ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **10** (2021), no.2, 981–990
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.2.27

CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS INVOLVING TOUCHARD POLYNOMIALS

T. Soupramanien[1], C. Ramachandran, and Khalifa Al-Shaqsi

ABSTRACT. This work provides an extraordinary view of establishing the connections in between the different set of subclasses that exists in the univalent analytic functions, thereby utilizing some unique operator related to the convolutions that involves the Touchard polynomials. Exactly saying the analytic univalent function classes along with the positive coefficients in the open unit disk $\mathbb{U}$ are investigated along with those connections.

1. INTRODUCTION AND DEFINITIONS

The application of special function on Geometric Function Theory is a current and interesting topic of research. It is frequently applied in various branches Mathematics, Physics, Sciences, Engineering and Technology. The impressing application of the common oriented hypergeometric functions proposed by L. de Branges [5] provides the solution for the well known Bieberbach conjecture. The expanded literature discussion arises here along with the analytical in addition with the geometric properties that consists of different kinds of unique

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Univalent functions, Starlike functions, Convex functions, Hadamard product, Touchard Polynomials.

Submitted: 07.02.2021; *Accepted:* 22.02.2021; *Published:* 26.02.2021.

functions that exists specially, particularly for the common approach related to the Gaussian hypergeometric functions [4, 9, 10, 14, 16].

The Touchard polynomials that is analysed and reviewed by Jacques Touchard [17], is known to be the exponential generating polynomials (see [2, 13, 15]) which is also called as Bell polynomials (see [1]) that comprised of a polynomial sequence with sub categorised binomial type for X. This is considered to be a Poisson distribution having a random variable along with the desired value l, hence the nth instant is $E(X_\kappa^n) = \mathcal{T}(\kappa, l)$, predominant to the form:

$$\mathcal{T}(\kappa, l) = e^{\kappa} \sum_{n=0}^{\infty} \frac{\kappa^n n^l}{n!}. \tag{1.1}$$

The coefficients of the Touchard polynomials that exists subsequent the second force is given as below.

Lately, introduce Touchard polynomials coefficients following the second force is given as

$$\Phi(l, \kappa, z) = z + \sum_{n=2}^{\infty} \frac{(n-1)^l \kappa^{n-1}}{(n-1)!} e^{-\kappa} z^n \qquad z \in \mathbb{D}, \tag{1.2}$$

where $l \geq 0$, $\kappa > 0$, and it is noted that the radius of convergence of above series is infinity when a ratio test is conducted. The class of the analytic functions in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ is considered to be $\mathcal{H}$.

It is assumed that $\mathcal{A}$ is considered to be the class of functions $f \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{U}. \tag{1.3}$$

It is also considered that $\mathcal{S}$ be the subclass of $\mathcal{A}$ which consists of other functions that are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in $\mathbb{U}$.

Denoted by $\mathcal{V}$ the subclass of $\mathcal{A}$ that consists of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geq 0.$$

For functions $f \in \mathcal{A}$ given by (1.3) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution)is defined as that of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \qquad z \in \mathbb{U}.$$

Then the linear operator is defined as

$$\mathcal{I}(l, m, z) : \mathcal{A} \to \mathcal{A}$$

and is defined by the hadamard product or convolution

$$\mathcal{I}(l, m, z)f = \Phi(l, m, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(n-1)^l m^{n-1}}{(n-1)!} e^{-m} a_n z^n,$$

where $\Phi_m^l(z)$ is the series given by (1.2).

The class $\mathcal{M}(\alpha)$ of starlike functions of order $1 < \alpha \leq \frac{4}{3}$:

$$\mathcal{M}(\alpha) = \left\{ f \in \mathcal{A} : \Re \left(\frac{zf'(z)}{f(z)} \right) < \alpha, \quad z \in \mathbb{U} \right\}$$

and the class $\mathcal{N}(\alpha)$ of convex functions of order $1 < \alpha \leq \frac{4}{3}$:

$$\mathcal{N}(\alpha) = \left\{ f \in \mathcal{A} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) < \alpha, \quad z \in \mathbb{U} \right\} = \left\{ f \in \mathcal{A} : zf' \in \mathcal{M}(\alpha) \right\},$$

were identified and discussed by Uralegaddi et al. [18] (see [6, 8]). Also let $\mathcal{M}^*(\alpha) \equiv \mathcal{M}(\alpha) \cap \mathcal{V}$ and $\mathcal{N}^*(\alpha) \equiv \mathcal{N}(\alpha) \cap \mathcal{V}$.

Couple of two novel subclasses of $\mathcal{S}$ namely $\mathcal{S}_P(\lambda, \alpha, \beta)$ and $\mathcal{UCV}(\lambda, \alpha, \beta)$ are introduced in this work for discussing with some of the inclusion properties.

For $0 \leq \lambda < 1$, $0 \leq \alpha < 1$ and $\beta \geq 0$, we let $\mathcal{S}_P(\lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}$ which consists of certain functions of the form (1.1) thereby satisfying the analytic criterion

$$\Re \left\{ \frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - \alpha \right\} > \beta \left| \frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1 \right|, \qquad z \in \mathbb{U},$$

and also, let $\mathcal{UCV}(\lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\Re \left\{ \frac{f'(z) + f''(z)}{f'(z) + \lambda zf''(z)} - \alpha \right\} > \beta \left| \frac{f'(z) + f''(z)}{f'(z) + \lambda zf''(z)} - 1 \right|, \qquad z \in \mathbb{U},$$

We also let $\mathcal{S}_P^*(\lambda, \alpha, \beta) \equiv \mathcal{S}_P(\lambda, \alpha, \beta) \cap \mathcal{V}$ and $\mathcal{UCV}^*(\lambda, \alpha, \beta) \equiv \mathcal{UCV}(\lambda, \alpha, \beta) \cap \mathcal{V}$.

Note that $\mathcal{S}_P(\lambda,\alpha,0) = \mathcal{M}(\lambda,\alpha)$, $\mathcal{UCV}(\lambda,\alpha,0) = \mathcal{N}(\lambda,\alpha)$; $\mathcal{M}^*(\lambda,\alpha)$, $\mathcal{N}^*(\lambda,\alpha)$ the subclasses are studied by Murugusundaramoorthy [11]. Also $\mathcal{S}_P(0,\alpha,0) = \mathcal{M}(\alpha)$, $\mathcal{UCV}(0,\alpha,0) = \mathcal{N}(\alpha)$; $\mathcal{M}^*(\alpha)$, $\mathcal{N}^*(\alpha)$ the subclasses are studied by Uralegaddi et al. [18].

The obtained results reveals that there exists an connection between the different subclasses of the analytic univalent functions with the help of hypergeometric functions (see [4, 9, 10, 14, 16]), the most needed and generally required conditions are obtained for the functions $\Phi(l,m,z)$ to be in the classes $\mathcal{S}_P(\lambda,\alpha,\beta)$, $\mathcal{UCV}(\lambda,\alpha,\beta)$ and the connections that exists in between $\mathcal{R}^\tau(A,B)$ by the means of applying convolution operator.

2. Preliminary Results

For proving our main results the following definitions and lemmas are mandatory.

Definition 2.1. *The l^{th} moment of the Poisson distribution is defined as*

$$\mu_l' = \sum_{n=0}^{\infty} \frac{n^l m^n}{n!} e^{-m}.$$

Lemma 2.1. *For $0 \le \lambda < 1$, $0 \le \alpha < 1$ and $\beta \ge 0$, and if $f \in \mathcal{V}$ then $f \in \mathcal{S}_P^*(\lambda,\alpha,\beta)$ satisfies*

$$\sum_{n=2}^{\infty} \left[n(1+\beta) - (\alpha+\beta)(1+n\lambda-\lambda)\right] |a_n| \le 1-\alpha.$$

Lemma 2.2. *For $0 \le \lambda < 1$, $0 \le \alpha < 1$ and $\beta \ge 0$, and if $f \in \mathcal{V}$ then $f \in \mathcal{UCV}^*(\lambda,\alpha,\beta)$ satisfies*

$$\sum_{n=2}^{\infty} n\left[n(1+\beta) - (\alpha+\beta)(1+n\lambda-\lambda)\right] |a_n| \le 1-\alpha.$$

3. Main Results

For convenience throughout in the sequel, we use the following notations:

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!} &= e^m-1,\\ \sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!} &= me^m,\\ \sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-3)!} &= m^2e^m.\end{aligned}$$

Theorem 3.1. *If* $m>0$ $(m\neq 0,-1,-2,\dots)$*,* $l\in\mathbb{N}_0$*, then* $\Phi(l,m,z)\in\mathcal{S}_P^*(\lambda,\alpha,\beta)$ *satisfies*

$$\begin{cases}(1+\beta-\lambda(\alpha+\beta))\mu'_{l+1}+(1-\alpha)\mu'_l\le 1-\alpha, & \textit{if } l\ge 1,\\ (1+\beta-\lambda(\alpha+\beta))me^m\le 1-\alpha, & \textit{if } l=0.\end{cases}\tag{3.1}$$

Proof. To prove that $\Phi(l,m,z)\in\mathcal{S}_P^*(\lambda,\alpha,\beta)$, then by virtue of Lemma 2.1, it suffices to show that

$$\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\right]\frac{(n-1)^l m^{n-1}}{(n-1)!}e^{-m}\le 1-\alpha.$$

Now,

$$\begin{aligned}&e^{-m}\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\right]\frac{(n-1)^l m^{n-1}}{(n-1)!}\\ &= e^{-m}\sum_{n=2}^{\infty}\left[n(1+\beta-\lambda(\alpha+\beta))-(\alpha+\beta)(1-\lambda)\right]\frac{(n-1)^l m^{n-1}}{(n-1)!}\\ &= e^{-m}\sum_{n=2}^{\infty}\left[(n-1)(1+\beta-\lambda(\alpha+\beta))+(1-\alpha)\right]\frac{(n-1)^l m^{n-1}}{(n-1)!}\\ &= e^{-m}\sum_{n=1}^{\infty}\left[n(1+\beta-\lambda(\alpha+\beta))+(1-\alpha)\right]\frac{n^l m^n}{n!}\\ &= e^{-m}\sum_{n=1}^{\infty}\left[(1+\beta-\lambda(\alpha+\beta))\frac{n^{l+1}m^n}{n!}+(1-\alpha)\frac{n^l m^n}{n!}\right]\\ &= \begin{cases}(1+\beta-\lambda(\alpha+\beta))\mu'_{l+1}+(1-\alpha)\mu'_l, & \text{if } l\ge 1\\ (1+\beta-\lambda(\alpha+\beta))m+(1-\alpha)(1-e^{-m}), & \text{if } l=0\end{cases}.\end{aligned}$$

But this expression is bounded above by $1-\alpha$ only if (3.1) holds. Thus the proof is complete. $\square$

Theorem 3.2. *If* $m>0$ $(m\neq 0,-1,-2,\cdots)$, $l\in\mathbb{N}_0$ *then* $\Phi(l,m,z)\in\mathcal{UCV}^*(\lambda,\alpha,\beta)$ *satisfies*

(3.2)
$$\begin{cases}(1+\beta-\lambda(\alpha+\beta))\mu'_{l+2}+(2-\alpha+\beta-\lambda(\alpha+\beta))\mu'_{l+1}+(1-\alpha)\mu'_l, & \text{if } l\geq 1\\ (1+\beta-\lambda(\alpha+\beta))m(m+1)e^m+(2-\alpha+\beta-\lambda(\alpha+\beta))me^m\leq 1-\alpha, & \text{if } l=0\end{cases}$$

Proof. To prove that $\Phi(l,m,z)\in\mathcal{UCV}^*(\lambda,\alpha,\beta)$, then by virtue of Lemma 2.2, it suffices to show that

$$\sum_{n=2}^{\infty} n\left[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\right]\frac{(n-1)^l m^{n-1}}{(n-1)!}e^{-m}\leq 1-\alpha.$$

Now,

$$\begin{aligned}
&e^{-m}\sum_{n=2}^{\infty} n\left[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\right]\frac{(n-1)^l m^{n-1}}{(n-1)!}\\
&=e^{-m}\sum_{n=2}^{\infty} n\left[n(1+\beta-\lambda(\alpha+\beta))-(\alpha+\beta)(1-\lambda)\right]\frac{(n-1)^l m^{n-1}}{(n-1)!}\\
&=e^{-m}\sum_{n=2}^{\infty}\big[(n-1)^2(1+\beta-\lambda(\alpha+\beta))+(n-1)(2-\alpha+\beta-\lambda(\alpha+\beta))\\
&\quad+(1-\alpha)\big]\frac{(n-1)^l m^{n-1}}{(n-1)!}\\
&=e^{-m}\sum_{n=1}^{\infty}\big[n^2(1+\beta-\lambda(\alpha+\beta))+n(2-\alpha+\beta-\lambda(\alpha+\beta))+(1-\alpha)\big]\frac{n^l m^n}{n!}\\
&=e^{-m}\sum_{n=1}^{\infty}\Big[(1+\beta-\lambda(\alpha+\beta))\frac{n^{l+2}m^n}{n!}+(2-\alpha+\beta-\lambda(\alpha+\beta))\frac{n^{l+1}m^n}{n!}\\
&\quad+(1-\alpha)\frac{n^l m^n}{n!}\Big]\\
&=\begin{cases}(1+\beta-\lambda(\alpha+\beta))\mu'_{l+2}+(2-\alpha+\beta-\lambda(\alpha+\beta))\mu'_{l+1}\\ \qquad+(1-\alpha)\mu'_l, \quad \text{if } l\geq 1\\ (1+\beta-\lambda(\alpha+\beta))m(m+1)+(2-\alpha+\beta-\lambda(\alpha+\beta))m\\ \qquad+(1-\alpha)(1-e^{-m}), \quad \text{if } l=0.\end{cases}
\end{aligned}$$

But this expression is bounded above by $1-\alpha$ only if (3.2) holds. Thus the proof is complete. $\square$

4. Inclusion Properties

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A,B)$, $\tau \in \mathbb{C}\backslash\{0\}$, $-1 \leq B < A \leq 1$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau - B[f'(z)-1]}\right| < 1, \qquad z \in \mathbb{U}.$$

This class was introduced by Dixit and Pal [7].

It is of interest to note that if

$$\tau = 1, \quad A = \beta \quad \text{and} \quad B = -\beta \; (0 < \beta \leq 1),$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \beta \qquad z \in \mathbb{U},$$

which was studied by (among others) Padmanabhan [12] and Caplinger and Causey [3].

Lemma 4.1. *[7] If $f \in \mathcal{R}^{\tau}(A,B)$ is of the form* (1.3)*, then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \qquad n \in \mathbb{N}\backslash\{1\}.$$

The result is sharp for the function

$$f(z) = \int_0^z \left(1 + (A-B)\frac{\tau t^{n-1}}{1+Bt^{n-1}}\right) dt, \qquad z \in \mathbb{U};\; n \in \mathbb{N}\backslash\{1\}.$$

Making use of the Lemma 4.1 we will study the action of the Poissons distribution series on the class $\mathcal{UCV}^*(\lambda,\alpha,\beta)$

Theorem 4.1. *Let $m > 0$ $(m \neq 0,-1,-2,\cdots)$, $l \in \mathbb{N}_0$. If $f \in \mathcal{R}^{\tau}(A,B)$, then $\mathcal{I}(l,m,z)f$ is in $\mathcal{UCV}^*(\lambda,\alpha,\beta)$ satisfies*

$$(A-B)\,|\tau| \begin{cases} (1+\beta-\lambda(\alpha+\beta))\mu'_{l+1} + (1-\alpha)\mu'_l \leq 1-\alpha, & \text{if } l \geq 1 \\ (1+\beta-\lambda(\alpha+\beta))me^m \leq 1-\alpha, & \text{if } l = 0 \end{cases}. \tag{4.1}$$

Proof. Let $f(z)$ be of the form (1.3) belong to the class $\mathcal{R}^{\tau}(A,B)$. By virtue of Lemma 2.2, it suffices to show that

$$\sum_{n=2}^{\infty} n\left[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\right]\frac{(n-1)^l m^{n-1}}{(n-1)!}e^{-m}|a_n| \leq 1-\alpha.$$

Since $f \in \mathcal{R}^{\tau}(A,B)$, then by Lemma 4.1, we have

$$|a_n| \leq \frac{(A-B)\,|\tau|}{n}.$$

Thus, we have

$$\begin{aligned}
&e^{-m}\sum_{n=2}^{\infty} n\left[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\right]\frac{(n-1)^l m^{n-1}}{(n-1)!}|a_n| \\
&\leq \quad (A-B)|\tau|e^{-m}\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\right]\frac{(n-1)^l m^{n-1}}{(n-1)!} \\
&= \quad (A-B)|\tau|e^{-m}\sum_{n=2}^{\infty}\left[(n-1)(1+\beta-\lambda(\alpha+\beta))+(1-\alpha)\right]\frac{(n-1)^l m^{n-1}}{(n-1)!} \\
&= \quad (A-B)|\tau|e^{-m}\sum_{n=1}^{\infty}\left[n(1+\beta-\lambda(\alpha+\beta))+(1-\alpha)\right]\frac{n^l m^n}{n!} \\
&= \quad (A-B)|\tau|e^{-m}\sum_{n=1}^{\infty}\left[(1+\beta-\lambda(\alpha+\beta))\frac{n^{l+1}m^n}{n!}+(1-\alpha)\frac{n^l m^n}{n!}\right] \\
&= \quad (A-B)|\tau|\begin{cases}(1+\beta-\lambda(\alpha+\beta))\mu'_{l+1}+(1-\alpha)\mu'_l, & \text{if } l \geq 1\\ (1+\beta-\lambda(\alpha+\beta))m+(1-\alpha)(1-e^{-m}), & \text{if } l=0\end{cases}
\end{aligned}$$

But this last expression is bounded by $1-\alpha$, if (4.1) holds. This completes the proof of Theorem 4.1. □

Theorem 4.2. *Let* $m>0$ $(m \neq 0,-1,-2,\cdots)$, $l \in \mathbb{N}_0$, *then the integral operator*

$$\mathcal{L}(l,m,z)=\int_0^z \frac{\mathcal{I}(l,m,t)}{t}dt$$

is in $\mathcal{UCV}^*(\lambda,\alpha,\beta)$ *if inequality* (3.1) *is satisfied.*

Proof. Since

$$\mathcal{L}(l,m,z)=z+\sum_{n=2}^{\infty}\frac{(n-1)^l m^{n-1}}{(n-1)!}e^{-m}\frac{z^n}{n}$$

By virtue of Lemma 2.2, we need only to show that

$$\sum_{n=2}^{\infty} n\left[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\right]\frac{(n-1)^l m^{n-1}}{n(n-1)!}e^{-m} \leq 1-\alpha.$$

or, equivalently

$$\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)(1+n\lambda-\lambda)\right]\frac{(n-1)^l m^{n-1}}{(n-1)!}e^{-m} \leq 1-\alpha.$$

The remaining part of the proof of Theorem 4.2 is similar to that of Theorem 3.1, and so we omit the details. □

Remark 4.1. *For $\mu = 0$, the Theorems 3.1 - 4.2 which correspond the results very recently reached by Murugusundaramoorthy and Saurabh Porwal [11].*

Remark 4.2. *For $\mu = 0$ and $\lambda = 0$, the Theorems 3.1 - 4.2 which correspond the results reached by Uralegaddi et al. [18].*

REFERENCES

[1] K. AL-SHAQSI: *On inclusion results of certain subclasses of analytic functions associated with generating function*, AIP Conf. Proc., **1830**(1) (2017), 1–6.

[2] K. N. BOYADZHIEV: *Exponential polynomials*, Stirling numbers, and evaluation of some gamma integrals, Abstr. Appl. Anal. **2009**, Art. ID 168672, 18 pp.

[3] T. R. CAPLINGER, W. M. CAUSEY: *A class of univalent functions*, Proc. Amer. Math. Soc., **39** (1973), 357–361.

[4] N. E. CHO, S. Y. WOO, S. OWA: *Uniform convexity properties for hypergeometric functions*, Fract. Calc. Appl. Anal., **5**(3) (2002), 303–313.

[5] L. DE BRANGES: *A proof of the Bieberbach conjecture*, Acta Math., **154**(1-2) (1985), 137–152.

[6] K. K. DIXIT, V. CHANDRA: *On subclass of univalent functions with positive coefficients*, Aligarh Bull. Math., **27**(2) (2008), 87–93.

[7] K. K. DIXIT, S. K. PAL: *On a class of univalent functions related to complex order*, Indian J. Pure Appl. Math., **26**(9) (1995), 889–896.

[8] K. K. DIXIT, A. L. PATHAK: *A new class of analytic functions with positive coefficients*, Indian J. Pure Appl. Math., **34**(2) (2003), 209–218.

[9] E. P. MERKES, W. T. SCOTT: *Starlike hypergeometric functions*, Proc. Amer. Math. Soc., **12** (1961), 885–888.

[10] A. O. MOSTAFA: *A study on starlike and convex properties for hypergeometric functions*, J. Inequal. Pure Appl. Math., **10**(3) (2009), Article 87, 8 pp.

[11] G. MURUGUSUNDARAMOORTHY, S. PORWAL: *Univalent Functions with Positive Coefficients Involving Touchard Polynomials*, arXiv:2007.05439v1

[12] K. S. PADMANABHAN: *On a certain class of functions whose derivatives have a positive real part in the unit disc*, Ann. Polon. Math. **23** (1970/71), 73–81.

[13] S. ROMAN: *The umbral calculus*, Pure and Applied Mathematics, 111, Academic Press, Inc., New York, 1984.

[14] H. SILVERMAN: *Starlike and convexity properties for hypergeometric functions*, J. Math. Anal. Appl., **172**(2) (1993), 574–581.

[15] Y. SIMSEK: *Special Numbers on Analytic Functions*, Applied Mathematics, **5** (2014), 1091–1098.

[16] H. M. SRIVASTAVA, G. MURUGUSUNDARAMOORTHY, S. SIVASUBRAMANIAN: *Hypergeometric functions in the parabolic starlike and uniformly convex domains*, Integral Transforms Spec. Funct., **18**(7-8) (2007), 511–520.

[17] J. TOUCHARD: *Sur les cycles des substitutions*, Acta Math., **70**(1) (1939), 243–297.

[18] B. A. URALEGADDI, M. D. GANIGI, S. M. SARANGI: *Univalent functions with positive coefficients*, Tamkang J. Math., **25**(3) (1994), 225–230.

DEPARTMENT OF MATHEMATICS
IFET COLLEGE OF ENGINEERING,
GANGARAMPALAYAM, VILLUPURAM - 605 108, TAMILNADU, INDIA
Email address: soupramani@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE OF ENGINEERING VILLUPURAM, ANNA UNIVERSITY
VILLUPURAM - 605 602, INDIA
Email address: crjsp2004@yahoo.com

DEPARTMENT OF INFORMATION TECHNOLOGY
NIZWA COLLEGE OF TECHNOLOGY
MINISTRY OF MANPOWER, NIZWA, OMAN
Email address: khalifa.alshaqsi@nct.edu.om