

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