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SINGULARLY PERTURBED LOGISTIC DIFFERENCE EQUATION

A. M. A. EL-Sayed, S. M. Salman, and A. M. A. Abo-Bakr[1]

ABSTRACT. This paper is concerned with the comparison of the dynamic behaviour between the singularly perturbed logistic diference equation with continuous argument and its difference equation with continuous argument counterpart.

1. INTRODUCTION

Singularly perturbed equations arises in applications where delays and perturbations play a role [1]. Delays arise in many mathematical models of population dynamics because any species need time to become mature or to digest their food for their activities [4, 5, 8]. The population represents humans, biological lifeforms in ecological systems, chemical compounds, farm lands [6].

Let $\epsilon \in (0,1]$ and consider the logistic difference equation with continuous argument $x(t) = \rho x(t-1)(1-x(t-1))$, $t \in [0,T]$, $x(t) = x_0$, $t \leq 0$ and its singularly perturbed equation

$$\epsilon \frac{dx}{dt} + x(t) = \rho x(t-1)(1-x(t-1)),\ t \in [0,T],\ x(t) = x_0,\ t \leq 0. \tag{1.1}$$

[1]*corresponding author*

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2. Main results

2.1. Stability and Bifurcation. There are two fixed points of (1.1) $(x_1)_{fix} = 0$ and $(x_2)_{fix} = 1 - \frac{1}{\rho}$.

The linearized equation at $(x_1)_{fix} = 0$ is $\frac{dx}{dt} = \frac{-1}{\epsilon}x(t) + \frac{\rho}{\epsilon}x(t-1)$. Assuming $x(t) = e^{\lambda t}$, the characteristic equation reads

$$\lambda + \frac{1}{\epsilon} - \frac{\rho}{\epsilon}e^{-\lambda} = 0.$$

The linearized equation at $(x_2)_{fix} = 1 - \frac{1}{\rho}$ is $\frac{dy}{dt} = \frac{-1}{\epsilon}y(t) + \frac{2-\rho}{\epsilon}y(t-1)$, where $y(t) = y(t) - (1 - \frac{1}{\rho})$. The characteristic equation reads

$$\lambda + \frac{1}{\epsilon} - \frac{2-\rho}{\epsilon}e^{-\lambda} = 0. \tag{2.1}$$

Theorem 2.1.

(1) *The fixed point $(x_1)_{fix} = 0$ of (1.1) is unstable if $\rho < \epsilon\rho_0$ or $\rho > 1$ where $\rho_0 = -\sqrt{\frac{1}{(\epsilon)^2} + (\xi)^2}$, $\xi = \frac{-1}{\epsilon}tan\xi$, $0 < \xi < \pi$, and is stable if $\epsilon\rho_0 < \rho < 1$,*

(2) *The fixed point $(x_2)_{fix} = 1 - \frac{1}{\rho}$ of (1.1) is local stable if $1 < \rho < 2 - \epsilon\rho_0$, and unstable if $\rho < 1$ or $\rho > 2 - \epsilon\rho_0$.*

Theorem 2.2. *When the parameter ρ passes through $\rho = \epsilon\rho_0 = -\epsilon\sqrt{\frac{1}{(\epsilon)^2} + \xi^2}$, $\xi = \frac{-1}{\epsilon}tan\xi$, $0 < \xi < \pi$, there is a Hopf Bifurcation from $(x_1)_{fix} = 0$ to a periodic orbit. When ρ passes through $\rho = 2 - \epsilon\rho_0$, there is a Hopf Bifurcation from $(x_2)_{fix} = 1 - \frac{1}{\rho}$ to a periodic orbit.*

Proof. Assume that $\lambda = i\omega_0$, $\omega_0 \in R^+$ is a pure imaginary solution of (2.1) for some parameter value $\rho = \rho_*$. This leads to the following equations

$$i\omega_0 + \frac{1}{\epsilon} - \frac{\rho_*}{\epsilon}e^{-i\omega_0} = 0, \quad \frac{1}{\epsilon} - \frac{\rho_*}{\epsilon}cos(\omega_0) = 0, \quad \frac{1}{\epsilon} = \frac{\rho_*}{\epsilon}cos(\omega_0),$$

$$\omega_0 - \frac{\rho_*}{\epsilon}sin(\omega_0) = 0, \quad \omega_0 = \frac{\rho_*}{\epsilon}sin(\omega_0), \quad \omega_0^2 + \frac{1}{\epsilon^2} = \frac{\rho_*}{\epsilon^2}[cos(\omega_0)^2 + sin(\omega_0)^2] = \frac{\rho_*}{\epsilon^2},$$

$$\rho_* = \pm\epsilon\sqrt{\frac{1}{(\epsilon)^2} + \omega_0^2}, \ \omega_0 = \frac{-1}{\epsilon}tan\omega_0.$$

By Theorem 2.1, $\rho_* = -\epsilon\sqrt{\frac{1}{\epsilon^2} + \omega_0^2}$ is the critical value of ρ, where ω_0 is the root of $\omega_0 = \frac{-1}{\epsilon}tan\omega_0$, $0 < \omega_0 < \pi$.

Now, we are left with the condition $\frac{d(Re(\lambda))}{d\rho} \mid_{\rho=\rho_*} \neq 0$. To show that this condition is satisfied, let $\lambda = k(\rho) + i\omega(\rho)$ and using (1.1) we get $k + i\omega + \frac{1}{\epsilon} - \frac{\rho}{\epsilon}e^{-k-i\omega} = 0$,

$$k + \frac{1}{\epsilon} - \frac{\rho}{\epsilon}e^{-k}cos\omega = 0, \tag{2.2}$$

$$\omega + \frac{\rho}{\epsilon}e^{-k}sin\omega = 0. \tag{2.3}$$

Differentiate (2.2) and (2.3) with respect to ρ, we obtain

$$\epsilon\frac{dk}{d\rho} - e^{-k}cos(\omega) + \rho e^{-k}cos(\omega)\frac{dk}{d\rho} + \rho e^{-k}sin(\omega)\frac{d\omega}{d\rho} = 0,$$

$$\epsilon\frac{d\omega}{d\rho} + e^{-k}sin(\omega) + \rho e^{-k}cos(\omega)\frac{d\omega}{d\rho} - \rho e^{-k}sin(\omega)\frac{dk}{d\rho} = 0.$$

Solving for $\frac{dk}{d\rho}$, we obtain

$$\begin{aligned}\frac{d(Re(\lambda))}{d\rho} \mid_{\rho=\rho_*} = \frac{dk}{d\rho} \mid_{k=0,\omega=\omega_0,\rho=\rho_*} &= \frac{\epsilon cos(\omega_0) + \rho_*}{(\epsilon + \rho_* cos(\omega_0))^2 + (\rho_* sin(\omega_0))^2} \\ &= \frac{\epsilon\rho_* cos(\omega_0) + \rho_*^2}{\rho_*[(\epsilon + \rho_* cos(\omega_0))^2 + (\rho_* sin(\omega_0))^2]} \\ &= \frac{\epsilon + \rho_*^2}{\rho_*[(\epsilon + \rho_* cos(\omega_0))^2 + (\rho_* sin(\omega_0))^2]} \neq 0.\end{aligned}$$

Similarly, we can prove that there is a Hopf Bifurcation from the fixed point $(x_{fix})_2 = 1 - \frac{1}{\rho}$ to a periodic orbit whenever ρ passes through the critical value $\rho = 2 - \epsilon\rho_0$. As $\epsilon \to 1$, we get the results as in [7]. □

2.2. **Discretization of (1.1).** Here, the method of steps [2] is used to get a discretized analogue of (1.1).

For $t \in (0, 1] : x_1(t) = x_0 e^{\frac{-t}{\epsilon}} + \frac{\rho}{\epsilon}\int_0^t e^{\frac{s-t}{\epsilon}} x(s-1)(1-(x(s-1)))ds$ as $t \longrightarrow 1$, $x_1 = x_0 e^{\frac{-1}{\epsilon}} + \rho x_0(1 - x_0)(1 - e^{\frac{-1}{\epsilon}})$.

For $t \in (1, 2] : x_2(t) = x_1 e^{\frac{-(t-1)}{\epsilon}} + \frac{\rho}{\epsilon}\int_1^t e^{\frac{s-t}{\epsilon}} x(s-1)(1-(x(s-1)))ds$ as $t \longrightarrow 2$, $x_2 = x_1 e^{\frac{-1}{\epsilon}} + \rho x_1(1 - x_1)(1 - e^{\frac{-1}{\epsilon}})$.

Repeating the process we deduce that for $t \in (n, n+1] : x_{n+1}(t) = x_n e^{\frac{-(t-n)}{\epsilon}} + \rho x_n(1 - x_n)(1 - e^{\frac{-(t-n)}{\epsilon}})$ as $t \longrightarrow n+1$,

$$x_{n+1} = x_n e^{\frac{-1}{\epsilon}} + \rho x_n(1 - x_n)(1 - e^{\frac{-1}{\epsilon}}). \tag{2.4}$$

2.3. **Stability of fixed points of (2.4).** The system (2.4) has two fixed points $(x_1)_{fix} = 0$ and $(x_2)_{fix} = 1 - \frac{1}{\rho}$.

(a) $(x_1)_{fix} = 0$ is stable if $|e^{\frac{-1}{\epsilon}} + \rho(1 - e^{\frac{-1}{\epsilon}})| < 1$,

(b) $(x_2)_{fix} = 1 - \frac{1}{\rho}$ is stable if $|e^{\frac{-1}{\epsilon}} + (1 - e^{\frac{-1}{\epsilon}})(2 - \rho) < 1$.

As $\epsilon \to 0$, we get the results in [3]. As $\epsilon \to 1$, we get the results in [7].

2.4. **Controlling the system (2.4).** The DFC algorithm is suitable for time discrete systems because even if the control loop is included, the dimension of the phase space stays finite [10]. We want to stabilize the nonzero fixed point $(x_2)_{fix} = 1 - \frac{1}{\rho}$ of (2.4) by using the DFC algorithm $x_{n+1} = x_n e^{\frac{-1}{\epsilon}} + \rho x_n(1 - x_n)(1 - e^{\frac{-1}{\epsilon}}) + k(x_n - x_{n-1})$, where k is the feedback gain. Let $y_n = x_{n-1}$ be an auxiliary variable, we get the 2D map:

$$x_{n+1} = x_n e^{\frac{-1}{\epsilon}} + \rho x_n(1 - x_n)(1 - e^{\frac{-1}{\epsilon}}) + k(x_n - y_n), \quad y_{n+1} = x_n. \tag{2.5}$$

This map has two fixed points $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1 - \frac{1}{\rho}, 1 - \frac{1}{\rho})$. (x_2, y_2) of (2.5) becomes stable when

$$\frac{(\rho - 2)(1 - e^{\frac{-1}{\epsilon}}) - (1 + e^{\frac{-1}{\epsilon}})}{2} < k < 1. \tag{2.6}$$

Let k_{op} be the optimal value of the feedback gain which leads to the fastest convergence towards the desired fixed point. To obtain its value, we linearize (2.5) around (x_2, y_2). The Jacobian matrix at (x_2, y_2) reads

$$J((x_2, y_2)) = \begin{pmatrix} e^{\frac{-1}{\epsilon}} + (2 - \rho)(1 - e^{\frac{-1}{\epsilon}}) + k & -k \\ 1 & 0 \end{pmatrix}.$$

$J((x_2, y_2))$ has two eigenvalues given by $\lambda_{1,2} = \frac{\sigma \pm (\sigma^2 - 4k)^{\frac{1}{2}}}{2}$, where $\sigma = e^{\frac{-1}{\epsilon}} + (2 - \rho)(1 - e^{\frac{-1}{\epsilon}}) + k$. When the magnitude of the leading eigenvalue is minimal, the feedback gain is optimal. If the discriminant is zero (i.e. $\sigma^2 = 4k$), we get the required minimal value and the feedback gain becomes optimal $k_{\pm} = 2 - e^{\frac{-1}{\epsilon}} + (2 - \rho)(1 - e^{\frac{-1}{\epsilon}}) \pm 2(1 - e^{\frac{-1}{\epsilon}} + (2 - \rho)(1 - e^{\frac{-1}{\epsilon}}))^{\frac{1}{2}}$. $k_{op} = k_{-}$ because the magnitudes of $|\lambda_{1,2}|$ are minimal at k_{-}. Stabilization of the desired fixed point for any initial conditions can be achieved by first checking the two conditions $|x_n - y_n| < \nu$ and $y_n > Y_{th}$, where Y_{th} is a threshold value. When these conditions are satisfied, we switch on the DFC force with k chosen in the interval of stability (2.6) and stabilize the fixed point. Y_{th} is defined as the intersection of $x_n =$

$y_n e^{\frac{-1}{\epsilon}} + \rho y_n(1-y_n)(1-e^{\frac{-1}{\epsilon}})$ with $x_n^+ = y_n + \nu$ and is given by

$$Y_{th} = \frac{e^{\frac{-1}{\epsilon}} + \rho(1-e^{\frac{-1}{\epsilon}} - 1 - ((e^{\frac{-1}{\epsilon}} + \rho(1-e^{\frac{-1}{\epsilon}} - 1)^2 - 4\rho(1-e^{\frac{-1}{\epsilon}}\nu)^{\frac{1}{2}}}{2\rho(1-e^{\frac{-1}{\epsilon}})}.$$

As $\epsilon \to 0$, we get the results in [9].

3. Numerical simulations

In Figure 1 we perform some numerical simulations to illustrate and confirm theoretical analysis obtained.

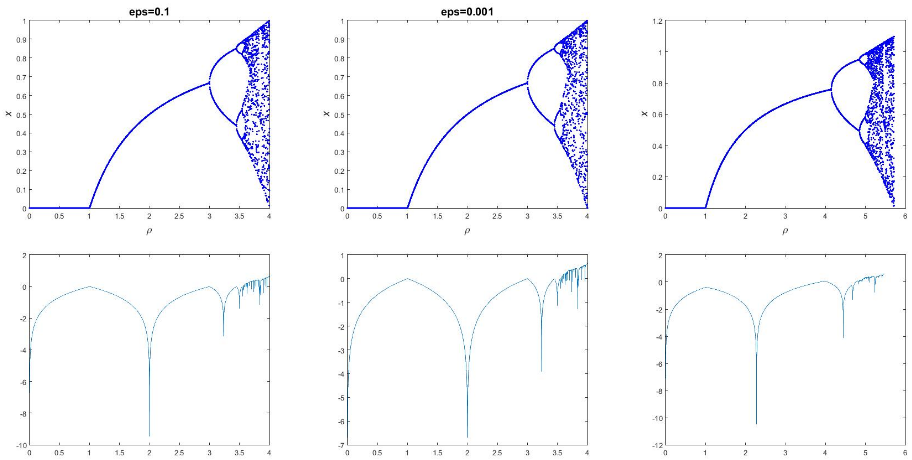

FIGURE 1. Bifurcation diagram and Lyapunov exponent of (2.4).

4. Conclusion

In this paper, stability, bifurcation, and chaos of the singularly perturbed Logistic difference equation with continuous argument are discussed. Local stability and Hopf bifurcation analysis of the fixed points of the singularly perturbed equation were investigated by analyzing the associated eigenvalues of the characteristic transcendental equation. The singularly perturbed system is discretized using the method of steps. Local stability of the fixed points of the discrete system is also investigated. The nonzero fixed point is stabilized using the delayed feedback control (DFC) algorithm. It is illustrated that the singularly perturbed logistic difference equation with continuous argument behaves as its logistic difference equation with continuous argument counterpart when the perturbation parameter $\epsilon \longrightarrow 0$ and behaves as the logistic delay differential equation when the perturbation parameter $\epsilon \longrightarrow 1$ within finite time intervals. Moreover, the theoretical analysis is confirmed by numerical simulations.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, FACULTY OF SCIENCE, ALEXANDRIA UNIVERSITY, EGYPT
Email address: amasayed@alexu.edu.eg

DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, ALEXANDRIA UNIVERSITY, EGYPT
Email address: samastars9@alexu.edu.eg

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, FACULTY OF SCIENCE, ALEXANDRIA UNIVERSITY, EGYPT
Email address: ali.abobakr@alexu.edu.eg