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NEW OSCILLATION CRITERION OF FIRST ORDER DIFFERENCE EQUATIONS WITH ADVANCED ARGUMENT

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ABSTRACT. In this paper, we obtain a new sufficient oscillation condition of the first-order linear advanced difference equation of the form

$$\Delta u(n) - p(n)u(\sigma(n)) = 0,\ n \geq n_0 > 0,$$

where $\{p(n)\}$ is a sequence of positive real numbers and $\{\sigma(n)\}$ is a nondecreasing sequence of integers with $\sigma(n) \geq n+2$. An example is provided to show the significance of our main result.

1. INTRODUCTION

In this paper, we present a new sufficient condition for the oscillation of all solutions of the advanced type difference equation

$$\Delta u(n) - p(n)u(\sigma(n)) = 0,\ n \geq n_0 > 0, \tag{1.1}$$

where $\{p(n)\}$ is a sequence of positive real numbers and $\{\sigma(n)\}$ is a nondecreasing sequence of integers such that $\sigma(n) \geq n+2$, for all $n \geq n_0$.

By a solution of (1.1), we mean a real sequence $\{u(n)\}$ defined and satisfies (1.1) for all $n \geq n_0$. A nontrivial solution $\{u(n)\}$ of (1.1) is called *oscillatory*

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if the terms $u(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is called *nonoscillatory*.

The oscillatory and asymptotic behavior of difference equations with advanced arguments received great interest in recent years because of the fact that such equations arise in many fields such as population dynamics, economics problems or mechanical control engineering, see for example [3, 4, 7].

The oscillatory behavior of (1.1) with $\sigma(n) = n + k,\ k \geq 2$ is studied in [6] and prove that if

$$\lim_{n\to\infty} \inf \sum_{s=n+1}^{n+k-1} p(s) > \left(\frac{k-1}{k}\right)^k,$$

then all solutions of (1.1) are oscillatory.

In 2012, Chatzarakis and Stavroulakis [1, 2], investigated the oscillatory behavior of (1.1) and proved that if

$$\limsup_{n\to\infty} \sum_{s=n}^{\sigma(n)-1} p(s) > 1,$$

or

$$\limsup_{n\to\infty} \sum_{s=n}^{\sigma(n)-1} p(s) > 1 - \left(1 - \sqrt{1-\alpha}\right)^2,$$

where

$$\alpha := \liminf_{n\to\infty} \sum_{s=n+1}^{\sigma(n)-1} p\,(s)\,,$$

then all solutions of (1.1) are oscillatory.

For further results concerning oscillation of first-order advanced type difference equations one can see [8–10] and the references cited there in.

The equation (1.1) can be looked upon as a discrete analogue of the first-order advanced differential equation

$$u'(t) - p(t)u(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1.2}$$

where $p(t) \in C([t_0, \infty), \mathbb{R}_+)$ and $\sigma(t) > t$.

In 1990, Hang and Driver [7], proved that if

$$\lim_{t\to\infty} \inf \int_t^{\sigma(t)} p(s)ds > \frac{1}{e} \tag{1.3}$$

then all solutions of (1.2) are oscillatory.

An interesting question then arises whether there exists the discrete analogue of condition (1.3) for (1.1). In the present paper a positive answer to the above question is given.

2. Oscillation Results

Here we present an oscillation result which is a discrete analogue of condition (1.3).

Theorem 2.1. *If*

$$\lim_{n\to\infty} \inf \sum_{n+1}^{\sigma(n)-1} p(s) > \frac{1}{e}, \tag{2.1}$$

then all solutions of (1.1) *are oscillatory.*

Proof. Assume to the contrary that $\{u(n)\}$ is a positive solution of (E) for all $n \geq n_0$. Then $u(n) > 0$ and $u(\sigma(n)) > 0$ for all $n \geq n_0$. Hence, from (1.1) we have

$$\Delta u(n) = p(n)u(\sigma(n)) > 0, \quad n \geq n_0, \tag{2.2}$$

which means that $\{u(n)\}$ is an increasing sequence.

Inequality (2.2) can be rewritten as

$$u(n+1) - u(n) \geq p(n)u(n+1) > 0$$

or

$$p(n) \leq 1 - \frac{u(n)}{u(n+1)}.$$

Summing up the above inequality from $n+1$ to $\sigma(n)-1$, we get

$$\sum_{s=n+1}^{\sigma(n)-1} p(s) \leq \sum_{s=n+1}^{\sigma(n)-1} \left(1 - \frac{u(s)}{u(s+1)}\right). \tag{2.3}$$

On the other hand, in view of (2.1), we can choose a positive constant β such that

$$\sum_{s=n+1}^{\sigma(n)-1} p(s) \geq \beta > \frac{1}{e}, \quad n \geq n_1,$$

for some integer $n_1 \geq n_0$. From (2.3) and the last inequality, we obtain

$$\beta \leq \sum_{s=n+1}^{\sigma(n)-1} \left(1 - \frac{u(s)}{u(s+1)}\right), \quad n \geq n_1. \tag{2.4}$$

The right side of (2.4) can be written as

$$\sum_{s=n+1}^{\sigma(n)-1} \left(1 - \frac{u(s)}{u(s+1)}\right) = \sum_{s=n+1}^{\sigma(n)-1} \left\{1 - \exp\left(\ln\left(\frac{u(s)}{u(s+1)}\right)\right)\right\}.$$

Since $\frac{u(n)}{u(n+1)} < 1$ and $e^{-z} \geq 1 - z (z > 0)$, we have

$$\begin{aligned} \sum_{s=n+1}^{\sigma(n)-1} \left(1 - \frac{u(s)}{u(s+1)}\right) &\leq \sum_{s=n+1}^{\sigma(n)-1} \left\{1 - \left(1 - \ln\left(\frac{u(s)}{u(s+1)}\right)\right)\right\} \\ &= \sum_{s=n+1}^{\sigma(n)-1} \ln\left(\frac{u(s+1)}{u(s)}\right) = \ln\left(\frac{u(\sigma(n))}{u(n+1)}\right). \end{aligned} \tag{2.5}$$

Combining (2.4) with (2.5) yields

$$\beta \leq \ln\left(\frac{u(\sigma(n))}{u(n+1)}\right),$$

which implies that

$$u(\sigma(n)) \geq e^{\beta} u(n+1). \tag{2.6}$$

Using (2.6) in (1.1) yields

$$\Delta u(n) - p(n) e^{\beta} u(n+1) \geq 0.$$

By repeating the above arguments m times, there exists an integer n_m such that

$$\frac{u(\sigma(n))}{u(n+1)} \geq e^{m\beta}, \tag{2.7}$$

for $n \geq n_m,\ m = 1, 2, \ldots$.

Now in view of (2.1), we can choose an integer $N > n_0$ such that

$$\sum_{s=n+1}^{\sigma(n)-1} p(s) \geq \beta, \tag{2.8}$$

for all $n \geq N$. Next, we will show that for each $n \geq N$, there exists an integer n^* with $N \leq n^* \leq n$ such that $\sigma(n^*) \geq n+2$, and

$$\sum_{s=n^*+1}^{n} p(s) < \frac{\beta}{2} \text{ and } \sum_{s=n^*}^{n} p(s) \geq \frac{\beta}{2}. \tag{2.9}$$

Indeed, (2.6) guarantees that

$$\sum_{s=N}^{\infty} p(s) = \infty.$$

In particular, it holds

$$\sum_{s=n^*}^{\infty} p(s) = \infty.$$

If $p(n) < \frac{\beta}{2}$, there always exists an integer n with $N \leq n^* < n$ so that (2.9) is satisfied. If $p(n) \geq \frac{\beta}{2}$, then $n^* = n \geq N$ so that

$$\sum_{s=n^*+1}^{n} p(s) = \sum_{s=n+1}^{n} p(s)(empty\ sum) = 0 < \frac{\beta}{2}$$

and

$$\sum_{s=n^*}^{n} p(s) = \sum_{s=n}^{n} p(s) = p(n) \geq \frac{\beta}{2}.$$

Thus, in both cases (2.9) is satisfied.

Now, we will show that $\sigma(n^*) \geq n+2$. Indeed, in the case where $p(n) \geq \frac{\beta}{2}$, since $n^* = n$, we have $\sigma(n^*) = \sigma(n) \geq n+2$. In the case when $p(n) < \frac{\beta}{2}$, then $n^* < n$. Assume for the sake of contradiction that $\sigma(n^*) < n+2$. Then $\sigma(n^*) \leq n+1$, and therefore

$$\sum_{s=n^*+1}^{\sigma(n^*)-1} p(s) \leq \sum_{s=n^*+1}^{n} p(s) < \frac{\beta}{2}. \tag{2.10}$$

On the other hand, in view of (2.8), we have

$$\sum_{s=n^*+1}^{\sigma(n^*)-1} p(s) \geq \beta > \frac{\beta}{2},$$

which contradicts (2.10). Thus, in both cases, we have $\sigma(n^*) \geq n+2$. Further, combining inequalities (2.8) and (2.9), we get

$$\sum_{s=n+1}^{\sigma(n^*)-1} p(s) = \sum_{s=n^*+1}^{\sigma(n^*)-1} p(s) - \sum_{s=n^*+1}^{n} p(s) \geq \beta - \frac{\beta}{2} = \frac{\beta}{2}. \tag{2.11}$$

Summing up (E) from n^* to n and taking into account the fact that $\{u(n)\}$ is increasing, we get

$$\begin{aligned} u(n+1) - u(n^*) &= \sum_{s=n^*}^{n} p(s)u(\sigma(s)) \\ &\geq \left(\sum_{s=n^*}^{n} p(s)\right) u(\sigma(n^*)) \geq \frac{\beta}{2} u(\sigma(n^*)), \end{aligned}$$

i.e.,

$$\frac{\beta}{2} u(\sigma(n^*)) \leq u(n+1). \tag{2.12}$$

Summing up (1.1) from $n+1$ to $\sigma(n^*)-1$ and using (2.11) yields

$$\begin{aligned} u(\sigma(n^*)) - u(n+1) &= \sum_{s=n+1}^{\sigma(n^*)-1} p(s)u(\sigma(s)) \\ &\geq \left(\sum_{s=n+1}^{\sigma(n^*)-1} p(s)\right) u(\sigma(n+1)) \geq \frac{\beta}{2} u(\sigma(n)), \end{aligned}$$

i.e.,

$$\frac{\beta}{2} u(\sigma(n)) \leq u(\sigma(n^*)). \tag{2.13}$$

Combining (2.12) and (2.13), we obtain

$$\left(\frac{\beta}{2}\right)^2 u(\sigma(n)) \leq u(n+1),$$

or

$$\frac{u(\sigma(n))}{u(n+1)} \leq \frac{4}{\beta^2}.$$

In view of (2.7), the above inequality takes the form

$$e^{m\beta} \leq \frac{u(\sigma(n))}{u(n+1)} \leq \frac{4}{\beta^2},\ m = 1, 2, \ldots$$

which is a contradiction as $m \to \infty$. The proof of the theorem is complete. $\square$

Example 1. *Consider the first-order advanced difference equation*

$$\Delta u(n) - \frac{1}{n}u(2n+1) = 0, \ n \geq 2. \tag{2.14}$$

Clearly, (2.14) is of type (1.1) with $p(n) = \frac{1}{n}$ and $\sigma(n) = 2n+1$.

Since $\frac{1}{n}$ is decreasing, and taking into account the fact that

$$\int_{b-1}^{b} f(x)dx \geq f(b) \geq \int_{b}^{b+1} f(x)dx,$$

where $f(x)$ is a decreasing positive function, we have

$$\begin{aligned}\sum_{s=n+1}^{2n} \frac{1}{s} &\geq \sum_{s=n+1}^{2n} \int_{s}^{s+1} \frac{dx}{x} \\ &= \sum_{s=n+1}^{2n} \ln\left(\frac{s+1}{s}\right) = \ln\left(\frac{2n+1}{n+1}\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{s=n+1}^{2n} \frac{1}{s} &\leq \sum_{s=n+1}^{2n} \int_{s-1}^{s} \frac{dx}{x} \\ &= \sum_{s=n+1}^{2n} \ln\left(\frac{s}{s-1}\right) = \ln\left(\frac{2n}{n}\right).\end{aligned}$$

Thus

$$\ln\left(\frac{2n+1}{n+1}\right) \leq \sum_{s=n+1}^{2n} \frac{1}{s} \leq \ln\left(\frac{2n}{n}\right),$$

which means that

$$\lim_{n\to\infty} \sum_{s=n+1}^{2n} \frac{1}{s} = \ln 2.$$

Therefore

$$\alpha := \liminf_{n\to\infty} \sum_{s=n+1}^{2n} \frac{1}{s} = \ln 2 > \frac{1}{e},$$

that is condition (2.1) of Theorem 2.1 is satisfied, and therefore all solutions of (2.14) are oscillatory.

Observe, however, that

$$\begin{aligned}\sum_{s=n}^{2n}\frac{1}{s} &\geq \sum_{s=n}^{2n}\int_{s}^{s+1}\frac{dx}{x}\\ &= \sum_{s=n}^{2n}\ln\left(\frac{s+1}{s}\right)=\ln\left(\frac{2n+1}{n}\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{s=n}^{2n}\frac{1}{s} &\leq \sum_{s=n}^{2n}\int_{s-1}^{s}\frac{dx}{x}\\ &= \sum_{s=n}^{2n}\ln\left(\frac{s}{s-1}\right)=\ln\left(\frac{2n}{n-1}\right).\end{aligned}$$

Thus

$$\ln\left(\frac{2n+1}{n}\right)\leq\sum_{s=n}^{2n}\frac{1}{s}\leq\ln\left(\frac{2n}{n-1}\right),$$

which means that

$$\lim_{n\to\infty}\sum_{s=n}^{2n}\frac{1}{s}=\ln 2.$$

Therefore

$$\limsup_{n\to\infty}\sum_{s=n}^{2n}\frac{1}{s}=\ln 2\leq 1$$

and

$$\begin{aligned}\limsup_{n\to\infty}\sum_{s=n}^{2n}\frac{1}{s} &= \ln 2<1-\left(1-\sqrt{1-\alpha}\right)^2\\ &= 1-\left(1-\sqrt{1-\ln 2}\right)^2\simeq 0.801.\end{aligned}$$

That is, none of the conditions (2.1) and (2.2) is satisfied.

3. Conclusion

In this paper, we have obtained a new condition for the oscillation of advanced difference equation (1.1). Our result is different from the existing oscillation criteria in the sense that it is easy to apply than the one already obtained in [1, 2, 9].

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